Optimal collision avoidance in swarms of active Brownian particles

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Collective motion

Living organisms

Flocking	Schooling	Swarming
		

Common approach: Observation → Goals/biological role

Optimization approach: Goals → Optimal behaviours
(Game theory, Machine learning)

Mazzolini, A., & Celani, A. (2020). Generosity, selfishness and exploitation as optimal greedy strategies for resource sharing. *Journal of theoretical biology*, *485*, 110041.
Pezzotta, A., Adorisio, M., & Celani, A. (2018). Chemotaxis emerges as the optimal solution to cooperative search games. *Physical Review E*, *98*(4), 042401.

Collective motion

Robots

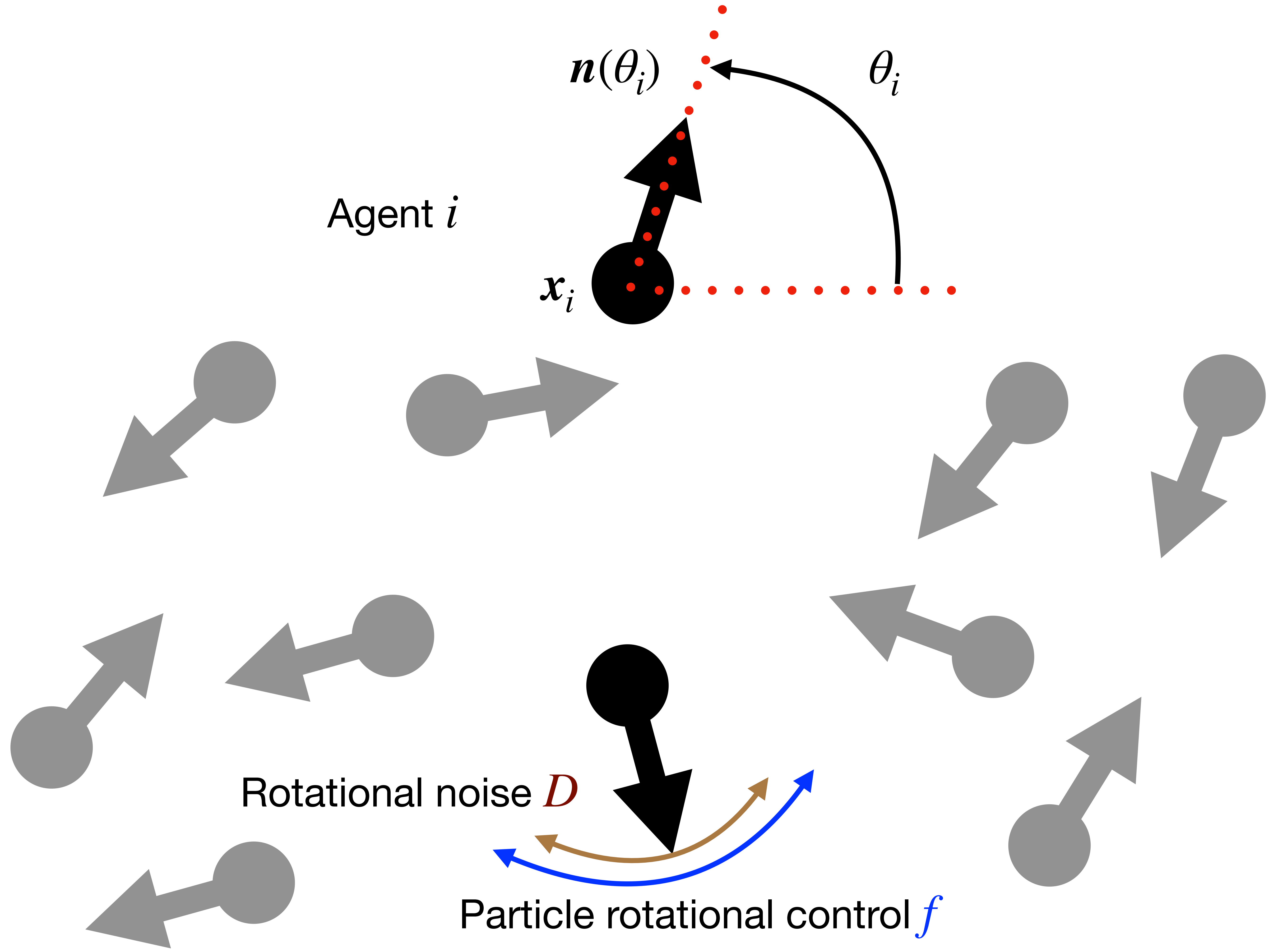
kilobots. Source: wikipedia. Author: asuscreative

Reinforcemente learning

Optimal control

Biomimetic algorithm

Our model: a system of N active brownian particles in 2D



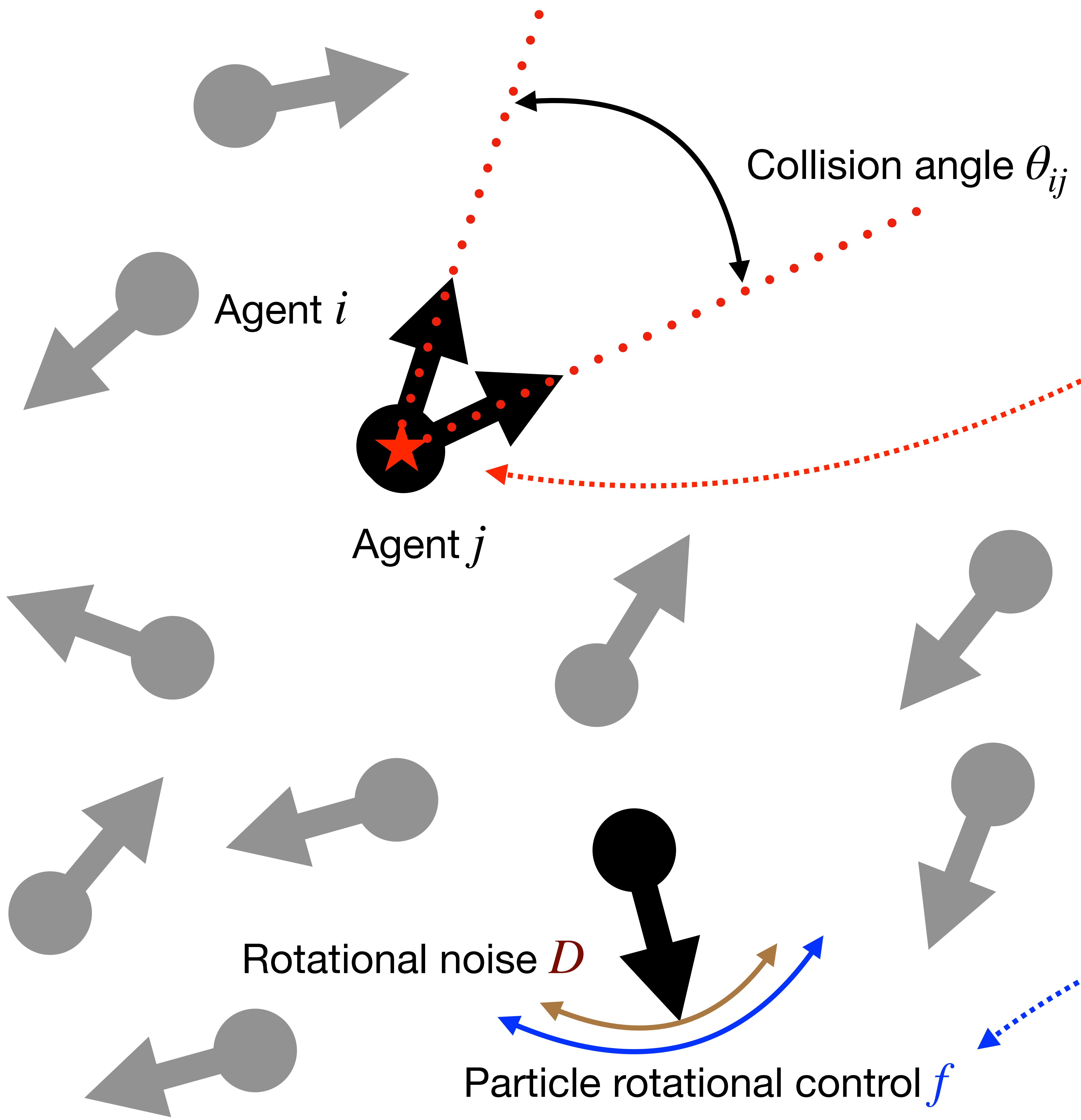
$$d\boldsymbol{x}_i = u_0\, \boldsymbol{n}(\theta_i)\, dt$$

$$d\theta_i = f_i(\boldsymbol{x}_i, \theta_i; \{\boldsymbol{x}_j, \theta_j\}_{j \neq i})\, dt + \sqrt{2\,D}\, d\xi_i$$

$$\langle \xi_i \rangle = 0 \qquad \langle \xi_i(t)\, \xi_j(t') \rangle = \delta_{ij}\, \delta(t - t')$$

