An extension of a theorem of Zermelo

Jouko Väänänen
100th anniversary of the death of Cantor

Georg Cantor 1845 (Saint Petersburg) — 1918 (Halle).
• Second order logic has great power in characterizing categorically mathematical structures: natural numbers, real numbers, cumulative hierarchy of sets up to the first inaccessible, etc.

• Which structures are second order characterizable?

• For which $\mathcal{A}$ is there a second order $\theta$ such that for all $\mathcal{B}$:

$$\mathcal{A} \cong \mathcal{B} \iff \mathcal{B} \models \theta$$
Sometimes infinitary second order logic can characterize “all” models.

Theorem (Hyttinen-Kangas-V. 2013)

Let $T$ be a countable complete first order theory and $\kappa$ an uncountable cardinal with certain not too uncommon properties. Then the following are equivalent:

1. Every model of $T$ of size $\kappa$ is $L^{2}_{\kappa, \omega}$-characterizable.
2. $T$ is superstable, shallow, without DOP or OTOP.

---

$^1$A regular cardinal such that $\kappa = \aleph_\alpha$, $\beth_1(|\alpha| + \omega) \leq \kappa$ and $2^\lambda < 2^\kappa$ for all $\lambda < \kappa$. 
Theorem (V. 2011)

1. If a model is second order characterizable, its isomorphism class is $\Delta_2$-definable in set theory.

2. A model class is second order definable\(^2\) if and only if it is $\Delta_2$-definable in set theory.

\(^2\)More exactly, second order $\Delta$-definable.
A hierarchy of second order characterizable models
Theorem (V. 2011)

1. Second order validity is \( \Pi_2 \)-complete in set theory.
2. The second order theory of a second order characterizable structure is always \( \Delta_2 \) in set theory.

Corollary

Second order validity cannot be second order defined in any second order characterizable structure.

This is a critique of structuralism as I understand that view.
Definition (Zermelo 1930)

1. The Second Order Zermelo-Fraenkel axioms $\text{ZFC}^2$ are as the first order one except that the Separation and Replacement Schemas are replaced by single second order axioms.

2. The Second Order Peano axioms $\text{P}^2$ are as the first order Peano axioms except that the Induction Schema is replaced by a single second order axiom.
Theorem (Zermelo 1930)

$(M, E)$ satisfies the second order Zermelo-Fraenkel axioms $ZFC^2$ if and only if $(M, E) \cong (V_\kappa, \in)$ for some strongly inaccessible $\kappa$.

Proof.

Zermelo shows that if $(M, E) \models ZFC^2$ and $\kappa$ is the smallest ordinal not “represented" in $(M, E)$, then $(M, E) \cong (V_\kappa, \in)$. □

Corollary

If $(M, \in_1)$ and $(M, \in_2)$ both satisfy the second order Zermelo-Fraenkel axioms $ZFC^2$, then $(M, \in_1) \cong (M, \in_2)$. 
Theorem (Dedekind 1888)

If $\langle M_1, +_1, \times_1 \rangle$ and $\langle M_2, +_2, \times_2 \rangle$ both satisfy the second order Peano axioms $P^2$, then $\langle M_1, +_1, \times_1 \rangle \simeq \langle M_2, +_2, \times_2 \rangle$.

Proof.
Dedekind essentially argues that if $\langle M, +, \times \rangle \models P^2$, then $\langle M, +, \times \rangle$ is isomorphic with the standard model $\langle \mathbb{N}, +, \times \rangle$. □
• Let us consider the vocabulary \{\in_1, \in_2\}, where both \in_1 and \in_2 are binary predicate symbols.

• \textit{ZFC}(\in_1) is the first order Zermelo-Fraenkel axioms of set theory when \in_1 is the membership relation and formulas are allowed to contain \in_2, too.

• \textit{ZFC}(\in_2) is the first order Zermelo-Fraenkel axioms of set theory when \in_2 is the membership relation and formulas are allowed to contain \in_1, too.
Theorem (V. 2018)

If $(M, \in_1, \in_2) \models ZFC(\in_1) \cup ZFC(\in_2)$, then $(M, \in_1) \cong (M, \in_2)$. 
Proof of the Theorem

- We work in \( \text{ZFC}(\in_1) \cup \text{ZFC}(\in_2) \).
- We alternate between \( \in_1 \)-set theory and \( \in_2 \)-set theory.
Proof of the Theorem

- Let $\text{tr}_i(x)$ be the formula $\forall t \in_i x \forall w \in_i t(w \in_i x)$. It says that $x$ is transitive in $\in_i$-set theory.

