Localization in the Discrete Non-Linear Schrödinger Equation: a ‘**Random First-Order**’ transition in the microcanonical ensemble

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Summary

1) Large Deviations and Localization

2) Discrete Non-Linear Schrödinger Equation (DNLSE)

3) DNLSE: State of the art and the problem of ensembles

4) Localization mechanism

5) Finite-size effects, negative temperature, participation ratio

6) A mixed-order transition, analogies with glasses

7) Differences with Many-Body Localization

8) Role of dimensionality (none)

9) Condensates and black holes

10) Conclusions

The ‘Linear Statistic’ problem

Linear Statistic Problem: probability distribution of a sum of random variables

$$P_N(M) = \int \prod_{i=1}^{N} dm_i \; p(m_1, \ldots, m_N) \; \delta\left(M - \sum_{i=1}^{N} m_i \right)$$

Simple case: independent identically distributed **random variables**

$$p(m_1, \ldots, m_N) = \prod_{i=1}^{N} p(m_i)$$

$\langle m \rangle < \infty$ Finite mean

$\langle m^2 \rangle < \infty$ Finite variance

$$|M - N\langle m \rangle| \sim \sqrt{N} \quad \Longrightarrow \quad P_N(M) = \frac{1}{\sqrt{2\pi\sigma N}} e^{-\frac{(X - N\langle m \rangle)^2}{2\sigma^2 N}}$$

Central Limit Theorem

$$|M - N\langle m \rangle| \sim N \quad \Longrightarrow \quad P_N(M) \sim e^{-N \, \underline{\mathcal{I}(m)}} \qquad m = M/N$$

Rate function

Large Deviations

‘Linear Statistic’ and Large Deviations

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Fat tailed distribution $e^{-m} < p(m) < \dfrac{1}{m^2}$ → ***Localization***

$$|M - N\langle m \rangle| \sim N \quad \Longrightarrow \quad P_N(M) \sim p(M)$$

Large Deviations

Whole sum is taken up by a single variable

‘Linear Statistic’ and Large Deviations

Mass transport model: stationary partition function

$$\mathcal{Z}_N(M) = \int_0^\infty \prod_{i=1}^N dm_i \prod_{i=1}^N p(m_i)\ \delta\left(M - \sum_{i=1}^N m_i\right)$$

Fat tailed distribution $$e^{-m} < p(m) < \frac{1}{m^2}$$ → ***Localization***

‘Nature of the condensate in mass transport models’,
Majumdar, Evans, Zia, *PRL* **94**, 180601 (2006)

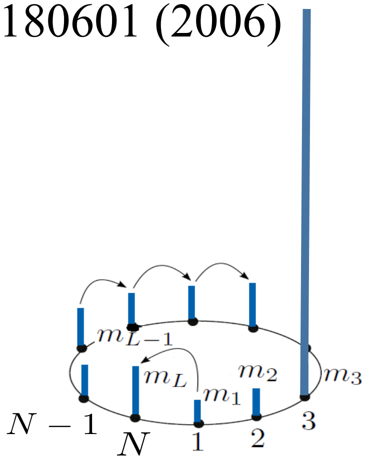
Participation Ratio

$$Y_2(M) = \left\langle \frac{\sum_{i=1}^n m_i^2}{\left(\sum_{i=1}^N m_i\right)^2} \right\rangle$$

Partition function

$$\mathcal{Z}_N(M) \sim p(M)$$

Whole sum is taken up by a single variable



$$M \sim m_i$$

$$M < N\langle m \rangle \implies Y_2(M) \sim 1/N$$

$$M > N\langle m \rangle \implies Y_2(M) = \mathcal{O}(1)$$

Discrete Non-Linear Schrödinger Equation (DNLSE)

Inspiration **‘A First-Order Dynamical Transition for a Driven Run-and-Tumble particle’** (G. Gradenigo, S. N Majumdar, JSTAT, **2019**)

$$\mathcal{Z}_N\left(z=\frac{M-N\langle m\rangle}{N^\alpha}\right)\sim e^{-N\mathcal{I}(\langle m\rangle)-N^{1-\alpha}\mathcal{C}(z)}$$

$$\alpha<1$$

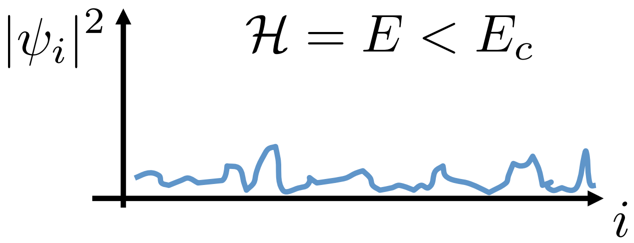
Key observation: the precise characterization of the transition comes from subleading corrections to the rate function.

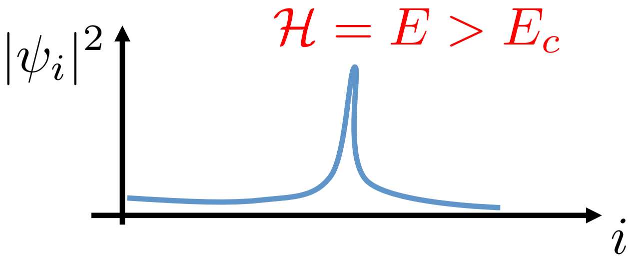
‘Localization in Discrete Non-Linear Schrödinger Equation’

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





‘Condensation transition and ensemble inequivalence in the discrete nonlinear Schrödinger equation’, G. Gradenigo, S. Iubini, R. Livi, S. N Majumdar, *EPJ-E* **44**, 1-6 (**2021**)

‘Localization transition in the discrete nonlinear Scrdinger equation: ensembles inequivalence and negative temperatures’, G. Gradenigo, S. Iubini, R. Livi, S. N Majumdar, *J. Stat. Mech.* 023201 (**2021**)

Discrete Non-Linear Schrödinger Equation (DNLSE)
A semiclassical Approximation

$$\hat{H} = \int d^3x \; \hat{\psi}^\dagger(\mathbf{x}) \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} \right] \hat{\psi}(\mathbf{x}) + \frac{4\pi\hbar^2 a_s}{2m} \int d^3x \; \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x})$$

'Discrete Breathers in Bose-Einstein Condensates', Franzosi, Livi, Oppo, Politi, *Nonlinearity.* **24**, R89 **(2011)**

Second-quantization Hamiltonian of interacting bosons condensate

$V(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$ **Repulsive contact interactions**

Bogoliubov approximation $\hat{\psi}(\mathbf{x}) = \Psi(\mathbf{x}) + \hat{\varphi}(\mathbf{x})$

$\Psi(\mathbf{x}) = \langle \hat{\psi}(\mathbf{x}) \rangle$ **Condensate wave-function** (c-number)

$\hat{\varphi}(\mathbf{x}) = \hat{\psi}(\mathbf{x}) - \langle \hat{\psi}(\mathbf{x}) \rangle$ **Deviation opeartor**

Expand the Hamiltonian up to second order in powers of $\hat{\varphi}(\mathbf{x}), \; \hat{\varphi}^\dagger(\mathbf{x})$
(small quantum fluctuations around the mean-field solution)

$\hat{H} = K_0 + \hat{K}_1 + \hat{K}_2 + \dots$ $\hat{K}_1 = \mathcal{O}(\hat{\varphi})$ $\hat{K}_2 = \mathcal{O}(\hat{\varphi}^2)$

Discrete Non-Linear Schrödinger Equation (DNLSE)
A semiclassical Approximation

$$\hat{K}_1 = 0 \quad \Longleftrightarrow \quad \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x})\right]\Psi(\mathbf{x}) - \frac{\nu}{2}|\Psi(\mathbf{x})|^2\Psi(\mathbf{x}) = 0$$

Gross-Pitaevskii Equation: non-linear equation for the 'order parameter' of a quantum transition (semiclassical approximation)

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Gross-Pitaevskii Equation: non-linear equation for the 'order parameter' of a quantum transition (semiclassical approximation)

$$V_{\rm ext}(\mathbf{x}) = \underbrace{\frac{\hbar^2\omega^2}{4E_r}\,\sin^2(k_{\rm L}x)}_{\text{Periodic modulation - x}} + \underbrace{\frac{m\Omega^2}{2}\,(y^2+z^2)}_{\text{Harmonic traps (y,z)-plane}}$$

Effectively on a
1-dimensional lattice

Hamiltonian system
on a lattice

$$\mathcal{H} = \sum_{i=1}^{N} \Psi_i^*\Psi_{i+1} + \Psi_{i+1}^*\Psi_i + \frac{\nu}{2}\sum_{i=1}^{N}|\Psi_i|^2$$

Canonical conjugate
variables (classical)

$$\{\Psi_i^*, \Psi_j\} = i\ \delta_{ij}/\hbar \qquad i\dot{\Psi}_i = -\frac{\partial\mathcal{H}}{\partial\Psi_i^*}$$

Poisson parentheses

Discrete Non-Linear Schrödinger Equation (DNLSE)
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HAMILTONIAN ⟺ **EQUILIBRIUM STATISTICAL MECHANICS**

Hamiltonian system on a lattice

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Discrete Non-Linear Schrödinger Equation (DNLSE)

Condensate wave-function (order parameter) $\langle\hat{\psi}\rangle = \psi(x_i,t) = \psi_i(t)$

$$i\,\frac{\partial\psi_i}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\psi_i^*} = -(\psi_{i+1}+\psi_{i-1}) - \nu|\psi_i|^2\psi_i$$

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N}(\psi_i^*\psi_{i+1}+\psi_i\psi_{i+1}^*) + \frac{\nu}{2}\sum_{i=1}^{N}|\psi_i|^4$$

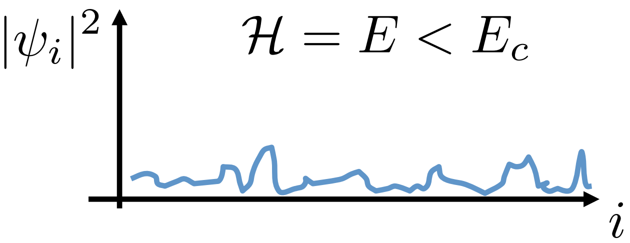
PARTICLES NUMBER (conserved)

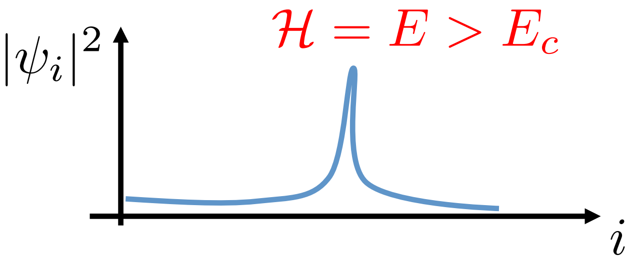
$$A = \sum_{i=1}^{N}|\psi_i|^2$$

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





1) WHICH KIND OF PHASE TRANSITION ?