$D =$ Rotational diffusivity
$u_0 =$ Linear velocity
$f =$ Control
$\theta_i =$ Heading direction

Avoiding collisions: an optimization problem with a *tradeoff*



Collision cost on contact

$$G_{ij} = \delta(\boldsymbol{x}_i - \boldsymbol{x}_j)\,(g_0 - g_1 \cos(\theta_{ij}))$$

General second order expansion in even harmonics of a general cost

$$G_{ij} = \delta(\boldsymbol{x}_i - \boldsymbol{x}_j)\; g(\theta_{ij})$$

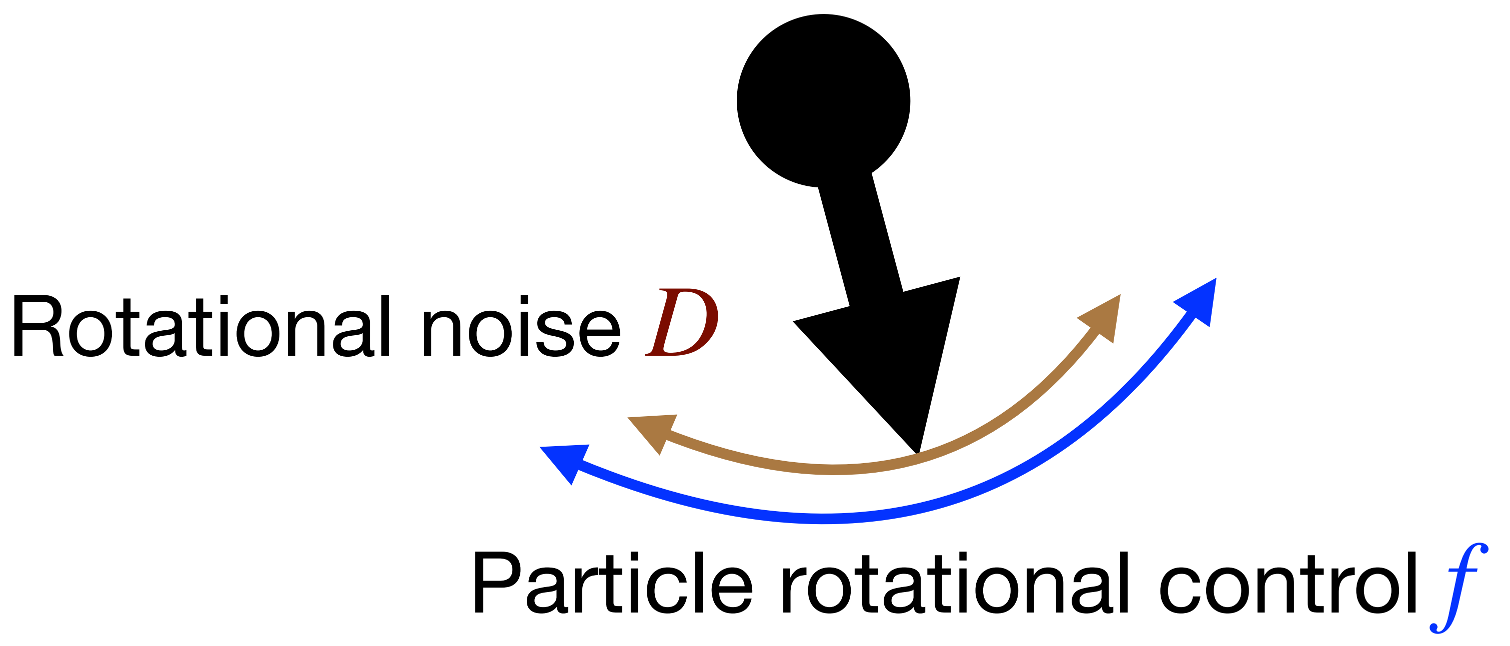
Use control f_i to avoid collisions

BUT…

Control cost

$$Y_i = \alpha\, f_i^2/2$$

Quadratic control costs

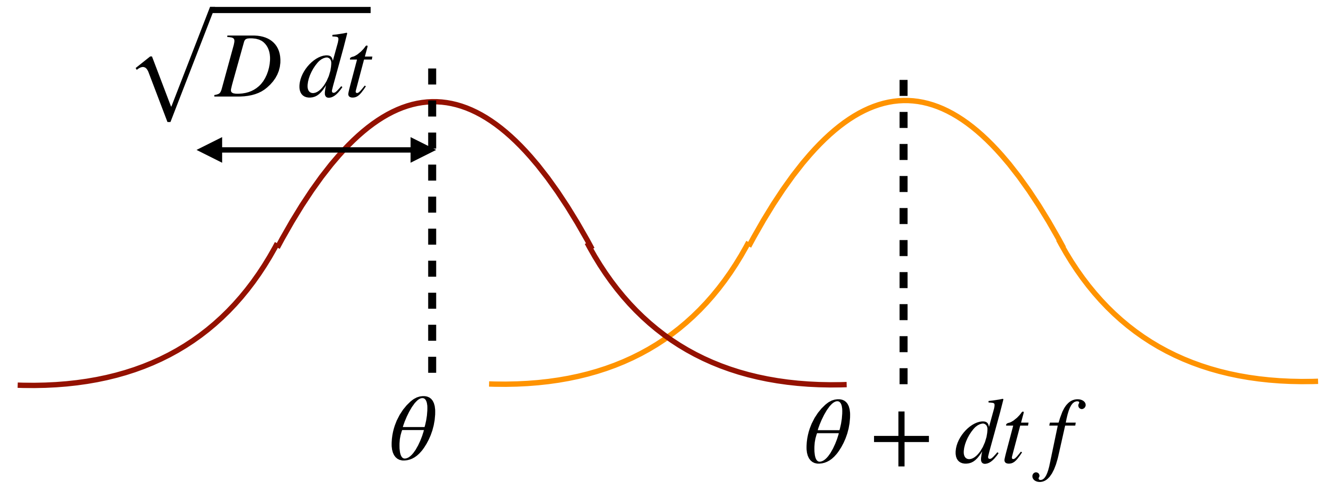


Control f_i

$$d\theta_i = f_i(\boldsymbol{x}_i, \theta_i; \{\boldsymbol{x}_j, \theta_j\}_{j \neq i})\, dt + \sqrt{2\,D}\, d\xi_i$$

★ Quadratic costs: a theoretically sound choice

$$Y_i = \alpha\, f_i^2 / 2$$



$$P_f(\theta + d\theta, t + dt\,|\,\theta, t) \propto \exp(-(d\theta - dt\,f)^2/(2\,D\,dt))$$

$$P_0(\theta + d\theta, t + dt\,|\,\theta, t) \propto \exp(-d\theta^2/(2\,D\,dt))$$

$$D[P_f || P_0] = \int d\theta\, P_0\, \log(P_f/P_0) \approx dt\, \frac{f^2}{2D}$$

Kullbak-Leibler distance from uncontrolled dynamics

Cost Interpretation

1) Cognitive cost ★

2) Power dissipation ★

3) Mechanical constraints

Ullmo, D., Swiecicki, I., & Gobron, T. (2019). Quadratic mean field games. *Physics Reports*, *799*, 1-35.

Todorov, E. (2009). Efficient computation of optimal actions. *Proceedings of the national academy of sciences*, *106*(28), 11478-11483.

