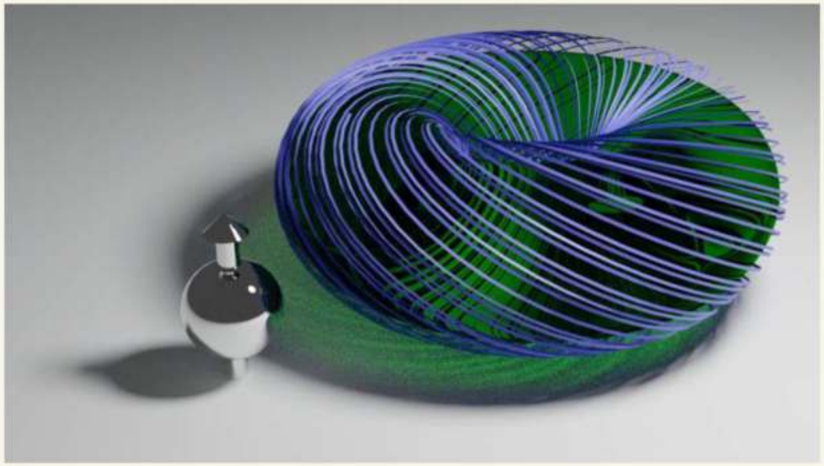
KAM stability for conserved quantities in quantum systems

Paolo Facchi - University of Bari & INFN Bari

Joint work with Daniel Burgarth
Hiromichi Nakazato
Saverio Pascazio
Kazuya Yuasa



Classical mechanics

Kolmogorov – Arnold – Moser (KAM) theorem

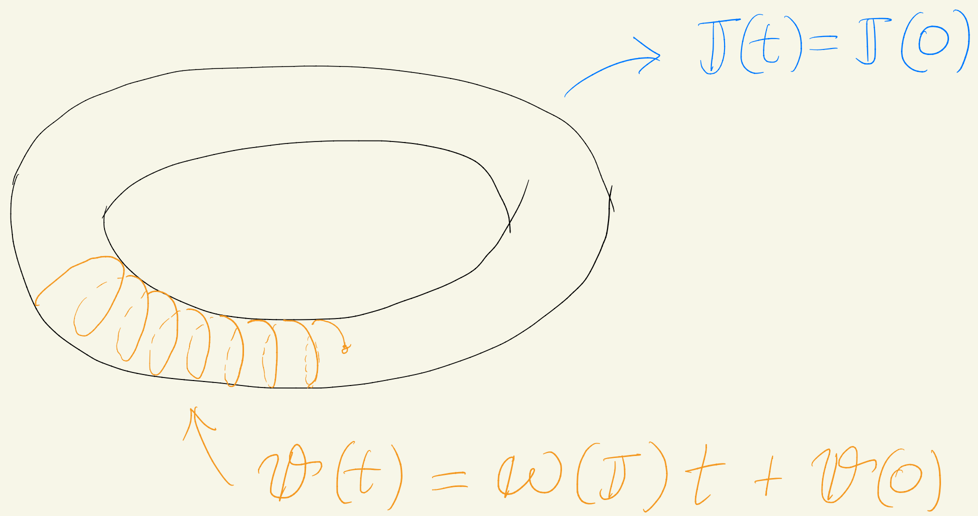
$H_0 = H_0(\mathbb{J})$ integrable system [action – angle variables $(\mathbb{J}, \vartheta) \in \mathbb{R}^d \times \mathbb{T}^d$]

Hamilton equations
$$\begin{cases} \dot{\vartheta} = \nabla_{\mathbb{J}} H_0(\mathbb{J}) = \omega(\mathbb{J}) \\ \dot{\mathbb{J}} = -\nabla_{\vartheta} H_0(\mathbb{J}) = 0 \end{cases}$$

$\Rightarrow$ $\mathbb{J}(t) = \mathbb{J}(0)$ is conserved (actions)

$\vartheta(t) = \vartheta(0) + \omega(\mathbb{J})\,t$ (angles)

(quasi-)periodic motion on the torus $\mathbb{T}^d$ at $\mathbb{J} = \text{const}$



Consider a small perturbation $\varepsilon << 1$

$$H_0(J) \longrightarrow H(J,\vartheta) = H_0(J) + \varepsilon V(J,\vartheta)$$

$$\dot{J}(t) = -\nabla_\vartheta H = \varepsilon \nabla_\vartheta V(J,\vartheta)$$ no longer conserved

$$\dot{\vartheta}(t) = \nabla_J H = \omega(J) + \varepsilon \nabla_J V(J,\vartheta)$$

Theorem (KAM) [1954 – 1967]

Assume H analytic and H_0 nondegenerate $\left[\left(\frac{\partial^2 H_0}{\partial J_j \partial J_k}(J)\right) \text{ invertible}\right]$

Assume that the torus $\mathcal{T}_0 = \{J_0\} \times \mathbb{T}^d$ has frequencies $\omega_0 = \omega(J_0)$ satisfying

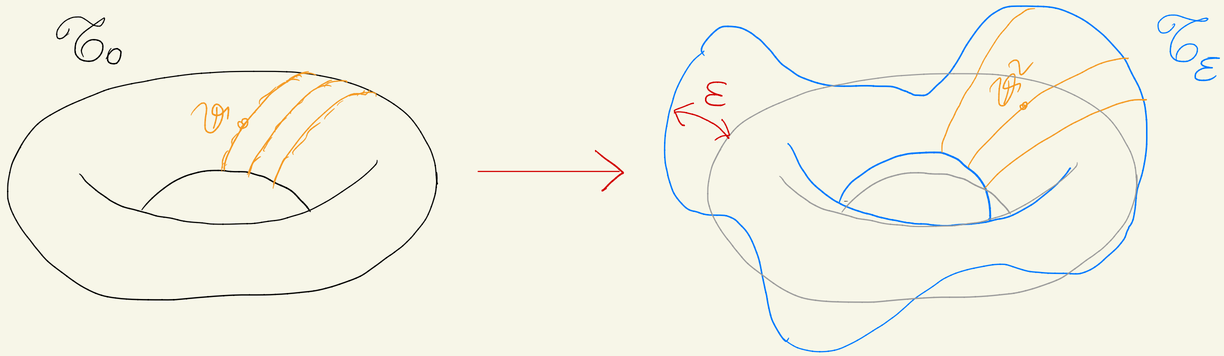
$$|\omega_0 \cdot m| \geq \frac{\gamma}{|m|^\mu} \quad \forall\, m \in \mathbb{Z}^d \setminus \{0\}, \text{ for some}$$

$\gamma > 0$, $\mu > d-1$ (Diophantine condition).