2) WHICH STATISTICAL ENSEMBLE?

3) LOCALIZATION COMES FROM INTEGRABILITY? (N integrals of motion)

4) IS DISORDER NECESSARY FOR LOCALIZATION?

Discrete Non-Linear Schrödinger Equation (DNLSE)

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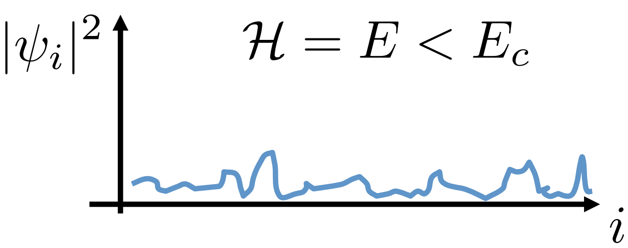
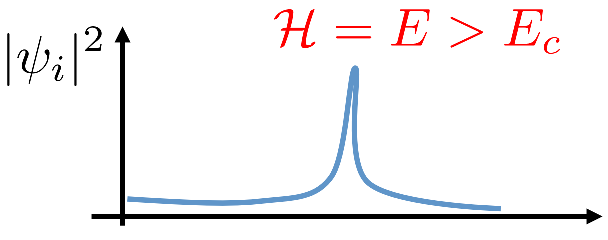




The 'Fundamental Ensemble' : MICROCANONICAL

Microcanonical Partition function

$$\Omega_N(A,E) = \int\prod_{i=1}^{N}d\psi_i\;\underbrace{\delta(A-\sum_{i=1}^{N}|\psi_i|^2)}_{\text{Particle number conservation}}\;\underbrace{\delta\left(E-\mathcal{H}[\psi_i^*,\psi_i]\right)}_{\text{Energy conservation}}$$

DNLSE theory: state of the art

'Statistical Mechanics of a Discrete Non-Linear System',
K.O. Rasmussen, T. Cretegny, P.G. Kevridis, N. Gronbech-Jensen, *Phys. Rev. Lett.* **84**, 3740 **(2000)**

Microcanonical →

Grand Canonical $\mathcal{Z}_N(\mu,\beta) = \int_0^\infty dA\; dE\; e^{-\beta E - \mu A}\; \Omega_N(A,E)$

Grand Canonical: **exact solution with trasfer matrix techniques!**

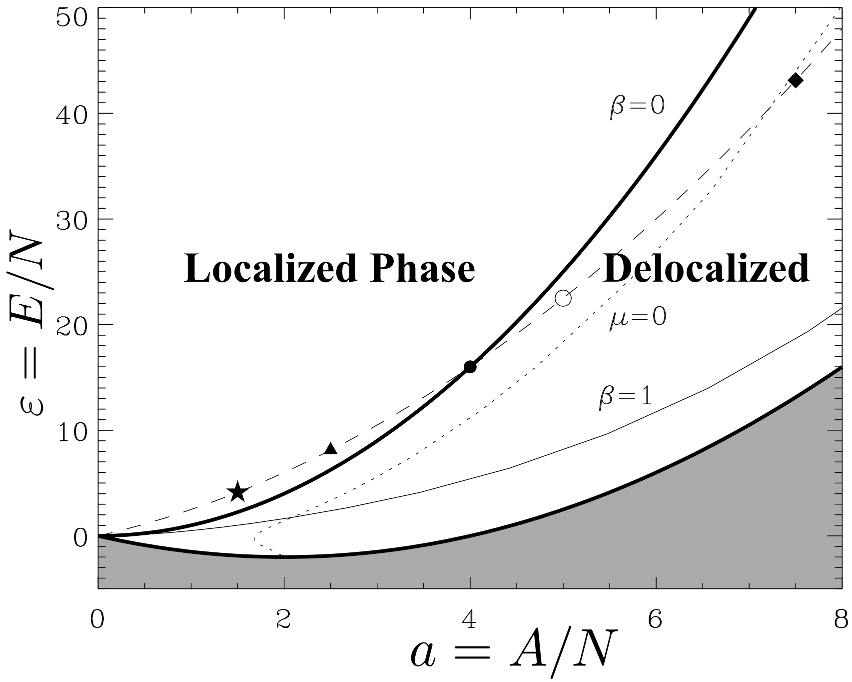
Transition line at **infinite** temperature: $\beta = 0$

$\varepsilon = 2\, a^2$

PROBLEM

Many numerical evidences that the localized phase has negative temperature, T<0

'Discrete Breathers and Negative-Temperature States',
S. Iubini, R. Franzosi, R. Livi, G.-L. Oppo, A. Politi,
New J. Phys. **15**, 023032 **(2013)**



HOW CAN β<0 BE CONSISTENT WITH $e^{-\beta\mathcal{H}}$ **?** → **IT CANNOT!**

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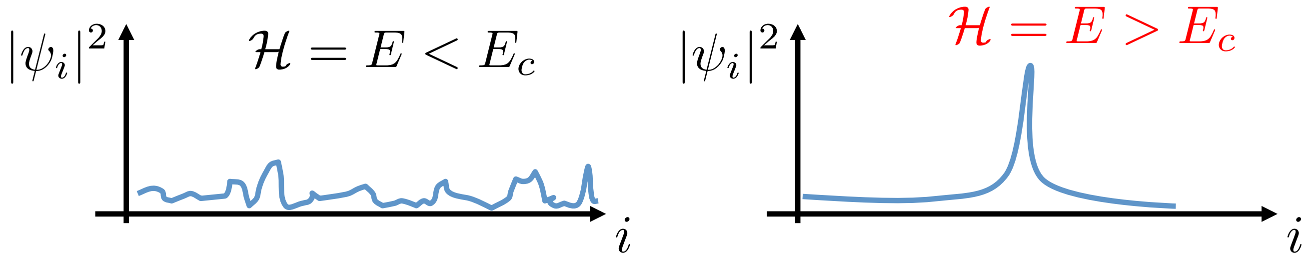


ONLY THE MICROCANONICAL IS CORRECT: GO FOR IT!

$$\Omega_N(A,E) = \int \prod_{i=1}^{N} d\psi_i\; \underbrace{\delta\left(A - \sum_{i=1}^{N}|\psi_i|^2\right)}_{\text{Particle number conservation}}\; \underbrace{\delta\left(E - \sum_{i=1}^{N}|\psi_i|^4\right)}_{\text{Energy conservation}}$$

Neglect hopping terms

(random-phase argument)

ENSEMBLES IN-EQUIVALENCE

$$\underbrace{\mathcal{Z}_N(\mu,\beta)}_{\textit{Grand-Canonical}} = \int_0^\infty \underbrace{dA\; dE\; e^{-\beta E-\mu A}}_{\textit{Laplace Transform}} \underbrace{\Omega_N(A,E)}_{\textit{Micro-Canonical}} = [z(\mu,\beta)]^N$$

$$\Omega_N(A,E) = \int_{\mu_0+i\infty}^{\mu_0-i\infty} d\mu \int_{\beta_0-i\infty}^{\beta_0-i\infty} d\beta\; e^{\mu A+\beta E+N\log z(\mu,\beta)}$$

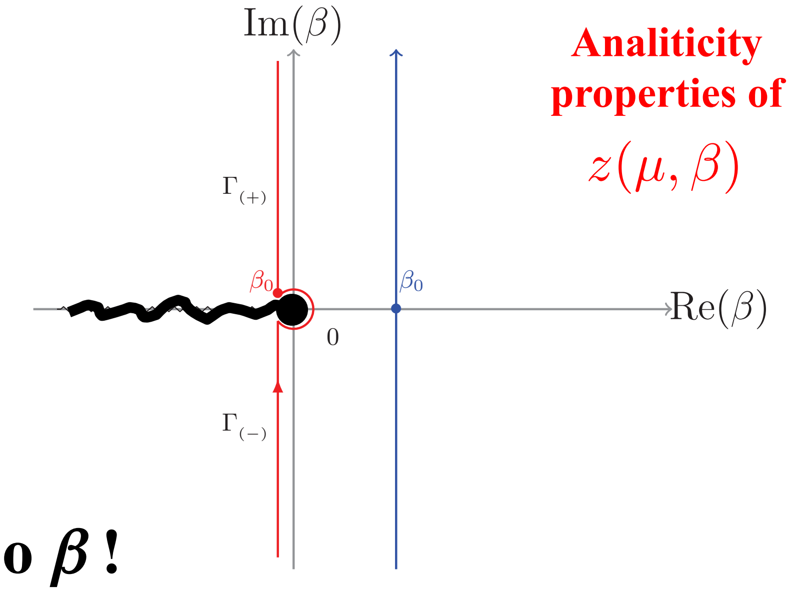
Inverse Laplace Transform

ENSEMBLES are equivalent when saddle-points equations have real solutions

Can I find real *β* and *μ* for ANY choice of E and A?

$$\frac{E}{N} = -\frac{\partial}{\partial\beta}\log[z(\mu,\beta)]$$

$$\frac{A}{N} = -\frac{\partial}{\partial\mu}\log[z(\mu,\beta)]$$



Analiticity properties of

$z(\mu,\beta)$

DNLSE: For *E* > *E*$_{th}$ there is no *β* !