Dvijotham, K., & Todorov, E. (2011). A unified theory of linearly solvable optimal control. *Artificial Intelligence (UAI)*, *1*.

Constrained minimization with a *tradeoff*

Find optimal control f_i and associated probability P

Control Collision

$$C = \frac{1}{2}\int dX\, d\boldsymbol{\Theta}\; P \left[\alpha \sum_i f_i^2 + \sum_{ij} G_{ij} \right]$$

Average cost

$$\mathscr{L}\, P = 0$$

Dynamics as constraint:
Fokker-Plank equation

Lagrange multiplier formulation (Pontryagin principle)

$$\mathscr{H} = C + \lambda \left(1 - \int dX\, d\boldsymbol{\Theta}\; P \right) - \int dX\, d\boldsymbol{\Theta}\; \Phi(\boldsymbol{X}, \boldsymbol{\Theta})\; \mathscr{L}\; P(\boldsymbol{X}, \boldsymbol{\Theta})$$

$\lambda =$ multiplier for normalization

$\Phi(\boldsymbol{X}, \boldsymbol{\Theta}) =$ multiplier for dynamics at $\boldsymbol{X}, \boldsymbol{\Theta}$

$\boldsymbol{X}, \boldsymbol{\Theta} = \{\boldsymbol{x}_i\}, \{\theta_i\}$

$P =$ Angle-position joint probability

$$\mathscr{L} = \sum_{i=1}^{N} \left[-u_0\, \partial_{\boldsymbol{x}_i}\, \boldsymbol{n}(\theta_i) - \sum_i \partial_{\theta_i}\, f_i + D \sum_i \partial^2_{\theta_i} \right]$$

Todorov, E. (2009). *Proceedings of the national academy of sciences*, *106*(28), 11478-11483.
Dvijotham, K., & Todorov, E. (2011) *Artificial Intelligence (UAI)*, *1*.

Constrained minimization with a *tradeoff*

$$\mathcal{H} = C + \lambda \left(1 - \int dX \, d\mathbf{\Theta} \; P \right) - \int dX \, d\mathbf{\Theta} \; \Phi(\mathbf{\Theta}, X) \; \mathcal{L} \; P(X, \mathbf{\Theta})$$

Stationarity w.r.t. λ	→	Normalization
Stationarity w.r.t. Φ	→	Dynamics
Stationarity w.r.t. P	→	Differential equation for Φ
Stationarity w.r.t. f_i	→	Control f_i as a function of Φ

Equation for Φ can be linearized and reduces to a ***many body quantum problem***: as typical in quadratic mean-field games

Mean field hypotheses

1) Agent-wise factorization

$$P(\{\boldsymbol{x}_i, \theta_i\}_{i=1}^{N}) = \prod_{i=1}^{N} p(\boldsymbol{x}_i, \theta_i)$$

Strong hypothesis: agents have to base their behaviour on collective observables

2) Spatial homogeneity

$$p(\boldsymbol{x}_i, \theta_i) = \frac{1}{V}\rho(\theta_i)$$

$$\delta = (N-1)/V$$

Lagrangian $$\mathscr{H} = C + \lambda\left(1 - \int d\theta\, \rho\right) - \int d\theta\, \Phi(\theta)\, \mathscr{L}\, \rho(\theta)$$

Cost $$C = \int d\theta\, \rho(\theta)\ \left[\frac{\delta}{2}[g_0 \underbrace{- g_1\, \boldsymbol{n}(\theta)}_{\textit{Collision}} \cdot \langle \boldsymbol{n} \rangle] \underbrace{+ \frac{\alpha}{2} f^2}_{\textit{Control}}\right]$$

Mean field cost and self-interaction

$$C = \int d\theta\, \rho(\theta)\ \left[\frac{\delta}{2}[g_0 - \underbrace{g_1\, \boldsymbol{n}(\theta)}_{\text{Collision}} \cdot \langle \boldsymbol{n} \rangle] + \underbrace{\frac{\alpha}{2} f^2}_{\text{Control}}\right]$$

Self-interaction with Average polarization

$$m\, \boldsymbol{n}(\bar{\theta}) = \int d\theta\, \boldsymbol{n}(\theta)\, \rho(\theta) = \langle \boldsymbol{n} \rangle$$

$$\boldsymbol{n}(\theta) \cdot \langle \boldsymbol{n} \rangle = m \cos(\theta - \bar{\theta})$$

Without loss of generality (rotational invariance) $\bar{\theta} = 0$

$$m = \int d\theta\ \cos(\theta)\, \rho(\theta)$$

$$C = \int d\theta\, \rho(\theta)\ \left[\frac{\delta}{2}[g_0 - \underbrace{g_1\, m}_{\text{Collision}}\ \cos(\theta)] + \underbrace{\frac{\alpha}{2} f^2}_{\text{Control}}\right]$$

Self consistent formulation

$$\mathscr{H} = C + \lambda\left(1 - \int d\theta\, \rho\right) - \int d\theta\, \Phi(\theta)\, \mathscr{L}\, \rho(\theta)$$

$$m = \int d\theta\, \rho(\theta)\, cos(\theta)$$

$$C = \int d\theta\, \rho(\theta)\, \left[\frac{\delta}{2}[g_0 - g_1 m\, \cos(\theta)] + \frac{1}{2} f^2\right]$$

Hopf-Cole transformation $\Phi =: 2\, D\, \log Z$ $Z =$ “desirability”

$$\frac{d}{df}\mathscr{H} = 0$$

$$\frac{d}{d\rho}\mathscr{H} = 0$$

$\Rightarrow$

$$f = 2\, D \frac{d}{d\theta} \ln Z$$

$$\left[\frac{\lambda}{2D} + \frac{\delta\, m\, g_1}{2D}\, \cos\theta\right] Z + D\, \frac{d^2}{d\theta^2} Z = 0$$

linearized Hamilton-Bellman-Jacobi equation

Mathieu equation and self-consistency condition

$$f = 2\,D\,\frac{d}{d\theta}\ln Z$$

$$\left[-\frac{d}{d\theta}f + D\,\frac{d^2}{d\theta^2}\right]\rho = 0$$

$\longrightarrow$ $Z^2 = \rho$

Assuming
-stationarity
-periodic BC

$$m = \int d\theta\, \underbrace{Z^2(\theta)}_{\rho(\theta)} \cos(\theta)$$

$$\left[\frac{\lambda}{2D} + \frac{\delta\, m\, g_1}{2D}\cos\theta\right] Z + D\,\frac{d^2}{d\theta^2} Z = 0$$

LJB equation = Schroedinger equation for ***quantum pendulum***

- the Mathieu equation -

with “energy” $E = -\lambda/4D$ and eigenfunction Z

We need a real and positive solution -> ground state eigenfunction

Mathieu equation and self-consistency condition

Mathieu eq. standard notation

$$\left[a(q) - q \cos(2y)\right] Z + Z'' = 0$$

$$y = \theta/2 \qquad y = \in [-\pi/2, \pi/2]$$

Fundamental parameters

$$h = \frac{g_1 \delta}{\alpha D^2} \qquad q = -h m$$

Mathieu equation: both eigenvalue $a = a(q)$ and eigenfunction $Z_{q(m)}(\theta)$ are known

The ground state is symmetric and depends on m

Self-consistency equation:

$$m = \int d\theta \, Z^2_{q(m)}(\theta) \cos(\theta) =: \mathcal{F}_h(m)$$

Universal behaviour in the tradeoff parameter

$$m = \mathscr{F}_h(m) \quad \longrightarrow \quad m = m(h)$$

h is therefore the fundamental parameter

$$C = C_0 + \frac{1}{2} D^2 \left[h\, m^2(h) + a(-m(h)\, h) \right]$$

with $C_0 = \delta\, g_0 / 2D^2 = const$

C is only a function of h

$$C - C_0 = \frac{1}{2} \int d\theta\, \rho(\theta) \left[g_1\, \delta\ m\ \cos(\theta - \bar{\theta}) + \alpha\, D^2 \left(2 \frac{d}{d\theta} \log Z \right)^2 \right]$$