Then $\exists\, \varepsilon_c > 0$ s.t. $\forall\, \varepsilon \in (-\varepsilon_c, \varepsilon_c)$, there exists a deformation of the torus $\mathcal{T}_0$ into an invariant torus $\mathcal{T}_\varepsilon$.

$$|\omega(J_0)\cdot m| \geq \frac{\gamma}{|m|^{\mu}} \qquad \forall\, m \in \mathbb{Z}^d \setminus \{0\}$$

Non-resonant Diophantine condition



adiabatic invariant eternal

- The motions on $\mathcal{T}_\varepsilon$ are quasi-periodic with the same frequency ω_0 of the motions on $\mathcal{T}_0$

- The set of Diophantine invariant tori is a Cantor set. Its complement is dense and has Lebesgue measure $O(\sqrt{\varepsilon})$.

Quantum systems

Conserved quantities vs robustness to perturbations

- fragile symmetries:

  small perturbations $\rightarrow$ large deviations

- robust symmetries:

  small perturbations $\rightarrow$ small deviations

  forever

All symmetries are conserved
but some symmetries are more conserved than others.

Setting

System Finite-dimensional quantum system (closed)

Hilbert space $\mathcal{H}$, $\dim \mathcal{H} = d < +\infty$

Hamiltonian $H = H^\dagger \in B(\mathcal{H})$

Evolution unitary group $t \mapsto e^{-itH}$

Observables $A = A^\dagger \in B(\mathcal{H})$

$t \mapsto A_t = e^{itH} A e^{-itH}$ Heisenberg picture

Conserved quantities

$$A_t = e^{itH} A e^{-itH} = A \qquad \forall t \in \mathbb{R}$$

$$\Updownarrow$$

$$e^{itH} A - A e^{itH} = [e^{itH}, A] = 0 \quad \forall t \in \mathbb{R}$$

$$\Updownarrow$$

$$[A, H] = 0$$

Conserved quantities are symmetries [E. Noether]

$$[A, H] = 0 \iff e^{isA} H e^{-isA} = H \quad \forall s \in \mathbb{R}$$

Hamiltonian invariant with respect to the symmetry transformation e^{isA} generated by A.

$$\boxed{\begin{aligned} &\text{Symmetry} \iff S = S_t \iff S \in \{H\}' \\ &\{H\}' = \{ A \in B(\mathcal{H}) : [A, H] = 0 \} \quad \text{commutant of } H \end{aligned}}$$

The shape of a symmetry

$$H = \sum_{k=1}^{m} e_k P_k \quad \textit{spectral resolution} \qquad m \leq d$$

$$\mathrm{spec}(H) = \{e_k\} \subset \mathbb{R} \quad \textit{spectrum} \qquad (e_k \neq e_j\,, \quad k \neq j)$$

$$P_k P_j = \delta_{kj} P_j\,, \quad P_k = P_k^\dagger\,, \quad \sum_{k=1}^{m} P_k = 1 \quad \textit{eigenprojections}$$

$$S \in \{H\}' \text{ symmetry iff } \quad S = \sum_{k=1}^{m} P_k S P_k$$

$$H = \begin{pmatrix} \boxed{\begin{matrix} e_1 & \\ & e_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} e_2 & & \\ & e_2 & \\ & & e_2 \end{matrix}} & \\ & & \boxed{\begin{matrix} e_3 & \\ & e_3 \end{matrix}} \end{pmatrix} \Rightarrow \quad S = \begin{pmatrix} \boxed{S_1} & & \\ & \boxed{S_2} & \\ & & \boxed{S_3} \end{pmatrix} \quad \textit{symmetry}$$

$$S \in \{H\}' \text{ symmetry iff } S = \sum_{k=1}^{m} P_k S P_k$$

$$[A, H] = 0 \quad \Rightarrow \quad P_k [A, H] P_j = 0$$

$$0 = P_k (AH - HA) P_j = P_k A \underbrace{H P_j}_{\ell_j P_j} - \underbrace{P_k H}_{\ell_k P_k} A P_j = (\ell_j - \ell_k) P_k A P_j$$

$$\Rightarrow \quad P_k A P_j = 0 \text{ if } k \neq j$$

$$\Rightarrow \quad A = \sum_k P_k A \sum_j P_j = \sum_{k,j} P_k A P_j = \sum_k P_k A P_k$$

$$H = \begin{pmatrix} \boxed{\begin{matrix} \ell_1 & \\ & \ell_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} \ell_2 & & \\ & \ell_2 & \\ & & \ell_2 \end{matrix}} & \\ & & \boxed{\begin{matrix} \ell_3 & \\ & \ell_3 \end{matrix}} \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} \boxed{S_1} & & \\ & \boxed{S_2} & \\ & & \boxed{S_3} \end{pmatrix} \quad \text{symmetry}$$

Zeno projection

$$S \in \{H\}' \iff S = \sum_{k=1}^{m} P_k S P_k$$

(it depends only on P_k not on e_k)

$$\Pi : B(\mathcal{H}) \longrightarrow \{H\}'$$
$$A \longmapsto \Pi(A) = \sum_{k=1}^{m} P_k A P_k$$

projection onto the commutant

$$S \in \{H\}' \iff S = \Pi(S)$$

$$V_Z = \Pi(V) = \sum_{k=1}^{m} P_k V P_k$$ "Zeno" Hamiltonian

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$$\left\| e^{-it(KH+V)} - e^{-it(KH+V_Z)} \right\| \leqslant 2\frac{\sqrt{m}}{\eta}\frac{1}{K}\|V\|\left(1+\|V\|t\right)$$

$$e^{-it(KH+V)} = e^{-it(KH+V_Z)} + O\left(\frac{1}{K}\right)$$ up to times $O(1)$

as $K \to \infty$ strong coupling limit

$$\left(P e^{-i\frac{t}{n}V}\right)^n = \underbrace{P e^{-i\frac{t}{n}V} P e^{-i\frac{t}{n}V} \cdots P e^{-i\frac{t}{n}V}}_{n \text{ times}} \longrightarrow e^{-itPVP} P$$

as $n \longrightarrow \infty$

quantum Zeno effect (frequent projective measurements)

What happens if

$$H \longrightarrow H + \varepsilon V \qquad \varepsilon << 1 \qquad (\|V\| = 1)$$

small perturbation ?