ENSEMBLES IN-EQUIVALENCE

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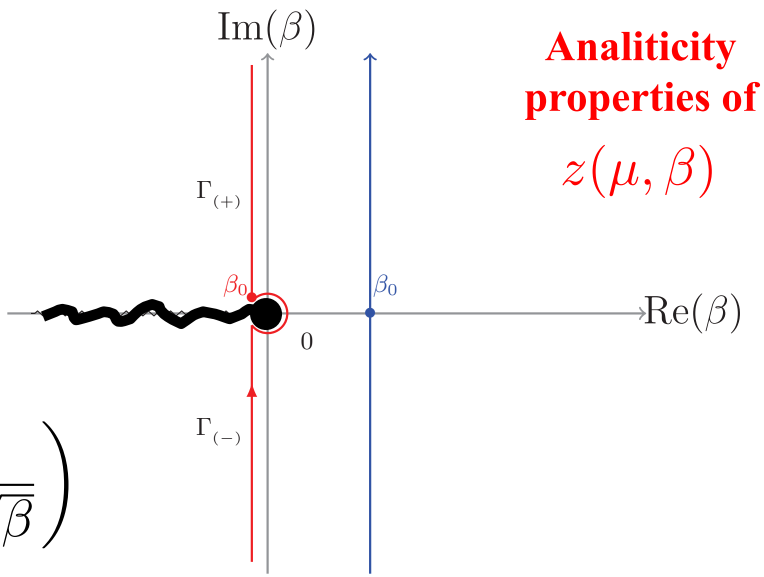
$$\frac{E}{N} = -\frac{\partial}{\partial\beta}\log[z(\mu,\beta)]$$

$$z(\mu,\beta) = \frac{\mu\sqrt{\pi}}{2\sqrt{\beta}}\ \exp\left(\frac{\mu^2}{4\beta}\right)\mathrm{Erfc}\left(\frac{\mu}{2\sqrt{\beta}}\right)$$



Analiticity properties of

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SKETCHY MECHANISM OF LOCALIZATION

$$\underset{\textit{Grand-Canonical}}{\underline{\mathcal{Z}_N(\mu,\beta)}} = \int_0^\infty \underbrace{dA\; dE\; e^{-\beta E-\mu A}}_{\textit{Laplace Transform}} \; \underset{\textit{Micro-Canonical}}{\underline{\Omega_N(A,E)}} = [z(\mu,\beta)]^N$$

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Inverse Laplace Transform

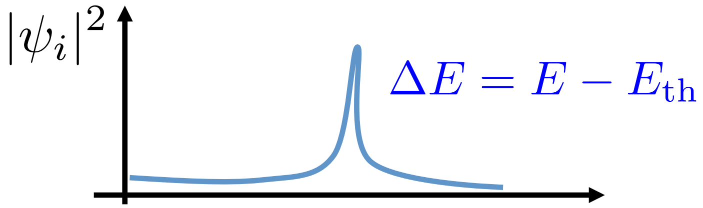
ENSEMBLES are equivalent when saddle-points equations have real solutions

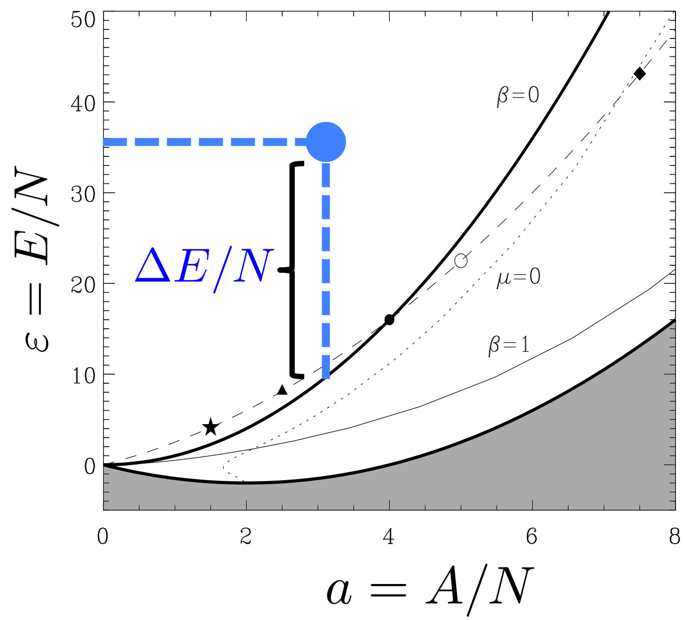
$$E > E_{\rm th}$$

1) Cannot reach such energy by equal sharing among d.o.f.

2) The amount $E_{\rm th}$ is identically distributed among the degrees of freedom (infinite temperature background)

3) Excess energy is put into the localized phase





THE LARGE DEVIATIONS APPROACH

Microcanonical Ensemble

$$\Omega_N(A,E) = \int \prod_{i=1}^{N} d\psi_i \; \delta\left(A - \sum_{i=1}^{N} |\psi_i|^2\right) \; \delta\left(E - \sum_{i=1}^{N} |\psi_i|^4\right)$$

Release constraint on 'particle number'

$$\Omega_N(\mu,E) = \int \prod_{i=1}^{n} d\psi_i \; e^{-\mu \sum_{i=1}^{N} |\psi_i|^2} \delta\left(E - \sum_{i=1}^{N} |\psi_i|^4\right)$$

Change of variables

$$\Omega_N(\mu,E) \approx \int \prod_{i=1}^{n} \left[d\varepsilon_i \, \frac{e^{-\mu\sqrt{\varepsilon_i}}}{\sqrt{\varepsilon_i}}\right] \delta\left(E - \sum_{i=1}^{N} \varepsilon_i\right)$$

1) $\psi = r e^{i\phi}$

2) $r_i^4 = \varepsilon_i$

Partition Function = Probability distribution of fat tailed variables sum

$$e^{-\varepsilon_i} < \frac{e^{-\mu\sqrt{\varepsilon_i}}}{\sqrt{\varepsilon_i}} < \frac{1}{\varepsilon_i^2}$$

→ ***Localization*** $E > N\langle \varepsilon \rangle_\mu = E_{\mathrm{th}}$

Slow decay of the energy per site probability distribution function

MATCHING ARGUMENT FOR LOCALIZATION

Gaussian regime $E - E_{th} \sim \sqrt{N}$

Extreme large deviations $E - E_{th} \sim N$

$$\Omega_N(A,E) \;\approx\; e^{-\frac{(E-E_{th})^2}{2\sigma^2 N}}$$

$$\Omega_N(A,E) \;\approx\; e^{-\sqrt{E-E_{th}}}$$

Matching regime (you set the scale)

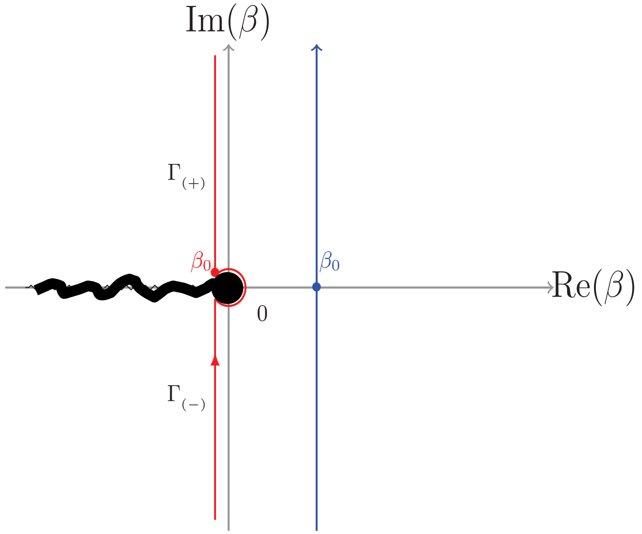
$$\frac{E - E_{th}}{N^{2/3}} = \zeta \sim 1$$

Zoom in the complex plane around the origin to propertly account for the cut contribution

$$\hat{\beta} = N^{1/3}\beta \sim 1$$

$$\int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta \; e^{\beta E + \underbrace{N \log[z(\beta,\mu)]}}$$

Expand this guy at the origin



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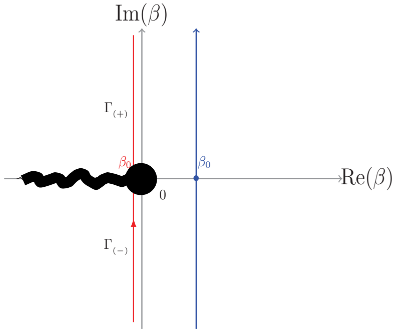
Matching regime (you set the scale)

$$\frac{E-E_{th}}{N^{2/3}} = \zeta \sim 1 \qquad \hat{\beta} = N^{1/3}\beta \sim 1$$

$$\int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta\; e^{\beta E + N\log[z(\beta,\mu)]} = \frac{1}{\sigma\sqrt{2\pi N}}\exp\left\{-\frac{(E-E_{th})^2}{2\sigma^2 N}\right\} + \underline{\mathcal{C}(E)}$$



Non-analiticity at the cut

MATCHING ARGUMENT FOR LOCALIZATION

Gaussian regime $E - E_{th} \sim \sqrt{N}$

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Extreme large deviations $E - E_{th} \sim N$

$$\Omega_N(A,E) \;\approx\; e^{-\sqrt{E-E_{th}}}$$

Matching regime (you set the scale)

$$\frac{E-E_{th}}{N^{2/3}} = \zeta \sim 1 \qquad \hat{\beta} = N^{1/3}\beta \sim 1$$

$$\Omega_N(A,E) \;\approx\; e^{N[1+\log(\pi a)]}\left[e^{-N^{1/3}\zeta^2/(2\sigma^2)} + \underline{e^{-N^{1/3}\chi(\zeta)}}\right]$$