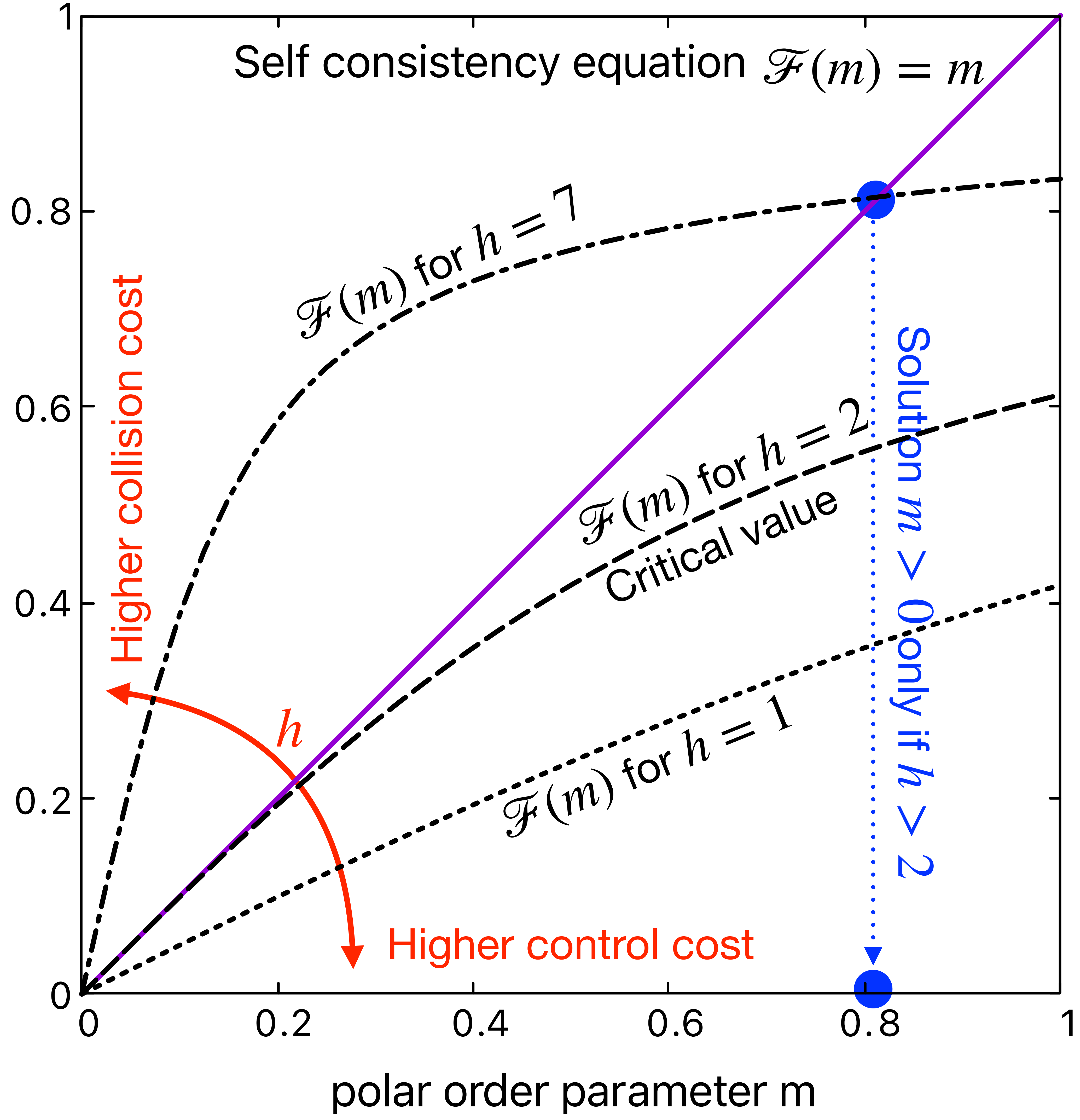
Collision

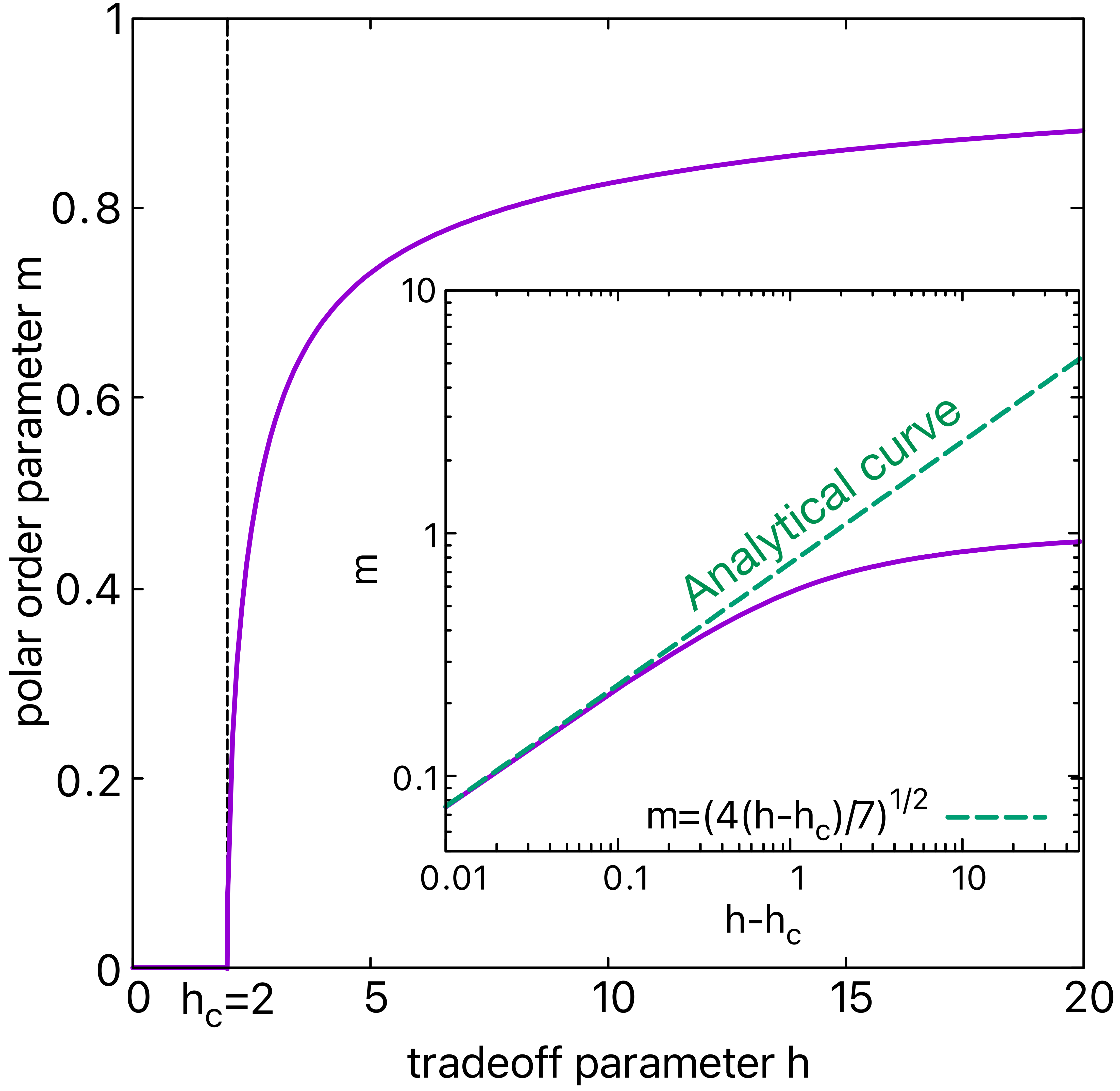
Control

$$h = \frac{g_1\, \delta}{\alpha\, D^2}$$

Solution: 2nd order phase transition

$$m = \int d\theta\, Z^2_{q(m)}(\theta)\, \cos(\theta) =: \mathscr{F}_h(m) \qquad h = \frac{g_1\, \delta}{\alpha\, D^2}$$