S symmetry, $S \in \{H\}'$

$$S_t^{\varepsilon} = e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} \overset{?}{=} S + \overbrace{O(\varepsilon)}^{\boxed{\text{independent of } t}} \quad \text{for all } \|V\| = 1$$

By perturbation theory this is true for small times. Is it stable for all times?

If this is the case, we call S a robust symmetry.

Divergence of two quantum evolutions

$$e^{it(H+\varepsilon V)} - e^{itH} = \left[e^{i(t-s)H} e^{is(H+\varepsilon V)} \right]_{s=0}^{s=t}$$

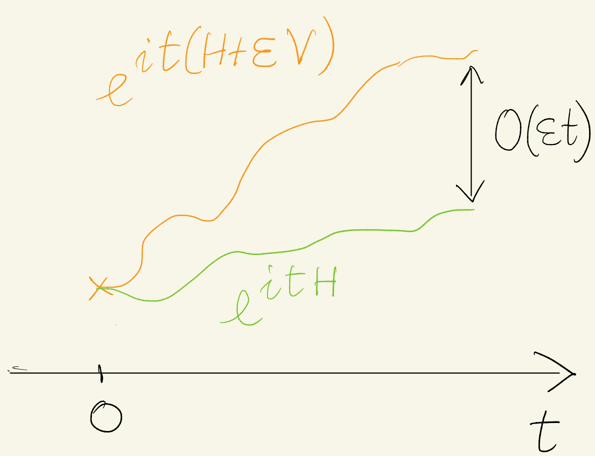
$$= \int_0^t ds \, \frac{d}{ds} \left[e^{i(t-s)H} e^{is(H+\varepsilon V)} \right]$$

$$= \int_0^t ds \; e^{i(t-s)H} \, i \varepsilon V \, e^{is(H+\varepsilon V)}$$

$$\| e^{it(H+\varepsilon V)} - e^{itH} \| \leq \int_0^t ds \; \varepsilon \, \| e^{i(t-s)H} V e^{is(H+\varepsilon V)} \|$$

$\|WAU\| = \|A\|$ $$\overset{(*)}{=} \int_0^t ds \; \varepsilon \|V\| = \varepsilon t \|V\|$$

$$\delta(t) = \| e^{it(H+\varepsilon V)} - e^{itH} \| \leq \varepsilon \|V\| t$$



$$(*) \quad \|WAU\| = \|A\| \qquad \forall\, W, U \text{ unitaries}$$

$$\|WAU\| = \sup_{\|\varphi\|=\|\psi\|=1} |\langle \varphi | WAU\psi\rangle|$$

$$= \sup_{\|\varphi\|=\|\psi\|=1} |\langle \underbrace{W^\dagger\varphi}_{\tilde{\varphi}} | A \underbrace{U\psi}_{\tilde{\psi}}\rangle|$$

$$= \sup_{\|\tilde{\varphi}\|=\|\tilde{\psi}\|=1} |\langle \tilde{\varphi} | A \tilde{\psi}\rangle| = \|A\|$$

Evolution of observables

$$\boxed{\|A_t^\varepsilon - A_t\| \leq 2\|A\|\,\|e^{it(H+\varepsilon V)} - e^{itH}\| = 2\|A\|\,\delta(t)}$$

$$\begin{aligned}
\|A_t^\varepsilon - A_t\| &= \|e^{it(H+\varepsilon V)} A\, e^{-it(H+\varepsilon V)} - e^{itH} A\, e^{-itH}\| \\
&= \|e^{itH}\left(e^{-itH} e^{it(H+\varepsilon V)} A - A\, e^{-itH} e^{it(H+\varepsilon V)}\right) e^{-it(H+\varepsilon V)}\| \\
&\overset{(*)}{=} \|[e^{-itH} e^{it(H+\varepsilon V)}, A]\| \\
&= \|[e^{-itH} e^{it(H+\varepsilon V)} \underbrace{- 1}, A]\| \qquad [1, A] = 0 \\
&\leq 2\|A\|\,\|e^{-itH} e^{it(H+\varepsilon V)} - 1\| \overset{(*)}{=} 2\|A\|\,\|e^{it(H+\varepsilon V)} - e^{itH}\|
\end{aligned}$$

$(*)$ $\|WAU\| = \|A\|$

$\|A_t^\varepsilon - A_t\| \leq 2\|A\|\,\delta(t) \leq 2\varepsilon t\,\|V\|\,\|A\| = O(\varepsilon)$ for $t = O(1)$

For a symmetry: $S_t = e^{itH} S e^{-itH} = S$

$$\|S_t^\varepsilon - S\| \le 2\varepsilon t \|V\| \|S\| = O(\varepsilon) \quad \text{for } t = O(1)$$

Hower, notice that

$$\|S_t^\varepsilon - S\| \le \|S_t^\varepsilon\| + \|S\| = 2\|S\|$$

maximum distance $= 2\|S\|$ obtained if $S_t^\varepsilon = -S$

A symmetry $S \in \{H\}'$ is fragile if there exists a perturbation $V = V^\dagger \in B(\mathcal{H})$ such that

$$\sup_{t \in \mathbb{R}} \| e^{+it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S \| = O(1) \qquad \forall \varepsilon > 0$$

however small

Eventually, for sufficiently long times, it becomes $O(1)$

$$\| S_t^\varepsilon - S \| = O(\varepsilon) \qquad \text{for times} \quad t = O(1)$$

$$= O(1) \qquad \text{for times} \quad t = O\left(\tfrac{1}{\varepsilon}\right)$$

the divergence can incremetally accumulate over time

but does it?