Non-analiticity at the cut

$$\Omega_N(A,E) \sim \exp\left\{N[1+\log(\pi a)] - N^{1/3}\Psi(\zeta)\right\}$$

$$\Psi(\zeta) = \min\left\{\zeta^2/(2\sigma^2), \chi(\zeta)\right\} \longrightarrow \frac{\zeta^2}{2\sigma^2} = \chi(\zeta) \quad \Longrightarrow \quad \zeta_c = \frac{E_c - E_{th}}{N^{2/3}}$$

MATCHING ARGUMENT FOR LOCALIZATION

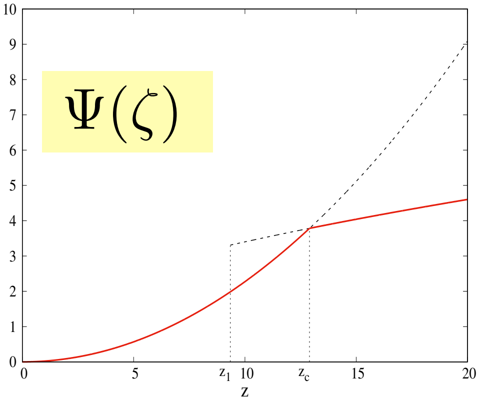
Gaussian regime $E - E_{th} \sim \sqrt{N}$

Extreme large deviations $E - E_{th} \sim N$

$$\Omega_N(A, E) \;\approx\; e^{-\frac{(E-E_{th})^2}{2\sigma^2 N}} \qquad\qquad \Omega_N(A, E) \;\approx\; e^{-\sqrt{E-E_{th}}}$$

Matching regime (you set the scale)

$$\frac{E - E_{th}}{N^{2/3}} = \zeta \sim 1 \qquad \hat{\beta} = N^{1/3}\beta \sim 1$$



$$\Omega_N(A, E) \sim \exp\left\{N[1 + \log(\pi a)] - N^{1/3}\Psi(\zeta)\right\}$$

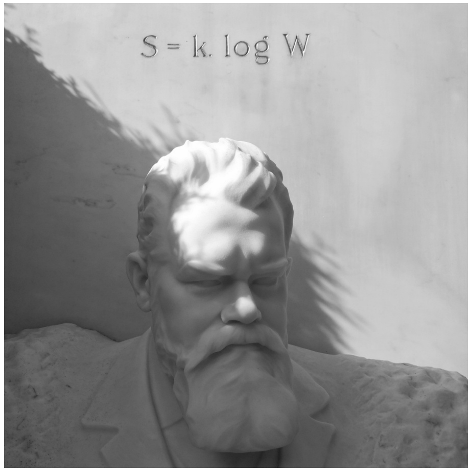
$$\Psi(\zeta) = \min\left\{\zeta^2/(2\sigma^2), \chi(\zeta)\right\}$$

$$\zeta \gg 1 \Longrightarrow \chi(\zeta) = \zeta^{1/2}\sqrt{\frac{2}{\langle\varepsilon\rangle}} - \frac{\sigma^2}{4\varepsilon}\frac{1}{\zeta} + \mathcal{O}(\zeta^{-5/2})$$

$$e^{-\varepsilon_i} \;<\; \frac{e^{-\mu\sqrt{\varepsilon_i}}}{\sqrt{\varepsilon_i}} \;<\; \frac{1}{\varepsilon_i^2}$$

THE MAIN RESULT: MICROCANONICAL ENTROPY

Microcanonical Entropy



$$S_N(A,E) = \mathrm{k}\ \log[\Omega_N(A,E)]$$

The first, the one … and the ONLY

$$E > E_{\mathrm{th}}$$

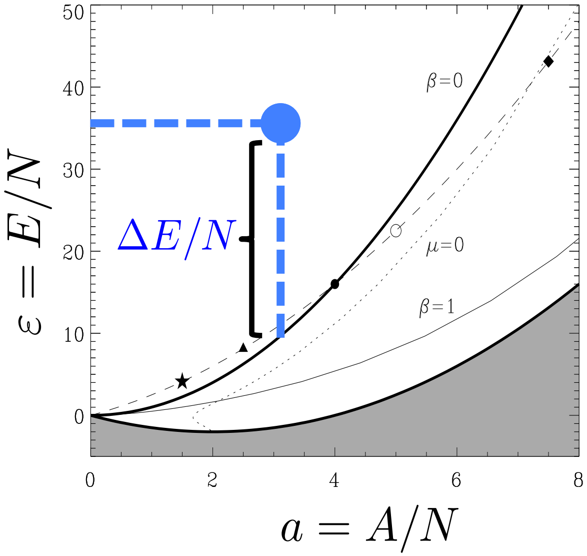
CONDENSATE ENTROPY (SUBEXTENSIVE)

$$S_N(A,E) = \underbrace{\Sigma_0(A)} + \overbrace{\Sigma_1(E,A)}$$

Background Entropy (energy indipendent)

$$\Sigma_0(A) = N[1+\log(\pi a)]$$

$$\varepsilon_{\mathrm{th}} = 2\ a^2$$



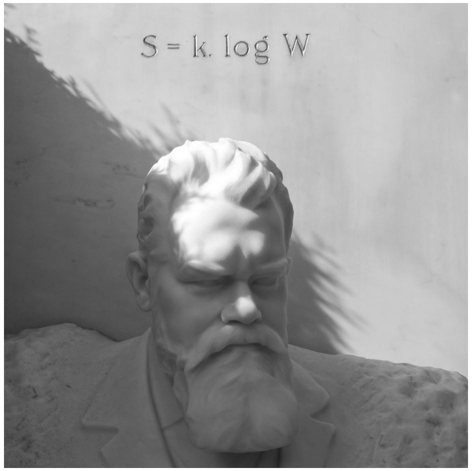
$$\Delta E = E - E_{\mathrm{th}}$$

THE MAIN RESULT: MICROCANONICAL ENTROPY

Microcanonical Entropy

$$S_N(A,E) = \mathrm{k}\ \log[\Omega_N(A,E)]$$

The first, the one … and the ONLY



Three regimes

$$\Sigma_1(E,A) = \begin{cases} -\dfrac{N}{2\sigma^2}(\varepsilon-\varepsilon_{\mathrm{th}})^2 \\ \\ -N^{1/3}\Psi(\zeta) \\ \\ -N^{1/2}\sqrt{\varepsilon-\varepsilon_{\mathrm{th}}} \end{cases}$$

CONDENSATE ENTROPY

Gaussian $\varepsilon-\varepsilon_{\mathrm{th}} \sim 1/\sqrt{N}$

Matching $\varepsilon-\varepsilon_{\mathrm{th}} \sim 1/N^{1/3}$

Large Deviations $\varepsilon-\varepsilon_{\mathrm{th}} \sim 1$

$$\varepsilon_{\mathrm{th}} = 2\ a^2 \qquad \zeta = N^{1/3}(\varepsilon-\varepsilon_{\mathrm{th}})$$

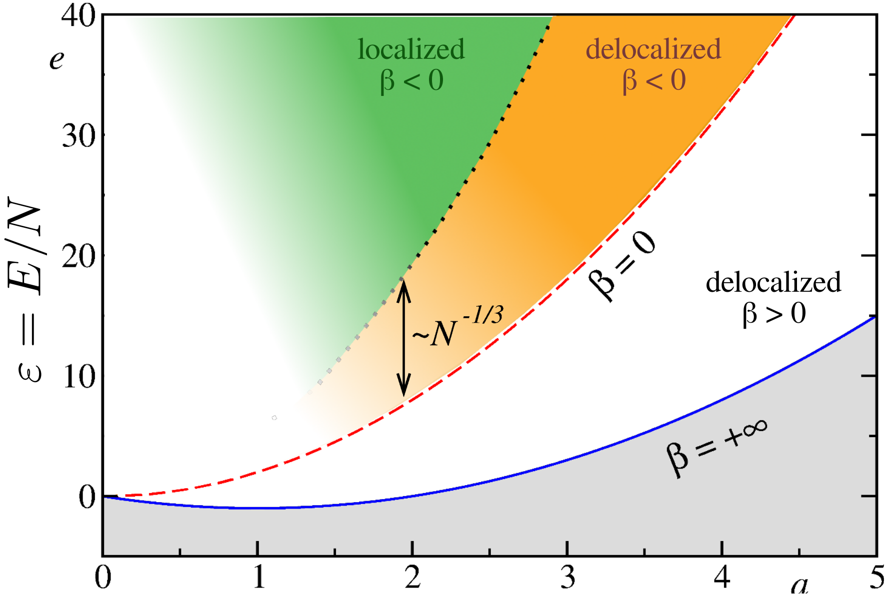
THE MAIN RESULT: MICROCANONICAL ENTROPY

$$\Psi'(\zeta_c) = \text{jump}$$

$$\zeta_c = N^{1/3}(\varepsilon_c - \varepsilon_{th})$$

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

Finite-size correction to the critical line



$$\Sigma_1(E,A) = \begin{cases} -\dfrac{N}{2\sigma^2}(\varepsilon - \varepsilon_{\rm th})^2 \\ -N^{1/3}\Psi(\zeta) \\ -N^{1/2}\sqrt{\varepsilon - \varepsilon_{\rm th}} \end{cases}$$

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Gaussian $\varepsilon - \varepsilon_{\rm th} \sim 1/\sqrt{N}$