Small Oscillation approximation

$h \gg 1$

$$m \sim 1 - \frac{1}{2\sqrt{h}}$$

Critical region: perturbative result in small m

Critical point $h \to h_c^+ = 2$

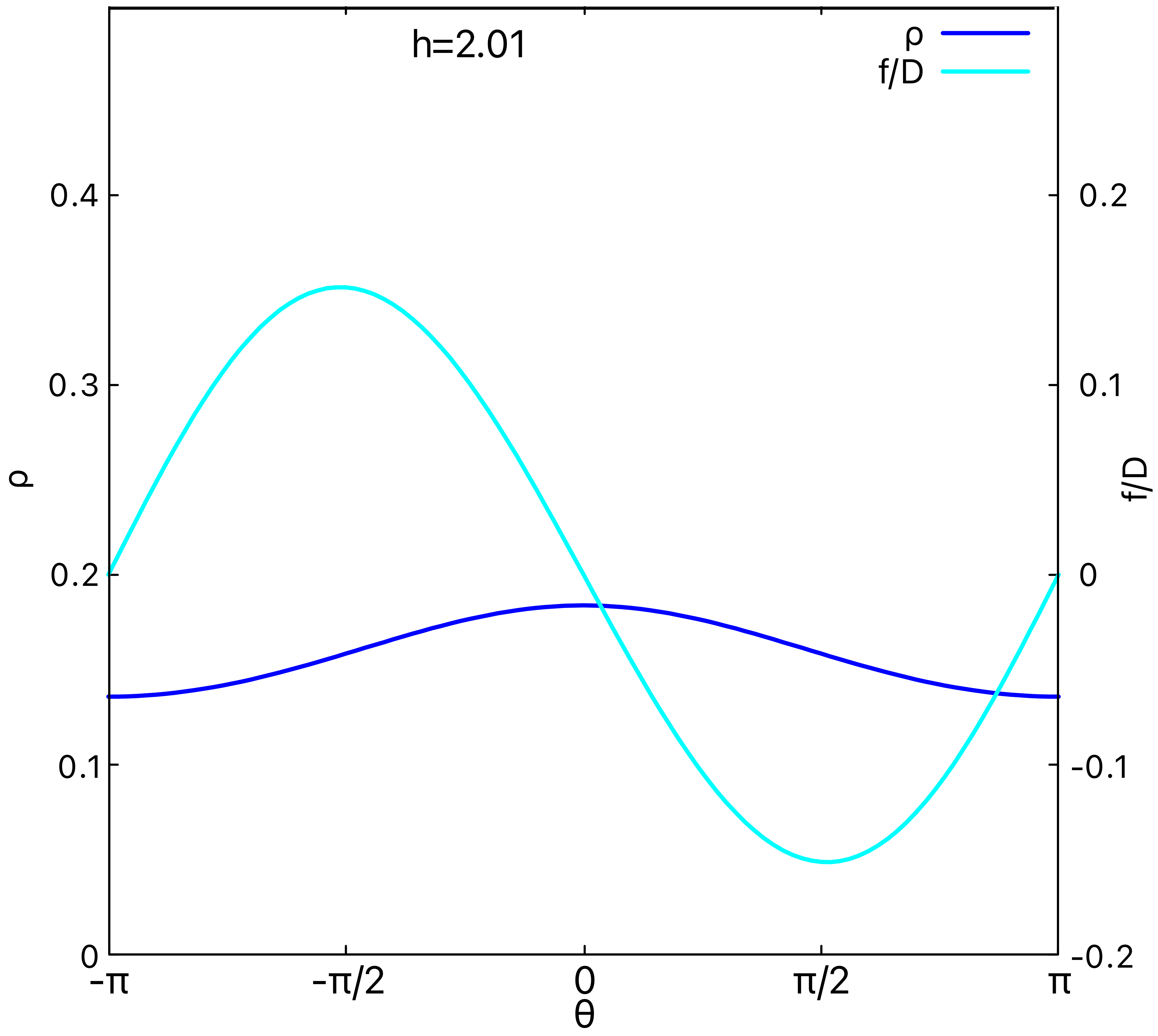
$$m \approx \sqrt{(4/7)\,(h - h_c)}$$

$$Z(\theta) \approx \frac{1}{\sqrt{2\pi}} \left[1 + \frac{h\,m(h)}{2}\,\cos\theta\right]$$

$$f \approx -D\,h_c\,m\,\sin(\theta)$$

Sinusoidal regime

$$C - C_0 \approx -(D^2/7)\,(h - h_c)^2$$



Perturbative result for strong coupling: Gaussian regime

Large coupling $h \gg h_c$

small oscillations

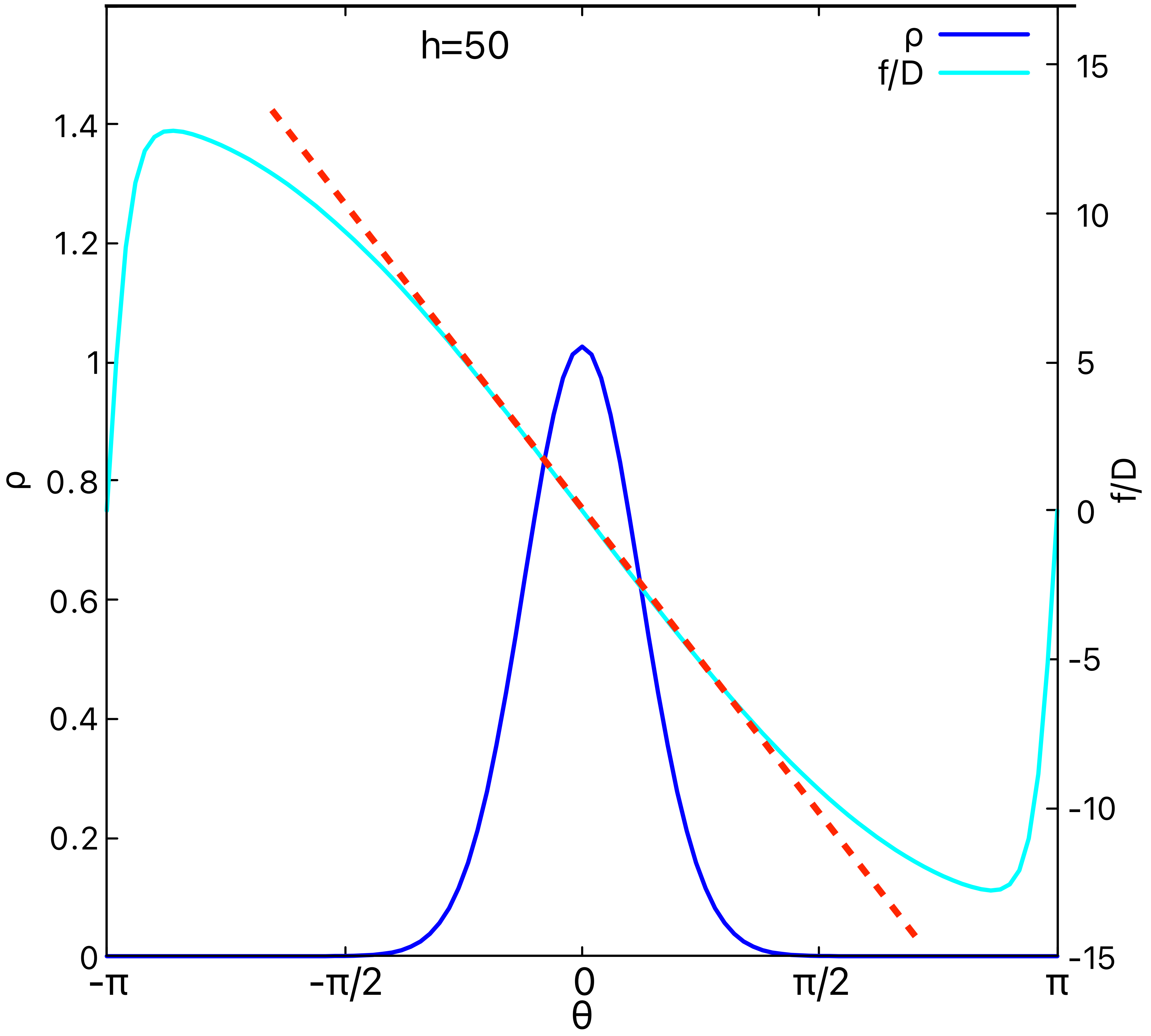
Quantum pendulum → Quantum harmonic oscillator

$$m \approx 1 - \frac{1}{2\sqrt{h}}$$

$$Z(\theta) \approx \left(2/\pi\sqrt{h\,m(h)}\right)^{1/4} \exp(-\sqrt{h\,m(h)}\,\theta^2)$$

$$f \approx -D\sqrt{h\,m(h)}\;\theta$$

$$C - C_0 \approx D^2\,[-h/2 + (3/4)\sqrt{h}]$$



The sinusoidal approximation and Vicsek model

Large coupling $h \gg h_c$

$$f \approx -D\sqrt{h\,m(h)}\ \theta$$

But $|\theta| \ll 1$

$$f \approx -D\sqrt{h\,m(h)}\ \sin(\theta)$$

Critical point $h \to h_c^+ = 2$

First order approximation!