$$H|u_1\rangle = e|u_1\rangle, \qquad H|u_2\rangle = e|u_2\rangle$$

$$H = \begin{pmatrix} \boxed{\begin{matrix} e_1 & \\ & e_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} e & & \\ & e & \\ & & e \end{matrix}} & \\ & & \boxed{\begin{matrix} e_3 & \\ & e_3 \end{matrix}} \end{pmatrix} \qquad S = \begin{pmatrix} \boxed{S_1} & & \\ & \boxed{\begin{matrix} s_1 & & \times \\ & s_2 & \times \\ \times & \times & \times \end{matrix}} & \\ & & \boxed{S_3} \end{pmatrix}$$

S breaks the degeneracy

$$S|u_1\rangle = s_1|u_1\rangle, \qquad S|u_2\rangle = s_2|u_2\rangle \qquad s_2 - s_1 > 0$$

$$V = |u_1\rangle\langle u_2| + |u_2\rangle\langle u_1| \qquad V = \begin{pmatrix} \boxed{} & & \\ & \boxed{\begin{matrix} & 1 & \\ 1 & & \\ & & \end{matrix}} & \\ & & \boxed{} \end{pmatrix}$$

$$|\psi\rangle = \alpha|u_1\rangle + \beta|u_2\rangle$$

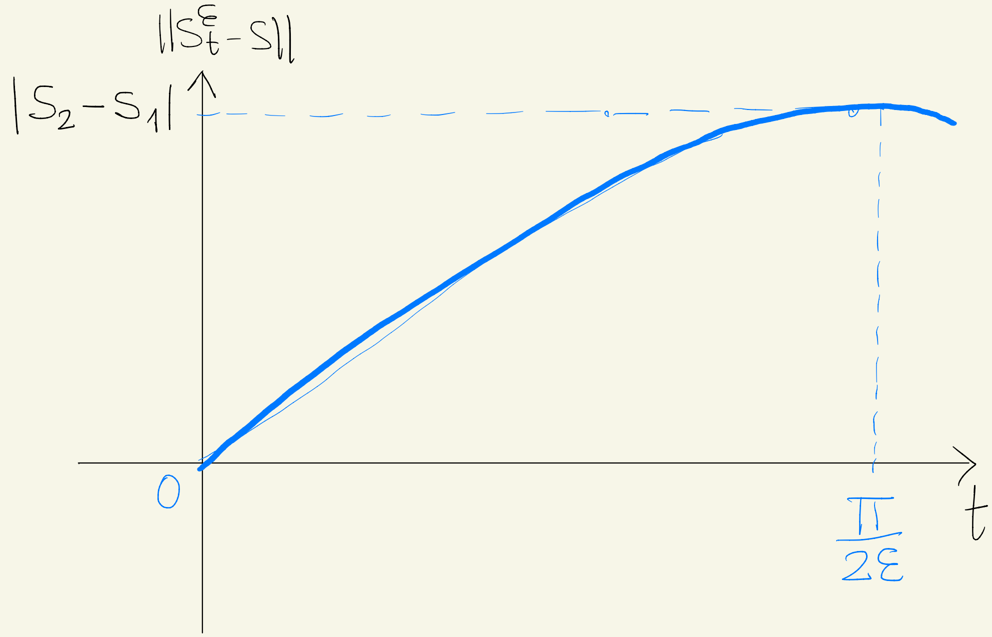
two-dimensional problem

$$|u_1(t)\rangle = e^{-it(H+\varepsilon V)}\,|u_1\rangle = e^{-ite}\left(\cos(\varepsilon t)\,|u_1\rangle - i\sin(\varepsilon t)\,|u_2\rangle\right)$$

$$|u_2(t)\rangle = e^{-it(H+\varepsilon V)}\,|u_2\rangle = e^{-ite}\left(-i\sin(\varepsilon t)\,|u_1\rangle + \cos(\varepsilon t)\,|u_2\rangle\right)$$

$$\|S_t^\varepsilon - S\| = |s_2 - s_1|\,|\sin \varepsilon t|$$

$$t = \frac{\pi}{2\varepsilon} = O\left(\frac{1}{\varepsilon}\right) \Rightarrow \|S_t^\varepsilon - S\| = O(1)$$



S is fragile over long times!

$$|u_1(t)\rangle = e^{-it(H+\varepsilon V)}|u_1\rangle = e^{-ite}\left(\cos(\varepsilon t)|u_1\rangle - i\sin(\varepsilon t)\,|u_2\rangle\right)$$

$$|u_2(t)\rangle = e^{-it(H+\varepsilon V)}|u_2\rangle = e^{-ite}\left(-i\sin(\varepsilon t)|u_1\rangle + \cos(\varepsilon t)\,|u_2\rangle\right)$$

$$S_t^\varepsilon = s_1|u_1(-t)\rangle\langle u_1(-t)| + s_2|u_2(-t)\rangle\langle u_2(-t)|$$

$$S_t^\varepsilon = \begin{pmatrix} s_1\cos^2(\varepsilon t) + s_2\sin^2(\varepsilon t) & i(s_1-s_2)\sin(\varepsilon t)\cos(\varepsilon t) \\ -i(s_1-s_2)\sin(\varepsilon t)\cos(\varepsilon t) & s_1\sin^2(\varepsilon t) + s_2\cos^2(\varepsilon t) \end{pmatrix}$$

$$S_t^\varepsilon - S = \begin{pmatrix} (s_2-s_1)\sin^2(\varepsilon t) & -i(s_2-s_1)\sin(\varepsilon t)\cos(\varepsilon t) \\ i(s_2-s_1)\sin(\varepsilon t)\cos(\varepsilon t) & -(s_2-s_1)\sin^2(\varepsilon t) \end{pmatrix}$$

$$S_t^\varepsilon - S = (s_2-s_1)\sin(\varepsilon t)\begin{pmatrix} \sin(\varepsilon t) & -i\cos(\varepsilon t) \\ i\cos(\varepsilon t) & -\sin(\varepsilon t) \end{pmatrix} = (s_2-s_1)\sin(\varepsilon t)\left(\cos(\varepsilon t)\sigma_y + \sin(\varepsilon t)\sigma_z\right)$$

$$\|\cos(\varepsilon t)\sigma_y + \sin(\varepsilon t)\sigma_z\| = 1$$

$$\|S_t^\varepsilon - S\| = |s_2-s_1|\,|\sin(\varepsilon t)| \leq 2\|S\|\,|\sin(\varepsilon t)|$$