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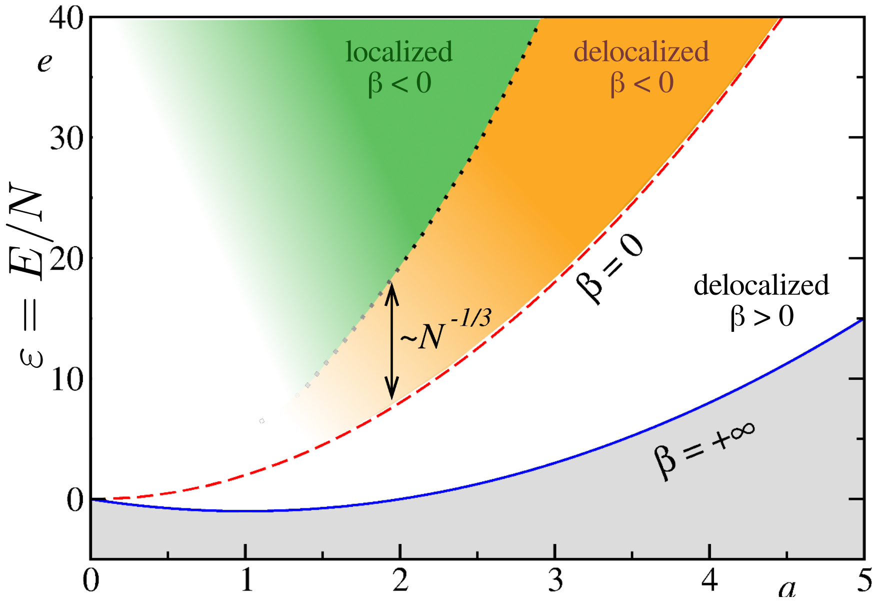
Large Deviations $\varepsilon - \varepsilon_{\rm th} \sim 1$

$$\varepsilon_{\rm th} = 2\ a^2 \qquad \zeta = N^{1/3}(\varepsilon - \varepsilon_{\rm th})$$

NEGATIVE TEMPERATURE – SUBEXTENSIVE ENTROPY



$\varepsilon_{\rm th} < \varepsilon < \varepsilon_c$ = **Uninteresting ?**

Not really…

$$\varepsilon > \varepsilon_{\rm th} \implies \frac{\partial S}{\partial E} = \frac{1}{T} < 0$$

NEGATIVE TEMPERATURE

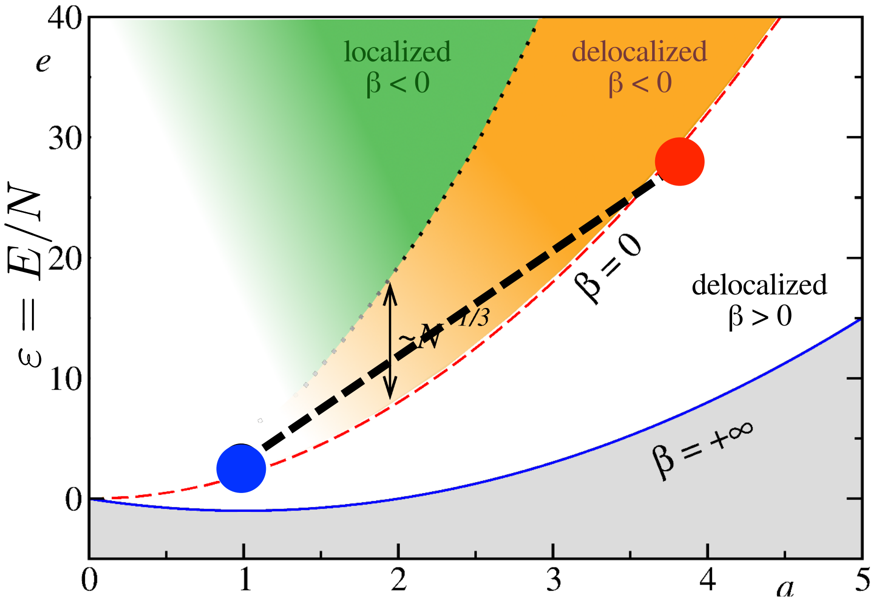
$$\Sigma_1(E,A) = \begin{cases} -\dfrac{N}{2\sigma^2}(\varepsilon - \varepsilon_{\rm th})^2 & \textbf{Gaussian} \quad \varepsilon - \varepsilon_{\rm th} \sim 1/\sqrt{N} \\ -N^{1/3}\Psi(\zeta) & \textbf{Matching} \quad \varepsilon - \varepsilon_{\rm th} \sim 1/N^{1/3} \\ -N^{1/2}\sqrt{\varepsilon - \varepsilon_{\rm th}} & \textbf{Large Deviations} \quad \varepsilon - \varepsilon_{\rm th} \sim 1 \end{cases}$$

CONDENSATE ENTROPY

$$\varepsilon_{\rm th} = 2\,a^2 \qquad \zeta = N^{1/3}(\varepsilon - \varepsilon_{\rm th})$$

$$T = N^{1/2}\sqrt{\varepsilon - \varepsilon_{\rm th}}$$

PROBING THE NEGATIVE TEMPEATURE



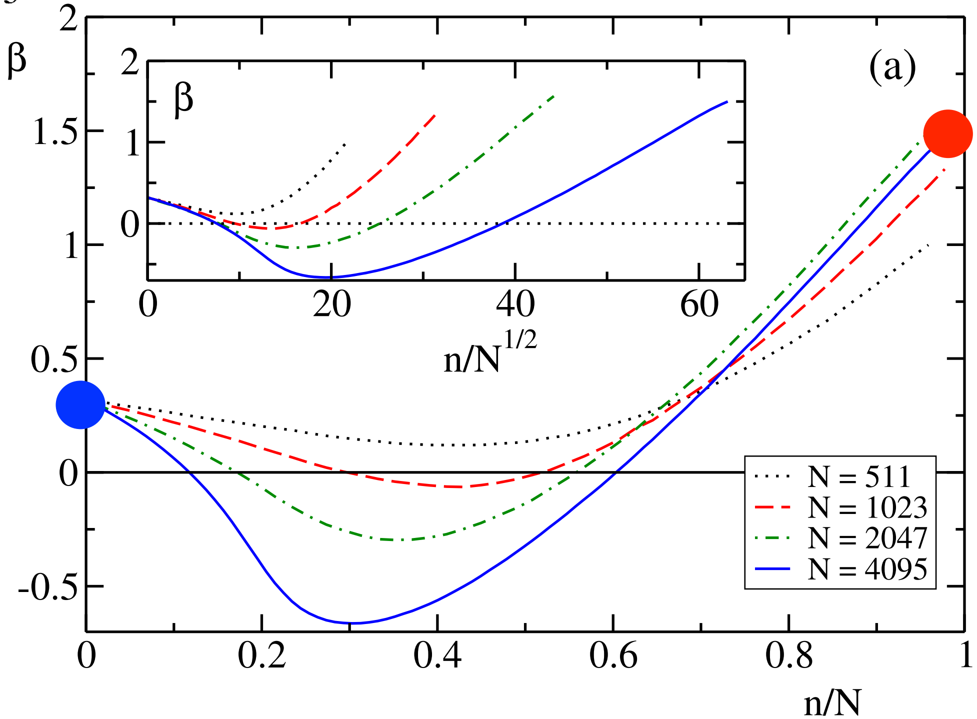
$$\varepsilon_{\mathrm{th}} < \varepsilon < \varepsilon_c \quad = \quad \textbf{Uninteresting ?}$$

Not really…

$$\varepsilon > \varepsilon_{\mathrm{th}} \quad \Longrightarrow \quad \frac{\partial S}{\partial E} = \frac{1}{T} < 0$$

NEGATIVE TEMPERATURE

Discrete Non-Linear Schrödinger Equation coupled at the boundaries with reservoirs at different temperature

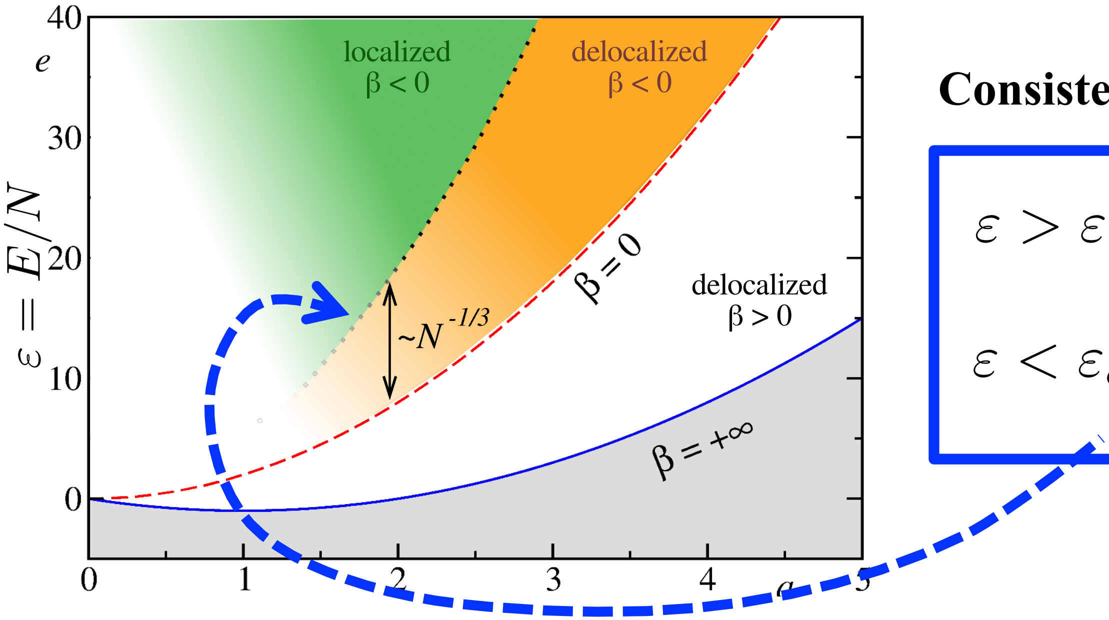


***'A chain, A bath, A sink and a Wall'*,**

S. Iubini, S. Lepri, R. Livi, G.-L. Oppo, A. Politi,

Entropy **(2017)**

ORDER PARAMETER: PARTICIPATION RATIO



Consistent with non-analyticity of Entropy

$$\varepsilon > \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N = c > 0$$

$$\varepsilon < \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N \sim 1/N$$

The order parameter jumps at the dotted blue line!

$$\Psi'(\zeta_c) = \text{jump}$$

$$\zeta_c = N^{1/3}(\varepsilon_c - \varepsilon_{th})$$

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

Finite-size correction to the critical line

ORDER PARAMETER: PARTICIPATION RATIO

$$\Psi'(\zeta_c) = \text{jump}$$

Order Parameter = Participation Ratio

$$\mathcal{P}_{_N} = \left\langle \frac{\sum_{i=1}^{N} \varepsilon_i^2}{\left(\sum_{i=1}^{N} \varepsilon_i\right)^2} \right\rangle_{\textbf{\textit{micro}}}$$

Consistent with non-analyticity of Entropy

$$\boxed{\begin{aligned} \varepsilon > \varepsilon_c \quad &\Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_{_N} = c > 0 \\ \varepsilon < \varepsilon_c \quad &\Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_{_N} \sim 1/N \end{aligned}}$$

‘Pseudo-condensate’ **Localization**

	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ensembles inequivalence

ORDER PARAMETER: PARTICIPATION RATIO

$$\Psi'(\zeta_c) = \text{jump}$$

Order Parameter = Participation Ratio

$$\mathcal{P}_{_N} = \left\langle \frac{\sum_{i=1}^{N} \varepsilon_i^2}{\left(\sum_{i=1}^{N} \varepsilon_i\right)^2} \right\rangle_{\textit{micro}}$$

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$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ergodicity breaking ?