$$f \approx -D\,h_c\,m\ \sin(\theta)$$

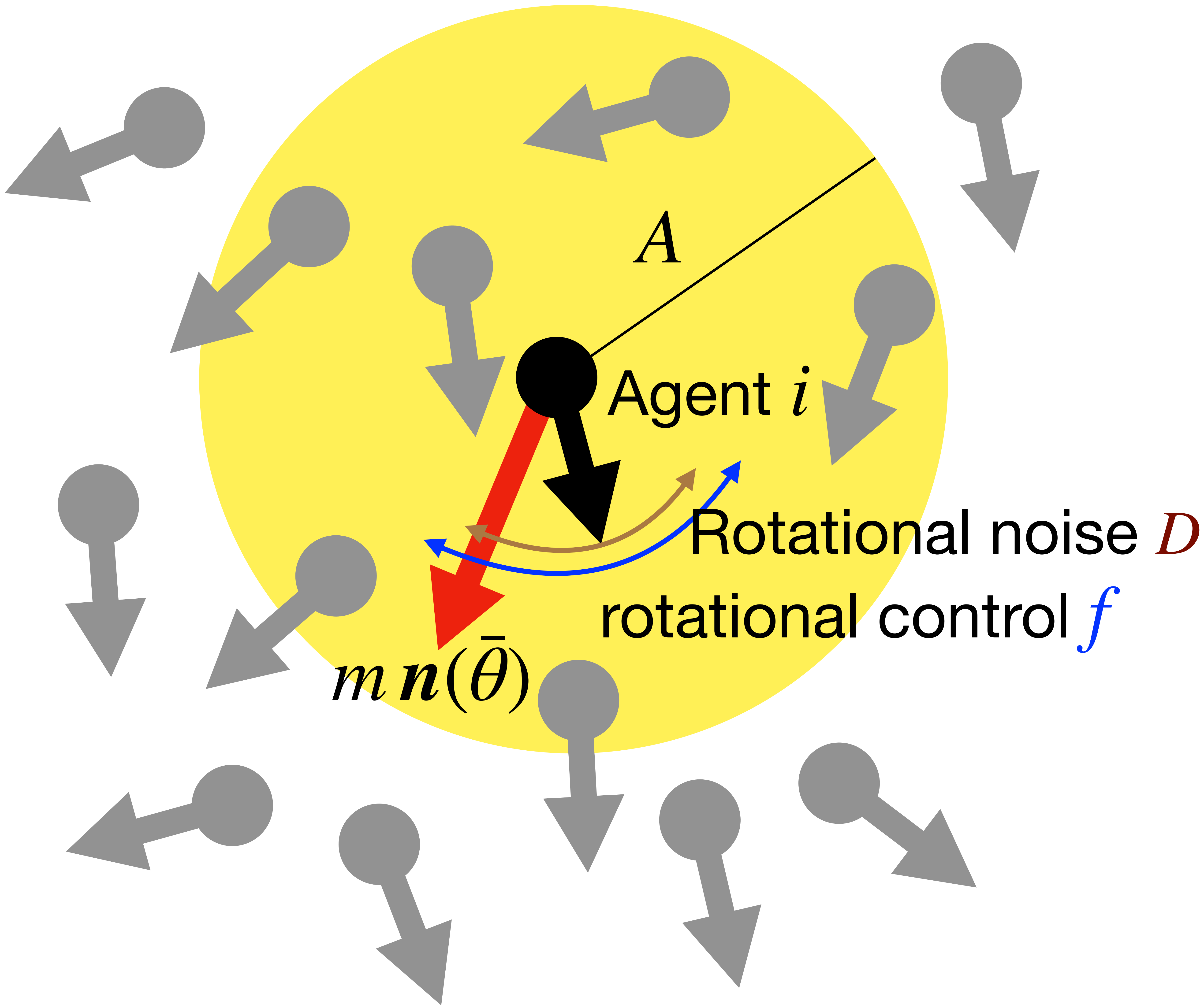
$$f \approx -D\,K(h)\ \sin(\theta)$$

Sinusoidal control in ***both*** asymptotic cases: Is it true in general?

Sinusoidal control corresponds to approximation of kinetic regime of stochastic Vicsek model

$$f \approx -R\,m\ \sin(\theta)$$

The sinusoidal approximation and stochastic Vicsek model (kinetic regime)

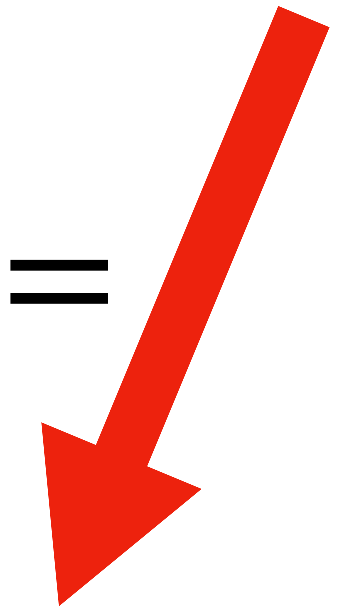


$$d\boldsymbol{x}_i = u_0\,\boldsymbol{n}(\theta_i)\,dt$$
$$d\theta_i = f_i\,dt + \sqrt{2\,D}\,d\xi_i$$
$$\langle \xi_i \rangle = 0 \quad \langle \xi_i(t)\,\xi_j(t') \rangle = \delta_{ij}\,\delta(t - t')$$

A = interaction radius

$$f_i \propto \boldsymbol{n}(\theta_i) \times \sum_{j \in [i]} \boldsymbol{n}(\theta_j)$$

$$\sum_{j \in [i]} \boldsymbol{n}(\theta_j) \approx \delta\,\pi\,A^2\,m\,\boldsymbol{n}(\bar{\theta}) =$$



$$f_i = -\,D\,R\,m\,\sin(\theta - \bar{\theta})$$

Peruani, F., Deutsch, A., & Bär, M. (2008). *The European Phys. Journal Special Topics*, *157*(1), 111-122.
Vicsek T and Zafeiris A 2012 Phys. Rep. 517 71–140
Ginelli F 2016 Eur. Phys. J. Spec. Top. 225 2099–2117
Chepizhko A and Kulinskii V 2010 Physica A 389 5347–5352
Chepizhko A and Kulinskii V 2009 The kinetic regime of the vicsek model AIP Conf. Proc. vol 1198 pp 25–33