Bicommutant $\{H\}''$

$$S = \sum_{k=1}^{m} s_k P_k \qquad S = f(H) \in \{H\}'' \subset \{H\}'$$

$$\begin{aligned}
\{H\}'' &= \{ A \in B(\mathcal{H}) : [A, B] = 0 \quad \forall B \in \{H\}' \} \\
&= \{ f(H) \in B(\mathcal{H}) \quad : \quad f : \mathbb{R} \to \mathbb{C} \} \\
&= \{ \sum_{k=1}^{m} f(e_k) P_k \quad : \quad f : \mathbb{R} \to \mathbb{C} \} \\
&= \{ \sum_{k=1}^{m} p_{m-1}(e_k) P_k \quad : \quad p_{m-1}(x) = \sum_{k=0}^{m-1} a_k x^k , \ a_k \in \mathbb{C} \} \\
&= \{ \sum_{k=0}^{m} a_k H^k \quad : \quad a_k \in \mathbb{C} \}
\end{aligned}$$

$S \in \{H\}''$ is robust ?

$$\left(f(e_1), \dots, f(e_m)\right) \longleftrightarrow \left(a_0, a_1, \dots, a_{m-1}\right)$$

$$\begin{cases} a_0 + a_1 e_1 + a_2 e_1^2 + \dots + a_{m-1} e_1^2 = f(e_1) \\ \vdots \\ a_0 + a_1 e_m + a_2 e_m^2 \dots + a_{m-1} e_m^2 = f(e_m) \end{cases}$$

$$V = \begin{pmatrix} 1 & e_1 & e_1^2 & \dots & e_1^{m-1} \\ 1 & e_2 & e_2^2 & \dots & e_2^{m-1} \\ \vdots & & & \ddots & \vdots \\ 1 & e_m & e_m^2 & \dots & e_m^{m-1} \end{pmatrix} = \left(e_j^{k-1} \right)_{1 \leq j, k \leq m}$$

$$\det V = \prod_{1 \leq j < k \leq m} (e_j - e_k) \neq 0$$

Vandermonde determinant

What if the perturbation is a symmetry?

$$V = \Pi(V) \in \{H\}' \qquad S = f(H) \in \{H\}'' \qquad S = \sum_{k=1}^{m} s_k P_k$$

$$\Downarrow$$

$$[S, V] = 0$$

$$S_t^{\varepsilon} = e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} = S = S_t = e^{itH} S e^{-itH}$$

$$S_t^{\varepsilon} = S_t = S$$ conserved for all times

For a generic $V \notin \{H\}'$ it is no longer true that $S_t^\varepsilon = S$.

However, we have

$$\|S_t^\varepsilon - S\| = \|[e^{it(H+\varepsilon V)}, S]\|$$

$$= \|[e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon\pi(V))}, S] + \underbrace{[e^{it(H+\varepsilon\pi(V))}, S]}_{=0}\|$$

$S \in \{H\}''$ $\quad H, \pi(V) \in \{H\}' \Rightarrow [e^{it(H+\varepsilon\pi(V))}, S] = 0$

$$\boxed{\|S_t^\varepsilon - S\| \leq 2\|S\| \, \|e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon\pi(V))}\| = 2\|S\| \, \delta_Z(t)}$$

$S \in \{H\}''$

$$\| S_t^{\varepsilon} - S \| \leqslant 2 \| S \| \, \delta_Z(t)$$

$$\delta_Z(t) = \| e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon \pi(V))} \|$$

One can prove that $\delta_Z(t) \leqslant \dfrac{2\sqrt{m}}{\eta} \varepsilon \|V\| (1 + \varepsilon \|V\| t)$

where $\eta = \min_{k \neq j} |e_k - e_j|$ *minimal spectral gap of H*

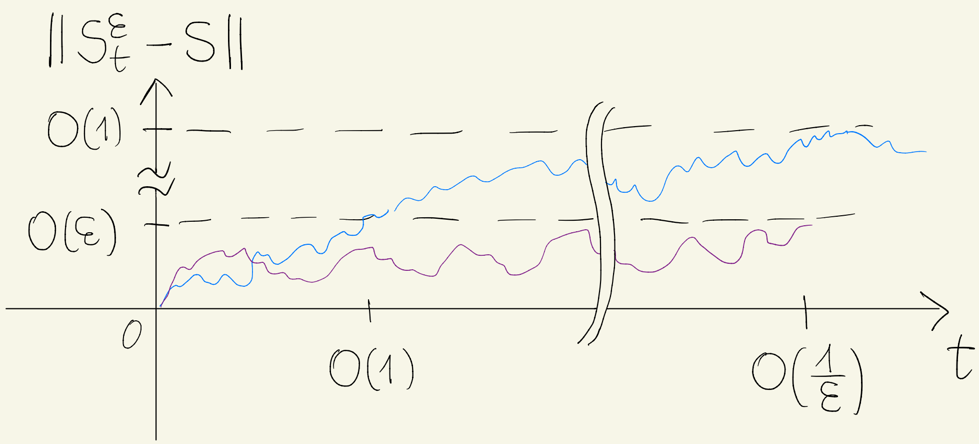
$$\delta_z(t) = O(\varepsilon)\big(1+O(\varepsilon t)\big)$$

The distance can accumulate over time.

Notice, however, that

$$\|S_t^\varepsilon - S\| = O(\varepsilon) \quad \text{up to times } t = O\left(\tfrac{1}{\varepsilon}\right) \;!$$

The symmetry S is conserved for longer times.