ORDER PARAMETER: PARTICIPATION RATIO

$$\varepsilon_c = \varepsilon_{th} + \frac{\zeta_c}{N^{1/3}}$$

In the thermodynamic limit the two values coincide and the order parameter is continuous at the transition

Consistent with non-analyticity of Entropy

$$\varepsilon > \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N = (\varepsilon - \varepsilon_{th})^2/\varepsilon^2$$

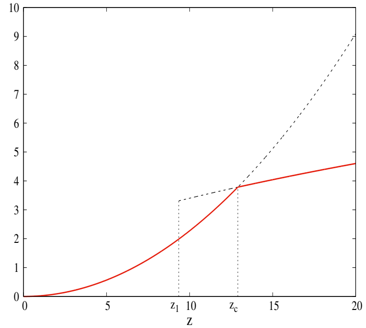
$$\varepsilon < \varepsilon_c \quad \Longrightarrow \quad \lim_{N\to\infty} \mathcal{P}_N \sim 1/N$$

'Pseudo-condensate' **Localization**

	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

Ergodicity breaking ?

ORDER PARAMETER: PARTICIPATION RATIO



Merging at N=∞ into a mixed-order transition?

Is there any known example of such a transition?

2^{nd} Order

1^{st} Order

‘Pseudo-condensate’

Localization

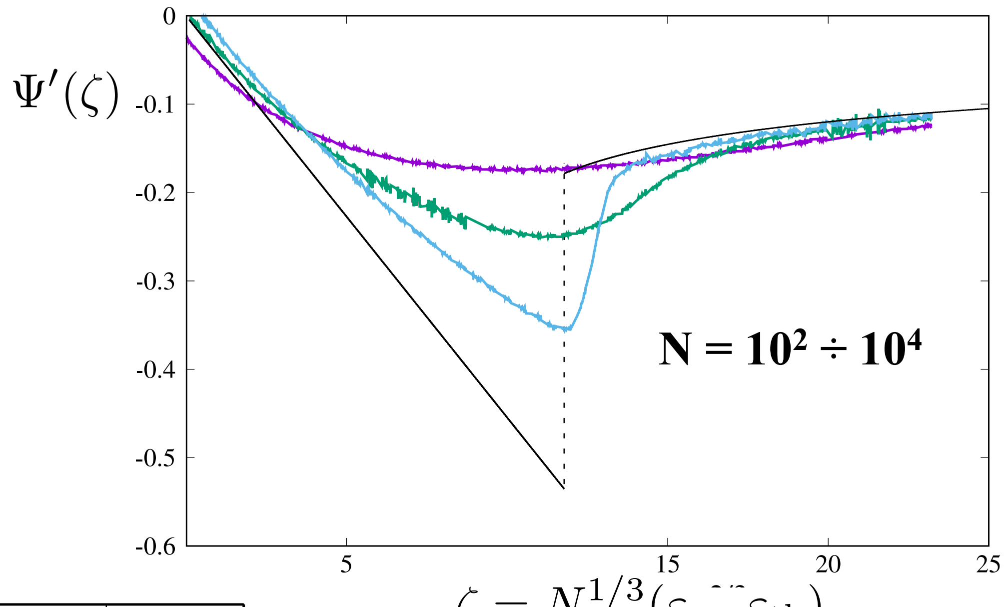
	$\varepsilon < \varepsilon_{th}$	$\varepsilon_{th} < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$\lim\limits_{N\to\infty} \mathcal{P}_N$	$1/N$	$1/N$	c
$T^{-1} = \partial S/\partial E$	> 0	< 0	< 0

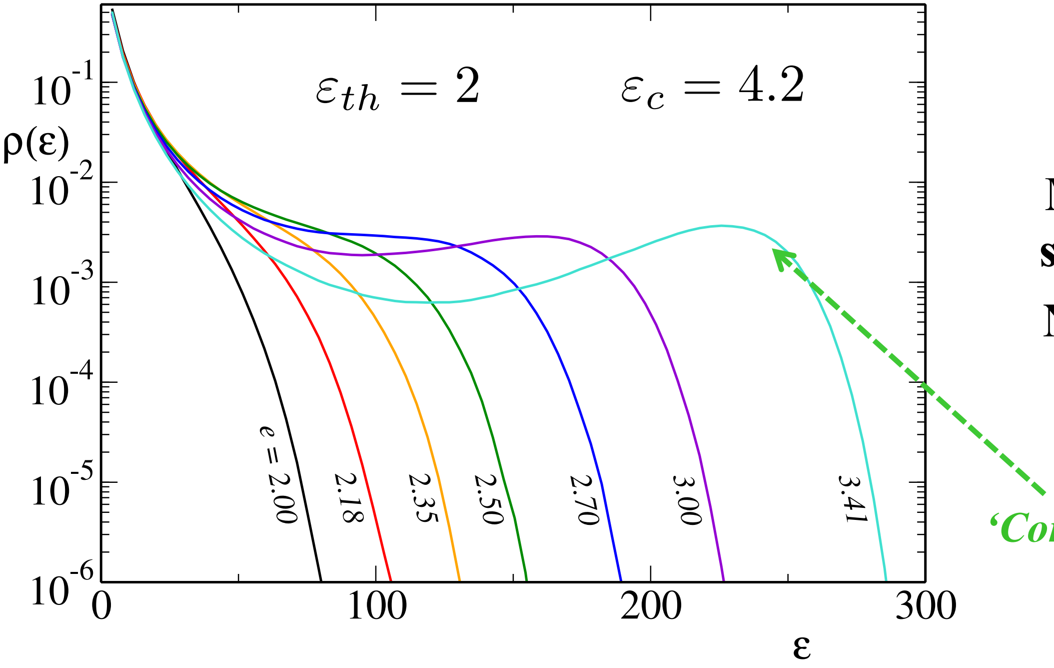
Ergodicity breaking ?

FINALLY SOME FIGURES!

Entropy of the condensate

As a function of size





'Condensate bump'

Marginal distribution on a single site (microcanonical)

N=128: pseudo-condensate

A VERY WELL KNOWN MIXED ORDER TRANSITION: RANDOM FIRST-ORDER or IDEAL GLASS TRANSITION

P-spin model $\mathcal{H} = -\sum_{ijkl} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l$ $\sum_{i=1}^{N}\sigma_i^2 = N$

#-interactions = N^4 $J_{ijkl} =$ **iid Gaussian variates** $\langle J^2\rangle \sim N^{-3}$

GLASS TRANSITION = ERGODICITY BREAKING TRANSITION

IMPORTANT SIMILARITIES WITH DNLS

- ✓ ***Locally unbounded continuous variables***
- ✓ ***Non-linear interactions***
- ✓ ***Global spherical constraint***

… NOT SHARED BY MODELS LIKE SHERRINGTON-KIRKPATRICK

- ✓ ***Discrete spins***
- ✓ ***Linear interactions***

A VERY WELL KNOWN MIXED ORDER TRANSITION: RANDOM FIRST-ORDER or IDEAL GLASS TRANSITION

P-spin model $\mathcal{H} = -\sum_{ijkl} J_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l$ $\sum_{i=1}^{N} \sigma_i^2 = N$

#-interactions = N^4 $J_{ijkl} =$ **iid Gaussian variates** $\langle J^2 \rangle \sim N^{-3}$

GLASS TRANSITION = ERGODICITY BREAKING TRANSITION

FIRST-ORDER FEATURES

Order Parameter: ***OVERLAP*** $=$ Similarity among two configurations chosen at random in the equilibrium ensemble

$$\boxed{q^{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{\alpha}\sigma_i^{\beta}}$$

$q \approx 0$ **different**

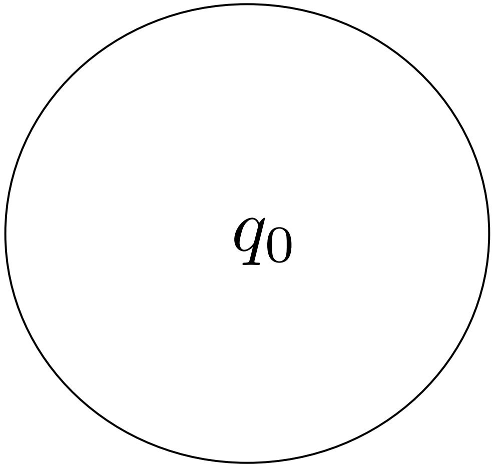
$q \approx 1$ **similar**

Can be measured in simulations

$$P(q) = m\ \delta(q - q_0) + (1-m)\ \delta(q - q_1)$$

Ergodicity Breaking: Parisi’s order parameter

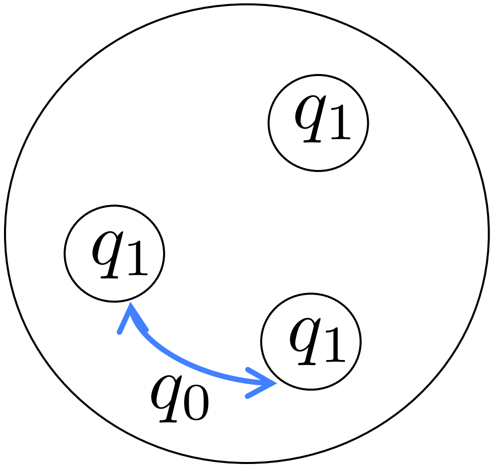
High Temperature



ERGODIC

all regions of
phase space are
equally available

Low Temperature

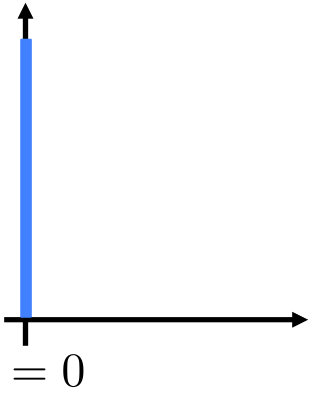


NON-ERGODIC

Phase-space partitioning in **disjoint ergodic components** with self overlap q_1 (mutual q_0)