Optimal vs sinusoidal control

Optimal control

Minimize cost C

by choosing control f amongst ***all possible*** control functional shapes

Best sinusoidal control

Find K such that control

$$f = -D\,K\,sin(\theta)$$

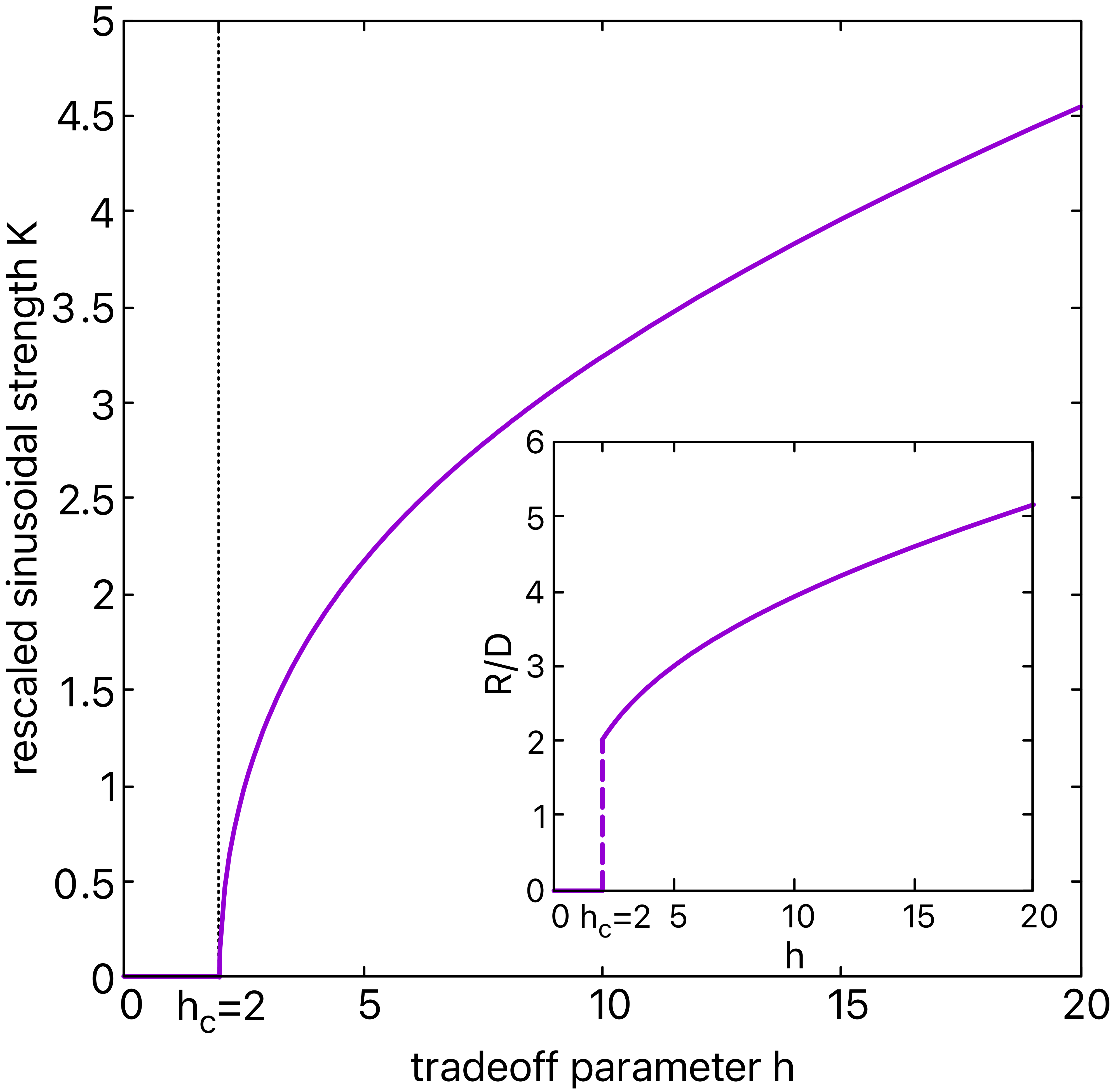
Minimizes cost

$$C - C_0 = \frac{D^2}{2}\int d\theta\,\rho_s\,[\underbrace{K^2\ \sin^2\theta}_{\text{Control}} - h\,\underbrace{m\ \cos(\theta)}_{\text{Collision}}]$$

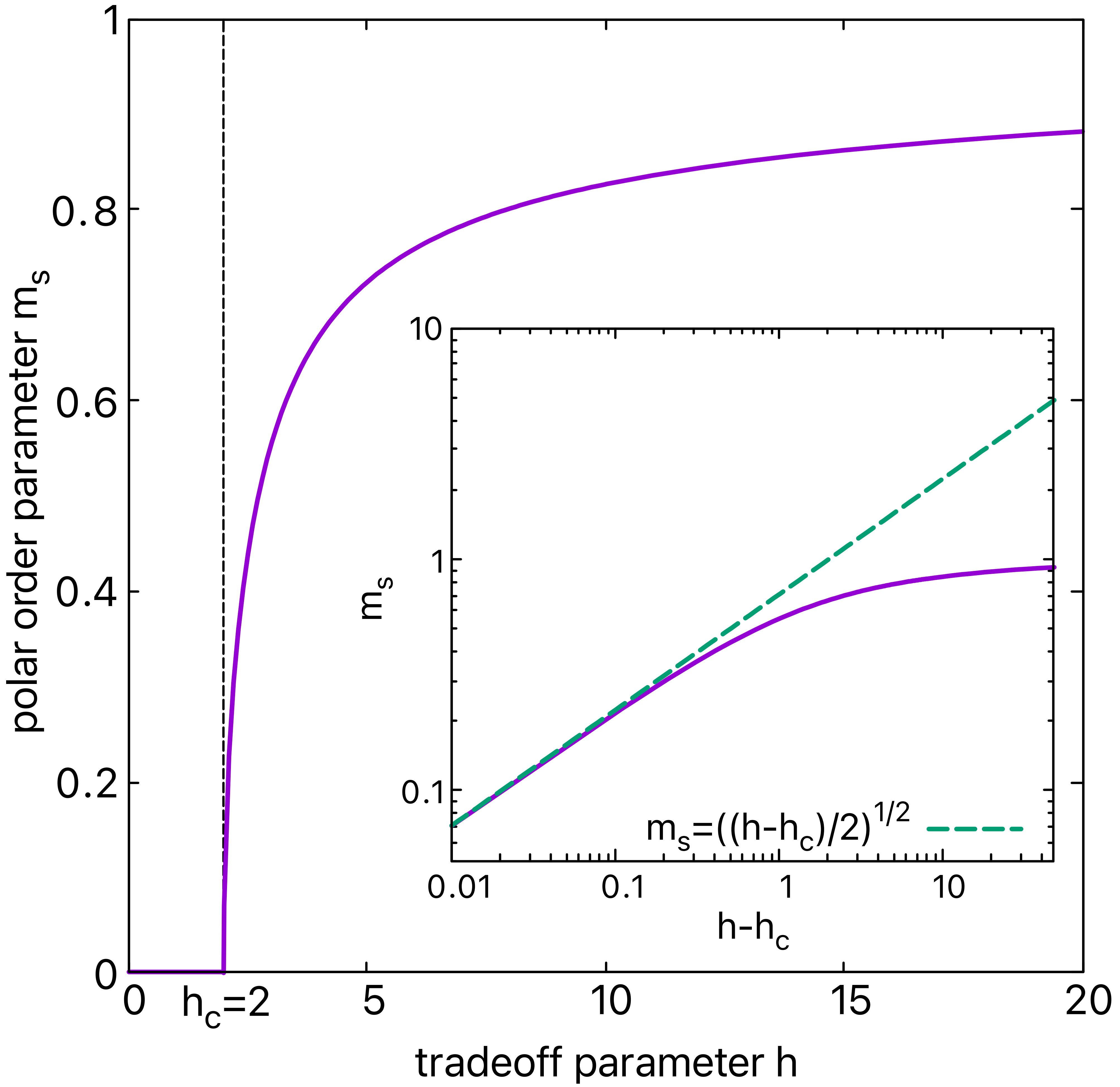
$$\frac{d}{d\theta}\left(-KD\sin\theta - D\frac{d}{d\theta}\right)\rho_s = 0$$