- Let $\text{TC}_i(x)$ be the unique $u$ such that $\text{tr}_i(u) \land x \in_i u \land \forall v((\text{tr}_i(v) \land x \in_i v) \rightarrow \forall w \in_i u(w \in_i v))$ (i.e. “$u$ is the $\in_i$-transitive closure of $x$”).

- Let $\varphi(x, y)$ be the formula $\exists f \psi(x, y, f)$, where $\psi(x, y, f)$ is the conjunction of the following formulas (where $f(t)$ and $f(w)$ are understood in the sense of $\in_1$):
Proof of the Theorem

\[ \psi(x, y, f) : \]

(1) In the sense of \( \mathcal{E}_1 \), the set \( f \) is a function with \( \text{TC}_1(x) \) as its domain.

(2) \( \forall t \in_1 \text{TC}_1(x)(f(t) \in_2 \text{TC}_2(y)) \)

(3) \( \forall t \in_2 \text{TC}_2(y) \exists \, w \in_1 \text{TC}_1(x)(t = f(w)) \)

(4) \( \forall t \in_1 \text{TC}_1(x) \forall \, w \in_1 \text{TC}_1(x)(t \in_1 w \leftrightarrow f(t) \in_2 f(w)) \)

(5) \( f(x) = y \)
Proof of the Theorem

Lemma

If $\psi(x, y, f)$ and $\psi(x, y, f')$, then $f = f'$.

Proof:
Lemma

If $\psi(x, y, f)$ and $x' \in_1 x$, then $\varphi(x', f(x'))$.

Proof:

![Diagram with points and arrows representing the proof steps.]
Proof of the Theorem

Lemma
If \( \psi(x, y, f) \) and \( y' \in y \), then there is \( x' \in x \) such that \( f(x') = y' \) and \( \varphi(x', y') \).

Proof:
Proof of the Theorem

Lemma

If $\varphi(x, y)$ and $\varphi(x, y')$, then $y = y'$.

Proof:
Proof of the Theorem

Lemma

If $\varphi(x, y)$ and $\varphi(x', y)$, then $x = x'$.

Proof:
Proof of the Theorem

Lemma

If $\varphi(x, y)$ and $\varphi(x', y')$, then $x' \in_1 x \leftrightarrow y' \in_2 y$.
Proof of the Theorem

- Let $\text{On}_1(x)$ be the $\in_1$-formula saying that $x$ is an ordinal i.e. a transitive set of transitive sets, and similarly $\text{On}_2(x)$.
- For $\text{On}_1(\alpha)$ let $V^1_\alpha$ be the $\alpha^{th}$ level of the cumulative hierarchy in the sense of $\in_1$, and similarly $V^2_\alpha$. 
Proof of the Theorem

Lemma

1. If \( \varphi(\alpha, y) \), then \( \text{On}_1(\alpha) \) if and only if \( \text{On}_2(y) \).

2. If \( \alpha \) is a limit ordinal then so is \( y \) i.e. if 
   \[ \forall u \in_1 \alpha \exists v \in_1 \alpha (u \in_1 v), \text{ then } \forall u \in_2 y \exists v \in_2 y(u \in_2 v). \]

3. Also vice versa.
Proof of the Theorem

Lemma

Suppose $\psi(\alpha, y, f)$. If $\text{On}_1(\alpha)$ (or equivalently $\text{On}_2(y)$), then there is $\bar{f} \supseteq f$ such that $\psi(V_1^1, V_2^2, \bar{f})$. 

\[ \begin{align*}
V_1^1 & \quad \text{and} \quad V_2^2 \\
\downarrow & \quad \bar{f} \\
& \quad \downarrow
\end{align*} \]
Proof of the Theorem

Lemma
\( \forall x \exists y \varphi(x, y) \) and \( \forall y \exists x \varphi(x, y) \).