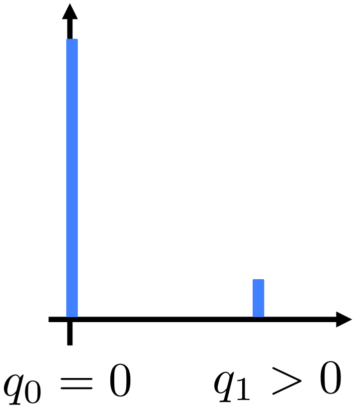
$P(q)$

‘First-order like’ behaviour

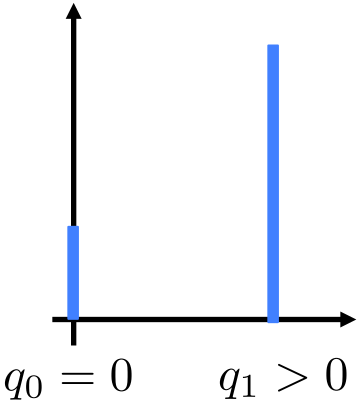


Ergodic T > T_K

Typically confs are different



T = T_K



Glass T < T_K

Typically confs are similar

Ergodicity Breaking: Parisi's order parameter

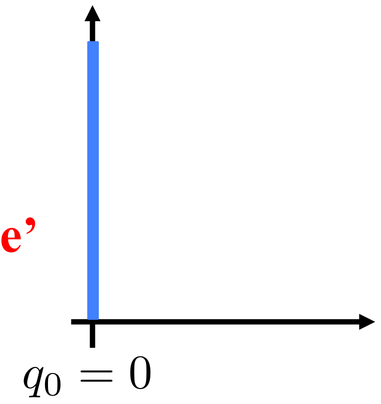
…BUT STILL IS NOT A FIRST-ORDER TRANSITION

- **NO LATENT HEAT AT THE CRITICAL TEMPERATURE T$_K$**
- **AVERAGE VALUE OF ORDER PARAMETER CONTINUOUS AT THE TRANSITION**

$$\int dq \; P(q) \; q = (1 - m) \; q_1$$

$P(q)$

'First-order like' behaviour



Ergodic T > T$_K$

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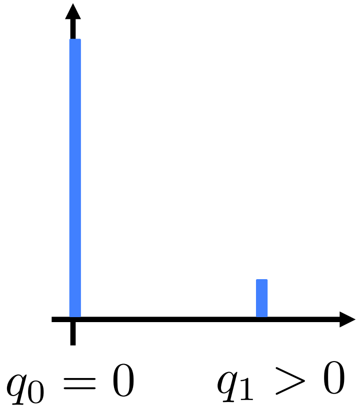
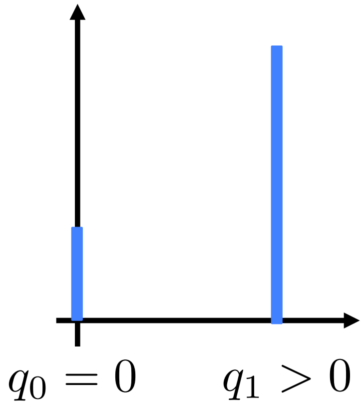


T = T$_K$



Glass T < T$_K$

Typically confs are similar

Ergodicity Breaking: Parisi's order parameter

RANDOM FIRST-ORDER TRANSITION

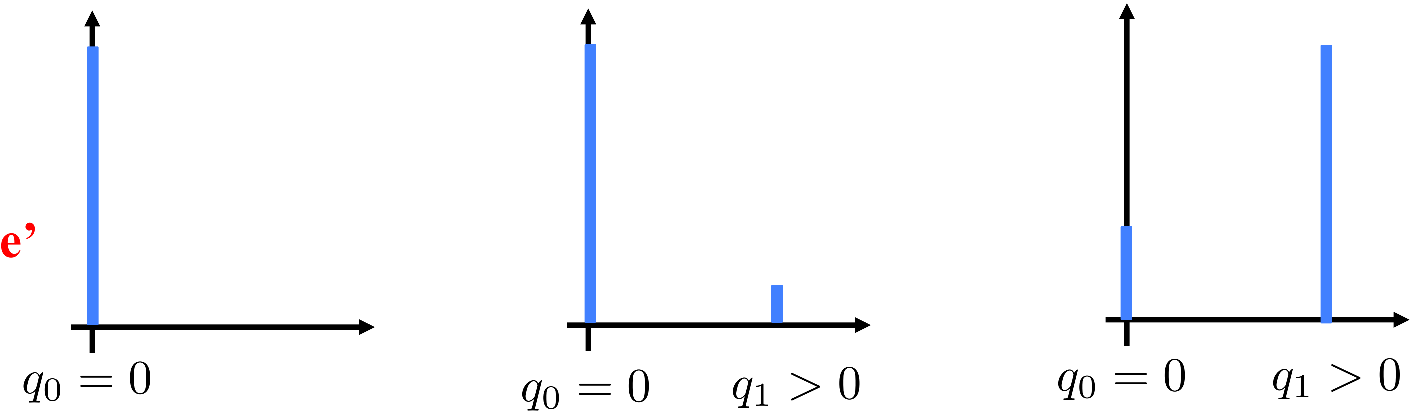
- NO LATENT HEAT AT THE CRITICAL TEMPERATURE T_K

- AVERAGE VALUE OF ORDER PARAMETER CONTINUOUS AT THE TRANSITION

$$\int dq \; P(q) \; q = (1 - m) \; q_1$$

$P(q)$

'First-order like' behaviour



Ergodic $T > T_K$

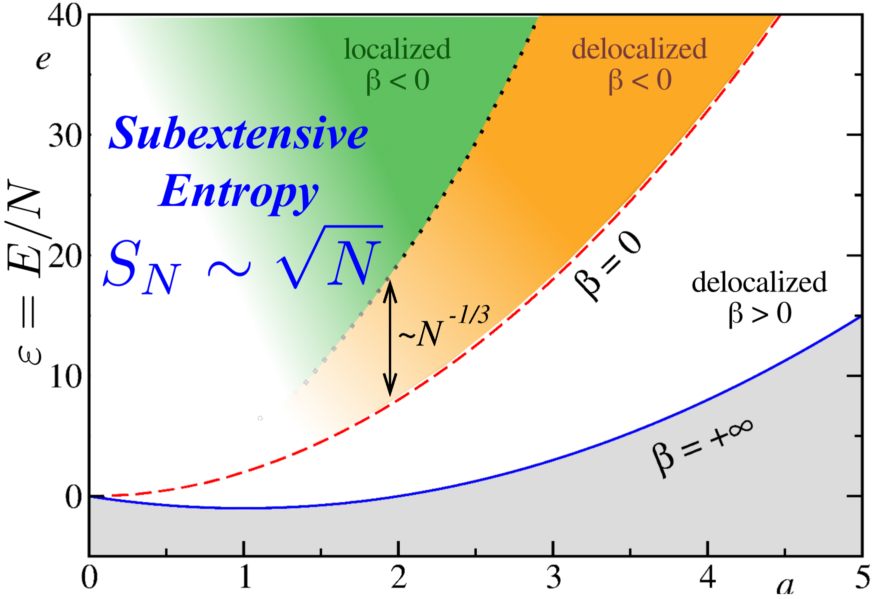
Typically confs are different

$T = T_K$

Glass $T < T_K$

Typically confs are similar

THE MAIN RESULT: MICROCANONICAL ENTROPY



1) **Microcanonical** and **canonical** ensembles are **not equivalent**

2) **Localization is a 'random first-order' transition in the microcanonical ensemble**

3) **Negative temperature ONLY in microcanonical ensemble (zero for N=∞).**

4) **Localized solution has subextensive entropy (area law?, entaglement?)**

Discrete Non-Linear Schrödinger Equation (DNLS)

Condensate wave-function (order parameter) $\langle\hat{\psi}\rangle = \psi(x_i,t) = \psi_i(t)$

$$i\,\frac{\partial\psi_i}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\psi_i^*} = -(\psi_{i+1}+\psi_{i-1}) - \nu|\psi_i|^2\psi_i$$

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N}(\psi_i^*\psi_{i+1} + \psi_i\psi_{i+1}^*) + \frac{\nu}{2}\sum_{i=1}^{N}|\psi_i|^4$$

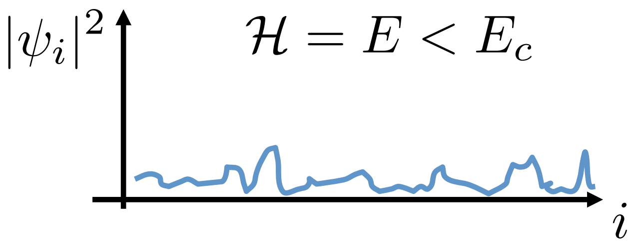
PARTICLES NUMBER (conserved)

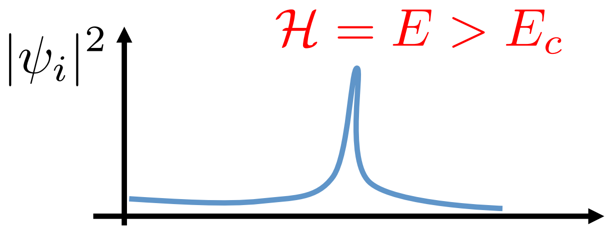
$$A = \sum_{i=1}^{N}|\psi_i|^2$$

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





RANDOM FIRST (MIXED) ORDER!