$S \in \{H\}' \setminus \{H\}''$

$S \in \{H\}''$

Improved strategy : generic $\hat{V} \in \{H\}'$ $\quad \hat{V} \neq \Pi(V)$

$$\|S_t^\varepsilon - S\| = \|[e^{it(H+\varepsilon V)}, S]\|$$

$$= \|[e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon \hat{V})}, S] + \underbrace{[e^{it(H+\varepsilon \hat{V})}, S]}_{=0}\|$$

$S \in \{H\}''$ $\quad H, \hat{V} \in \{H\}' \Rightarrow [e^{it(H+\varepsilon \hat{V})}, S] = 0$

$$\|S_t^\varepsilon - S\| \leq 2\|e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon \hat{V})}\| \, \|S\| \leq 2\|S\|\delta_\infty$$

$$\delta_\infty = \sup_{t\in\mathbb{R}} \|e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon \hat{V})}\| = O(\varepsilon) \ ?$$

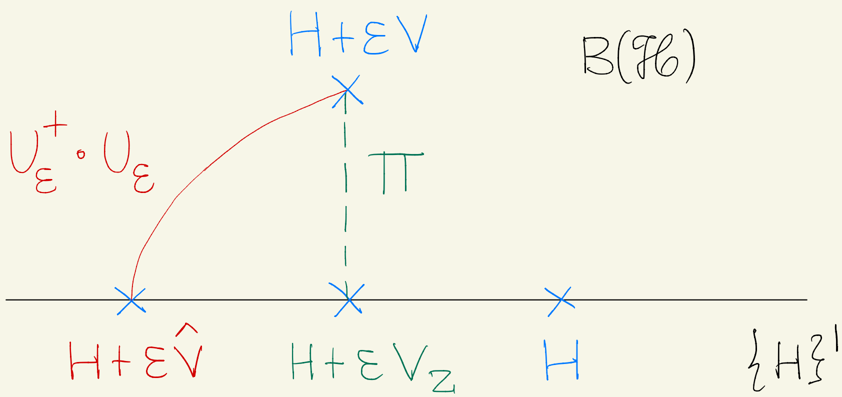
MAIN IDEA:

Suppose there exists a unitary U_ε, close to the identity $U_\varepsilon = 1 + O(\varepsilon)$, such that

$$\boxed{H + \varepsilon\hat{V} = U_\varepsilon^\dagger (H+\varepsilon V) U_\varepsilon}$$

isospectral

$$
\begin{aligned}
\delta_\infty &= \sup_{t\in\mathbb{R}} \left\| e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon \hat{V})} \right\| \\
&= \sup_{t\in\mathbb{R}} \left\| e^{it(H+\varepsilon V)} - U_\varepsilon^{+} e^{it(H+\varepsilon V)} U_\varepsilon \right\| \\
&= \sup_{t\in\mathbb{R}} \left\| \left[U_\varepsilon , e^{it(H+\varepsilon V)} \right] \right\| \\
&= \sup_{t\in\mathbb{R}} \left\| \left[U_\varepsilon - 1 , e^{it(H+\varepsilon V)} \right] \right\| \\
&\leq 2 \sup_{t\in\mathbb{R}} \left\| U_\varepsilon - 1 \right\| \left\| e^{it(H+\varepsilon V)} \right\| = 2 \left\| U_\varepsilon - 1 \right\| = O(\varepsilon) \quad !
\end{aligned}
$$



$$\mathrm{spec}\,(H+\varepsilon\hat{V}) = \mathrm{spec}\,(H+\varepsilon V)$$

Isospectral perturbations

$H = H^{+}$ perturbation $\tilde{H} = H + O(\varepsilon)$, $\varepsilon << 1$

divergence $\delta(t) = \| e^{it\tilde{H}} - e^{itH} \|$

spectral decompositions : $H = \sum_{k=1}^{m} e_k P_k \qquad \tilde{H} = \sum_{k=1}^{m} \tilde{e}_k \tilde{P}_k$

$\tilde{e}_k \neq \tilde{e}_j$ for $k \neq j$ (but it may be $e_k = e_j$)

$\tilde{P}_k = P_k + O(\varepsilon) \qquad \tilde{e}_k = e_k + O(\varepsilon)$ [Kato]

$$e^{it\tilde{H}} - e^{itH} = \sum_k \left(e^{it\tilde{e}_k} \tilde{P}_k - e^{ite_k} P_k \right)$$
$$= \underbrace{\sum_k e^{it\tilde{e}_k} (\tilde{P}_k - P_k)}_{\text{eigenprojections}} + \underbrace{\sum_k \left(e^{it\tilde{e}_k} - e^{ite_k} \right)}_{\text{eigenvalues}} P_k$$

$\delta_P(t) = \| \sum_k e^{it\tilde{e}_k} (\tilde{P}_k - P_k) \| \leq \sum_k \| \tilde{P}_k - P_k \| = O(\varepsilon)$ uniformly in time

$$\delta_e(t) = \| \sum_k \left(e^{it\tilde{e}_k} - e^{ite_k} \right) P_k \| = \max_k | e^{it\tilde{e}_k} - e^{ite_k} | = 2 \max_k \left| \sin\left(t \, \frac{\tilde{e}_k - e_k}{2} \right) \right|$$

$$\delta(t) = \delta_e(t) + \delta_P(t) = 2 \max_k \left| \sin\left(t \, \frac{\overbrace{\tilde{e}_k - e_k}^{O(\varepsilon)}}{2} \right) \right| + O(\varepsilon)$$

$$\delta(t) = O(\varepsilon) \qquad \text{for } t = O(1)$$

but $$\delta(t) = 2 + O(\varepsilon) \qquad \text{for } t = \frac{\pi}{\tilde{e}_k - e_k} = O\left(\frac{1}{\varepsilon}\right)$$