Proof: Consider

\[
\forall \alpha (\text{On}_1(\alpha) \rightarrow \exists y \varphi(\alpha, y)) \tag{1}
\]

\[
\forall y (\text{On}_2(y) \rightarrow \exists \alpha \varphi(\alpha, y)). \tag{2}
\]

Case 1: \((1) \land (2)\). The claim can be proved.

Case 2: \(\neg(1) \land \neg(2)\). Impossible!

Case 3: \((1) \land \neg(2)\). Impossible!

Case 4: \(\neg(1) \land (2)\). Impossible!
$O_{n_\alpha}^1 \quad O_{n_\alpha}^2$

QED

Impossible

Impossible

Impossible
Proof of the Theorem

Lemma
The class defined by $\varphi(x, y)$ is an isomorphism between the $\in_1$-reduct and the $\in_2$-reduct.

Proof.
By the previous Lemmas.

The Theorem is proved.
Theorem
If $(M, +_1, \times_1, +_2, \times_2) \models P(+_1, \times_1) \cup P(+_2, \times_2)$, then $(M, +_1, \times_1) \cong (M, +_2, \times_2)$.

Proof.
(Sketch) Let 0₁ and 1₁ be the first elements of $(M, +_1, \times_1)$, and respectively 0₂, 1₂. Let $\psi(x, u, v)$ say that $x$ codes, using $+_1$ and $\times_1$, an initial segment $I$ with the last element $u$, of $(M, +_1, \times_1)$, an initial segment $I'$ with the last element $v$, of $(M, +_2, \times_2)$, and a function $f : I \to I'$ such that $f(0₁) = 0₂$, $f(y +_1 1₁) = f(y) +_2 1₂$ for all $y \in I \setminus \{u\}$, and $f(u) = v$. Let $\varphi(u, v)$ be the formula $\exists x \psi(x, u, v)$. This formula defines the desired isomorphism. \qed
D. Martin 2018 (unpublished) argues informally that if $(M_1, \in_1)$ and $(M_2, \in_2)$ both satisfy the axioms of set theory with informal Full Comprehension and their classes of ordinals are isomorphic, then $(M, \in_1) \cong (M, \in_2)$.

Our result is a formalization of Martin’s informal result. We need not assume that the ordinals are isomorphic.
• Early researchers (Dedekind, Frege, Russell, Hilbert, Zermelo, Gödel, Mostowski) axiomatized mathematics using second order logic or its extension simple theory of types.

• Then ZFC emerged as a first order theory.

• Later philosophers (e.g. S. Shapiro) claimed second order logic would be better (can characterize mathematical structures) and first order logic is flawed (cannot characterize mathematical structures).

• I suggest: Zermelo’s and Dedekind’s second order categoricity results are actually first order at heart.

• The difference between second order logic or first order set theory is not as clear as what was previously thought.
Second order logic is praised for its categoricity results, i.e. its ability to characterize structures.

But what is universal second order truth — a problem!

Best understood in terms of provability i.e. truth in all Henkin (rather than “full”) models.

But Henkin models seem to ruin the categoricity results.

However, categoricity can be proved for Henkin models, too, in the form of internal categoricity, which implies full categoricity in full models.
• Note that \((M, \in_1)\) and \((M, \in_2)\) above can be models of 
\(V = L, \quad V \not\equiv L, \quad CH, \quad \neg CH, \) even of \(\neg Con(ZF).\)

• It is easy to construct such pairs of models using classical methods of Gödel and Cohen.

• Not all of them can be models of second order set theory.
Continuum Hypothesis (CH)

- What if \((M, \in_1) \models CH\) and \((M, \in_2) \models \neg CH\)?
- Then either \((M, \in_1)\) or \((M, \in_2)\) does not satisfy the Separation Schema or the Replacement Schema if formulas are allowed to mention the other membership-relation.
• An internal categoricity result.
• A strong robustness result for set theory.
• The model cannot be changed “internally”.
• To get non-isomorphic models one has to go “outside” the model.
• But going “outside” raises the potential of an infinite regress of meta theories.
• Should we think of second order logic or first order set theory as the foundation of classical mathematics?

• The answer: We need a new understanding of the difference between the two. The difference is not as clear as what was previously thought.

• The nice categoricity results of second order logic can be seen already on the first order level, revealing their inherent limitations.
Thank you!