1) WHICH KIND OF PHASE TRANSITION ?

MICROCANONICAL

2) WHICH STATISTICAL ENSEMBLE?

3) LOCALIZATION COMES FROM INTEGRABILITY? (N integrals of motion) ***NO!***

4) IS DISORDER NECESSARY FOR LOCALIZATION? ***NO!***

Discrete Non-Linear Schrödinger Equation (DNLS)

QUITE OFTEN LOCALIZATION IS RELATED TO INTEGRABILITY

'Integrals of motion in the many-body localized phase', Valentina Ros, M. Müller, A. Scardicchio, *Nuclear Physics B* **891**, 420-465 **(2015)**

They compute explicitly the N integrals of motion!

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N} (\psi_i^* \psi_{i+1} + \psi_i \psi_{i+1}^*) + \frac{\nu}{2} \sum_{i=1}^{N} |\psi_i|^4$$

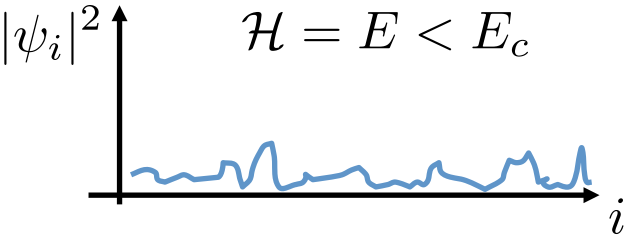
PARTICLES NUMBER (conserved)

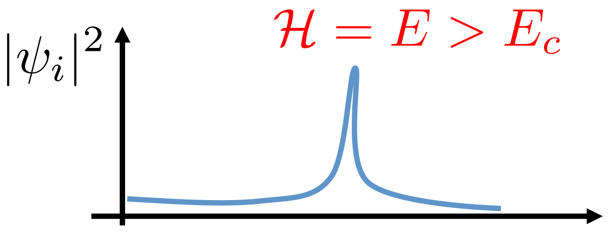
$$A = \sum_{i=1}^{N} |\psi_i|^2$$

PHENOMENON

Condensate wavefunction localized at high enegies

(numerical evidences)





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Discrete Non-Linear Schrödinger Equation (DNLS)

Anderson Localization

One-body localization due to quenched disorder

$$\mathcal{H} = J \sum_{\langle ij \rangle} \hat{c}_i^\dagger \hat{c}_j + \sum_{i=1}^{N} h_i \; \hat{c}_i^\dagger \hat{c}_i$$

Many-body Localization (MBL)

Disorder + **WEAK many-body interactions**.

$$\mathcal{H} = J \sum_{\langle ij \rangle} \hat{c}_i^\dagger \hat{c}_j + \sum_{i=1}^{N} h_i \; \hat{c}_i^\dagger \hat{c}_i + k \sum_{i=1}^{N} \hat{c}_i^\dagger \hat{c}_i \; \hat{c}_{i+1}^\dagger \hat{c}_{i+1}$$

STATE of THE ART

1) Localized phase is stable with respect to (weak) non-linearities.

2) Role of disorder in presence of many-body interactions?

3) Does localization survives without disorder?

Many-Body Localization is well understood pertubatively:

In jergon: 'a sort of quantum KAM theorem' (B. Altshuler)

OUR WORK
(strong coupling regime)

1) We do find localization in absence of disorder! (known numerically)

2) NON-LINEAR terms (many-body) are the source of localization!
(outcome of the exact calculation)

OUR RESULT IS ROBUST WITH RESPECT TO DIMENSIONALITY

ENERGY (conserved)

$$\mathcal{H} = \sum_{i=1}^{N} (\psi_i^* \psi_{i+1} + \psi_i \psi_{i+1}^*) + \frac{\nu}{2} \sum_{i=1}^{N} |\psi_i|^4$$

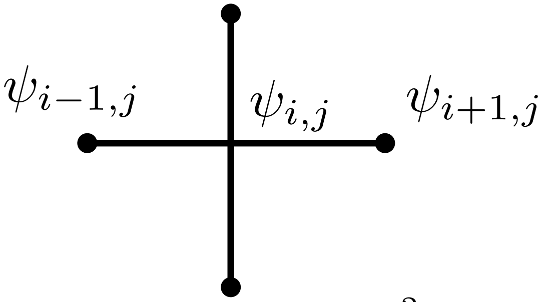
PARTICLES NUMBER (conserved)

$$A = \sum_{i=1}^{N} |\psi_i|^2$$

Everything relies upon neglecting the hopping terms at infinite temperature … very reasonable!

$$\Omega_N(A, E) = \int \prod_{i=1}^{N} d\psi_i \; \delta\left(A - \sum_{i=1}^{N} |\psi_i|^2\right) \; \delta\left(E - \sum_{i=1}^{N} |\psi_i|^4\right)$$

THE RESULT HOLDS IN ANY DIMENSION (consider 2d for example)



Discrete Laplacian:

- All information on dimensionality is here

- It does not play any role in localization

Entropy of the condensate

$$S_{micro} \sim N^{1/2} = L$$

$$\mathcal{H} = \sum_{ij}^{N=L^2} \overbrace{\left(\psi_{ij}^* \psi_{i+1,j} + \psi_{ij} \psi_{i+1,j}^* + \psi_{ij}^* \psi_{i,j+1} + \psi_{ij} \psi_{i,j+1}^*\right)} + \frac{\nu}{2} \sum_{ij} |\psi_{ij}|^4$$

OUR RESULT IS ROBUST WITH RESPECT TO DIMENSIONALITY

Exact results on Many-body Localization: severly tight to one-dimensional systems

SLOW DYNAMICS – ERGODICY BREAKING – LOCALIZATION – QUASI-INTEGRABILITY:
Perturbative approaches with results strongly attached to D=1
(consider for instance the Fermi-Pasta-Ulam problem)

QUANTUM DYNAMICS IN D=1 ~ CONFORMAL FIELD THEORIES IN D=2 (INTEGRABLE)

↓

By leaving the perturbative regime and exploiting the non-equivalence of ensembles

THE RESULT HOLDS IN ANY DIMENSION (consider 2d for example)

✓Localization in the strong coupling regime
✓Non-perturbative approach
✓Straighforward extension to D > 1

Entropy of the condensate

$$S_{micro} \sim N^{1/2} = L$$

$$\mathcal{H} = \sum_{ij}^{N=L^2} \left(\psi^*_{ij}\psi_{i+1,j} + \psi_{ij}\psi^*_{i+1,j} + \psi^*_{ij}\psi_{i,j+1} + \psi_{ij}\psi^*_{i,j+1} \right) + \frac{\nu}{2}\sum_{ij} |\psi_{ij}|^4$$

Localization and Ensemble Inequivalence

(in more 'exotic' systems, just an analogy)

NON-LINEAR FIELD EQUATIONS

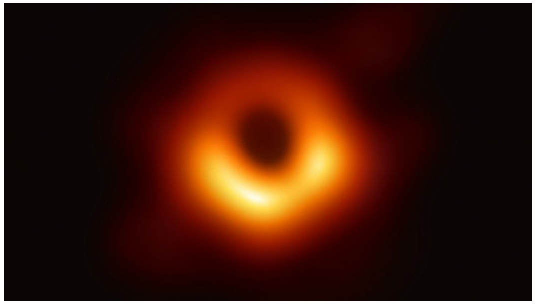
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Discrete Non-Linear Schrödinger

$$i \frac{\partial \psi_i}{\partial t} = -\frac{\partial \mathcal{H}}{\partial \psi_i^*} = -(\psi_{i+1} + \psi_{i-1}) - \nu |\psi_i|^2 \psi_i$$

LOCALIZED Schwarzschild SOLUTION (the Breather in the DNLS)

$$ds^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \frac{1}{\left(1 - \frac{2MG}{r}\right)} dr^2 - r^2 d\Omega$$



LOCALIZED SOLUTION PROPERTIES

- It adsorbs any extra amount of energy fed to the system, increasing its mass (like the Breather)
- True curvature singularity in the Black Hole, mass singularity in the DNLS
- Subextensive growth of the Entropy (counting of microstates)

$V \sim N$

$S_{\mathrm{Bh}} \sim N^{2/3}$

Non-Linear Schrodinger

$S_{\mathrm{micro}} \sim N^{1/2}$

CONCLUSIONS - PERSPECTIVES

1) We provided the first fully consistent description of the **localization transition** in the Discrete Non-Linear Schrödinger Equation **(DNLS)**

2) Localization in the DNLS can only described within the **Microcanonical Ensemble**

3) We put in evidence the existence, at large but finite N, of a delocalized (presumably non ergodic) state at negative temperature, the **pseudo-condensate** (relevant for experiments).

Further investigations: multifractal wave function: $I(q) = N\langle|\psi_i|^{2q}\rangle$

4) We clarified that the transition has a **mixed first/second order**, similarly to the **ergodicity breaking** transition in **glasses** (not spin glasses!): **Random First-Order transition**.

Further investigations: localization in models of glasses (in progress).

5) We clarified a mechanism for localization/ergodicity-breaking in the strong-coupling regime:

- Not related to integrability (only two conserved quantities, perhaps **emergent** integrability?)
- Straighforwad extension to D > 1 (further investigations)
- **DNLSE on dense random graph → Talk Next Week Tuesday 30th at 11.15AM**

« Localization in the Discrete Non-Linear Schrodinger Equation and the geometric properties of the Microcanonical surface »,

C. Arezzo, F. Balducci, R. Piergallini, A. Scardicchio, C. Vanoni,

arXiv:2102.10298

THANKS FOR YOUR ATTENTION