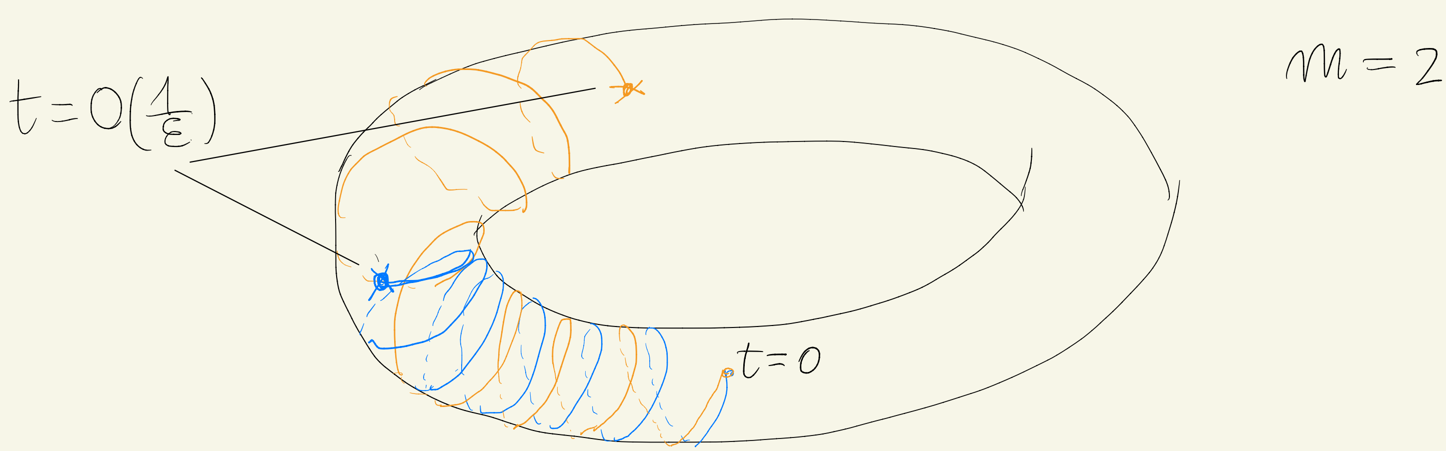
$\delta_e(t)$ secular drift.

$$\delta_\infty = \sup_t \delta(t) = 2$$

$$\boxed{\begin{aligned} \delta_\infty = O(\varepsilon) \quad &\text{iff} \quad \tilde{e}_k = e_k \quad \forall k \\ &\text{iff} \quad \operatorname{spec}(H) = \operatorname{spec}(\tilde{H}) \end{aligned}}$$

$$\sum_k \left(e^{it\tilde{e}_k} - e^{ite_k}\right) P_k \longmapsto \underbrace{\left(e^{it\tilde{e}_1}, e^{it\tilde{e}_2}, \ldots, e^{it\tilde{e}_m}\right)}_{\in\, \mathbb{T}^m} - \underbrace{\left(e^{ite_1}, e^{ite_2}, \ldots e^{ite_m}\right)}_{\in\, \mathbb{T}^m}$$

m-torus



KAM *iteration scheme*

$$H + \varepsilon \hat{V}(\varepsilon) = U_\varepsilon^+ (H + \varepsilon V) U_\varepsilon \qquad \text{with} \quad \hat{V}(\varepsilon) = \Pi(\hat{V}(\varepsilon))$$

$$\Pi(A) = \sum_{k=1}^{m} P_k A P_k$$

$$U_\varepsilon = e^{iK(\varepsilon)} \qquad K(\varepsilon) = \varepsilon K_1 + O(\varepsilon^2) \qquad K_1 = K_1^+$$

$$\hat{V}(\varepsilon) = V_0 + O(\varepsilon) \qquad V_0 = V_0^+ \qquad V_0 = \Pi(V_0)$$

$$H + \varepsilon \hat{V}(\varepsilon) = (1 - i\varepsilon K_1)(H + \varepsilon V)(1 + i\varepsilon K_1) + O(\varepsilon^2)$$

$$\boxed{V_0 = i[H, K_1] + V}$$

$$\Pi([H, K_1]) = \sum_k P_k (HK_1 - K_1 H) P_k = \sum_k P_k (e_k K_1 - K_1 e_k) P_k = 0$$

$$\Rightarrow \quad V_0 = \Pi(V_0) = \Pi(V) = V_Z \qquad 1) \; V_0 = \Pi(V) = V_Z$$

$$i[H, K_1] = -(V - \Pi(V)) = -V_{off} \qquad 2) \; i[H, K_1] = -V_{off}$$

$$V_{off} = V - \Pi(V) = \sum_{k,l : k \neq l} P_k V P_l = \sum_k P_k V (1 - P_k)$$

off-diagonal part of V

Homological equation for K_1

$$i[H, K_1] = V_{off} \qquad (*)$$

$$i P_k (H K_1 - K_1 H) P_j = - P_k V P_j \qquad k \neq j$$

$$i (e_k - e_j) P_k K_1 P_j = - P_k V P_j$$

$$P_k H P_j = i \frac{1}{e_k - e_j} P_k V P_j$$

$$K_1 = i \sum_{k,j : k \neq j} \frac{1}{e_k - e_j} P_k V P_j + \pi(K_1) = i \sum_j S_j V P_j + \pi(K_1)$$

$$S_j = \sum_{k : k \neq j} \frac{1}{e_k - e_j} P_k$$

reduced resolvent at e_j:

$$(H - e_j) S_j = S_j (H - e_j) = 1 - P_j$$

Equation $(*)$ has a unique solution with $\pi(K_1) = 0$:

$$K_1 = i \sum_{j=1}^{m} S_j V P_j$$

$$U_\varepsilon = 1 + i\varepsilon K_1 + O(\varepsilon^2)$$

$$\delta_\infty = \sup_{t\in\mathbb{R}} \| e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon \hat{V})} \| \leq 2\|U_\varepsilon - 1\| = 2\varepsilon \|K_1\| + O(\varepsilon^2)$$

$$\|K_1\|^2 = \|K_1 K_1^+\| = \|\sum_j S_j V P_j V S_j\| \leq \sum_j \|S_j V P_j V S_j\| \leq \sum_j \|S_j\|^2 \|V\|^2$$

$$\|S_j\| = \|\sum_{k:k\neq j} \frac{1}{e_k - e_j} P_k\| = \max_{k:k\neq j} \left| \frac{1}{e_k - e_j} \right| \leq \frac{1}{\eta}$$

$$\eta = \min_{k,j:\, k\neq j} |e_k - e_j|$$ *minimal spectral gap*

$$\|K_1\|^2 \leq \sum_{j=1}^{m} \frac{1}{\eta^2} \|V\|^2 = \frac{m}{\eta^2}\|V\|^2$$

$$\delta_\infty \leq 2\varepsilon \frac{\sqrt{m}}{\eta} \|V\| + O(\varepsilon^2) \leq \frac{7\sqrt{m}}{\eta} \varepsilon \|V\|$$

Conclusions

Definition

- $S \in \{H\}'$ *fragile* if $\exists\, V = V^{+} \in B(\mathcal{H})$ s.t.

$$\sup_{t\in\mathbb{R}} \| e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S \| = O(1)$$

$$\left[\inf_{\varepsilon>0} \sup_{t\in\mathbb{R}} \| e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S \| > 0 \right]$$

- $S \in \{H\}'$ *robust* if it is not fragile.

$$\left[\forall\, V = V^{+} \in B(\mathcal{H}) : \sup_{t\in\mathbb{R}} \| e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S \| = O(\varepsilon) \right]$$

Theorem [robust symmetries]

$S \in \{H\}'$ symmetry of $H = H^\dagger$. Then the following assertions are equivalent:

1. S is robust $\left[\sup_{t \in \mathbb{R}} \| e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S \| = O(\varepsilon) \quad \forall\, V = V^\dagger,\ \|V\| = 1 \right]$

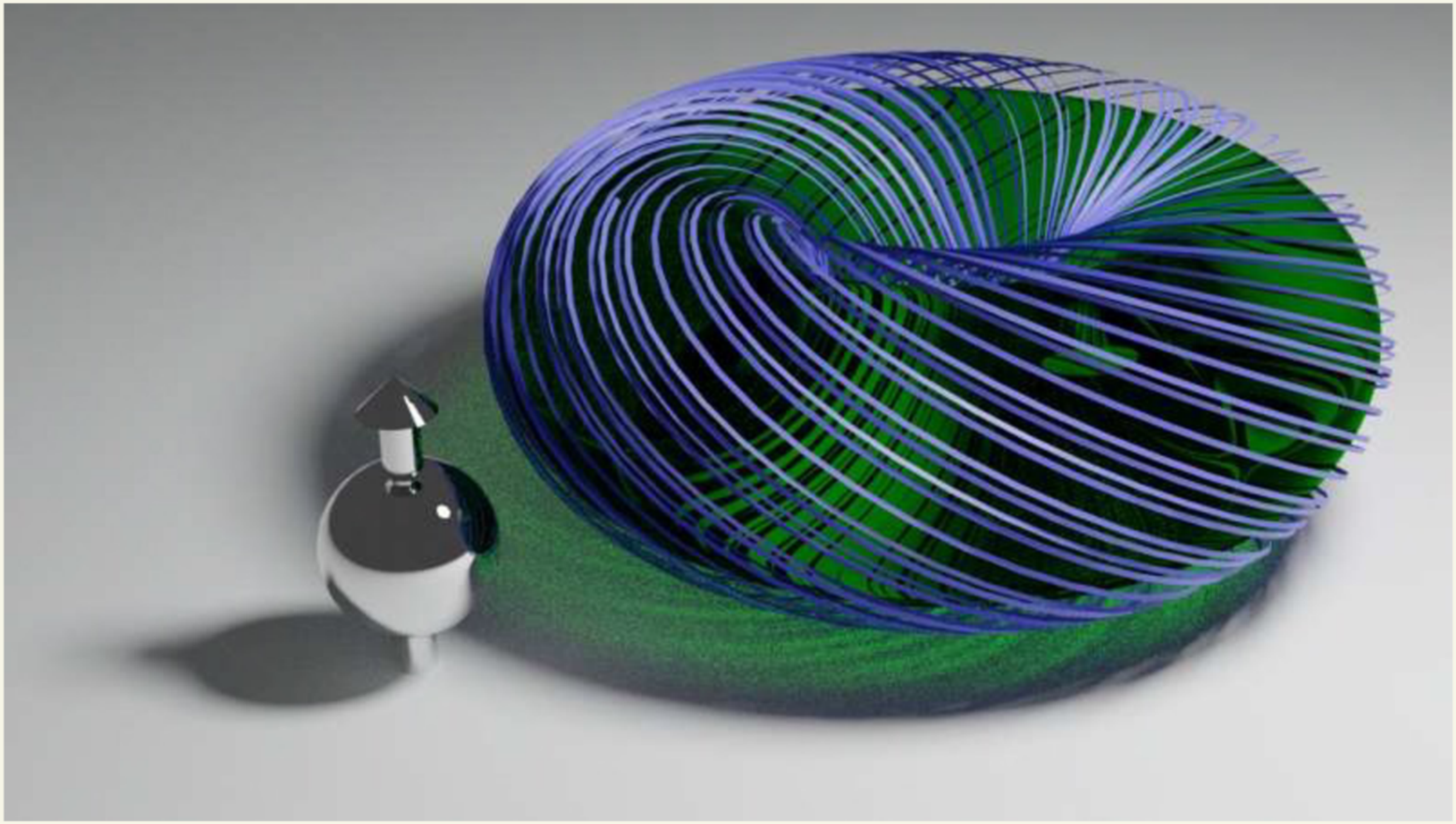
2. $S \in \{H\}''$

3. $\exists\, c = c_H > 0$ s.t. $\forall\, V = V^\dagger \in B(\mathcal{H})$:

$$\sup_{t \in \mathbb{R}} \| e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)} - S \| \leq c_H\, \varepsilon \|V\| \|S\|$$

$\left[c_H < 14 \frac{\sqrt{m}}{\eta} \quad , \quad H = \sum_{k=1}^{m} e_k P_k \quad , \quad \eta = \min_{k \neq j} |e_k - e_j| \right]$

Thank you!



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