$$\rho_s \propto \exp(-K\ \cos(\theta))$$

Von-Mises



Sinusoidal control shape



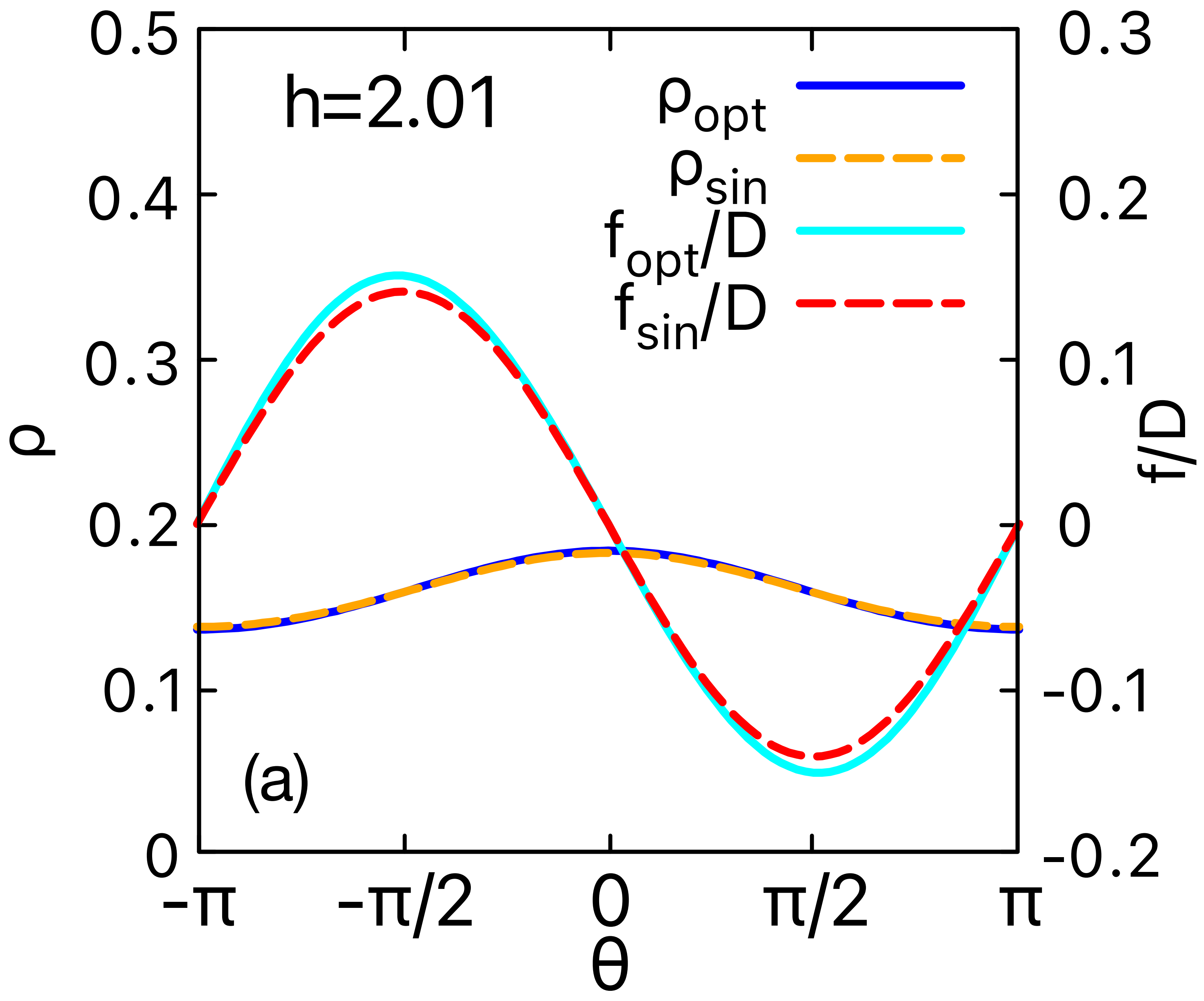
Control strength minimizes cost

$$m = \frac{I_1(K)}{I_0(K)}$$

$$f = -D\,K\,sin(\theta) = -R\,m\ \sin(\theta)$$

$$K = \arg\min_K C$$

Best sinusoidal control depends on the same tradeoff parameter h, i.e. ***K***=***K***(h)



Critical point

Critical point $h \to h_c^+ = 2$

$$m_{sin} \approx \sqrt{(1/2)\,(h-2)} \quad \text{Vs} \quad m_{opt} \approx \sqrt{(4/7)\,(h-2)}$$

$$m \approx \beta\,(h-h_c)^{\gamma}$$

Same exponent $\gamma = 1/2$ and h_c but different prefactors β: prefactor is a **third order** effect

Identical controls up to redefinition of m as a function of h

$$f = -2\,D\,m_{opt/sin}\,\sin(\theta)$$

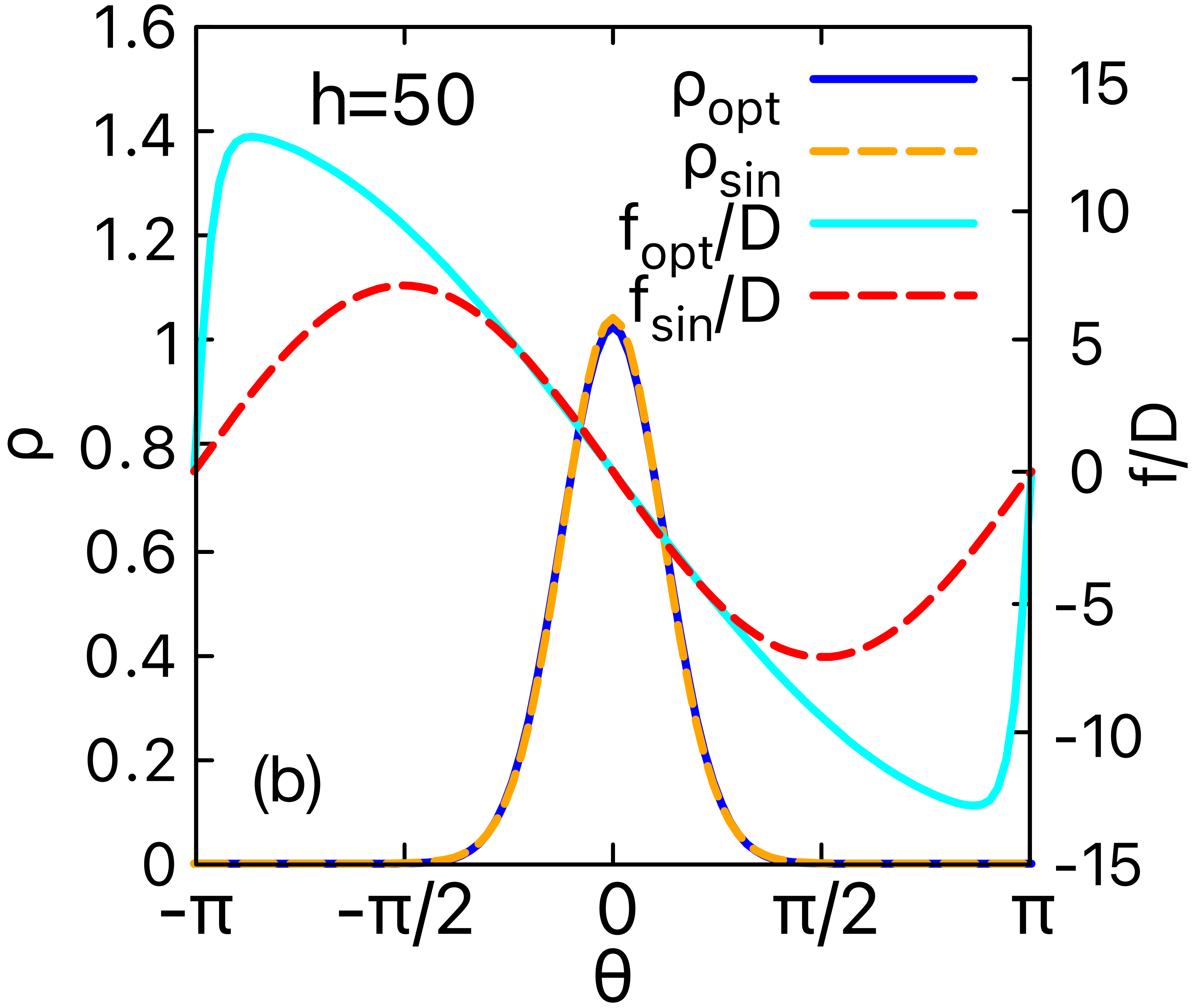
Cost discrepancy

$$C_{opt} - C_0 \approx = -(D^2/7)\,(h-h_c)^2 \;<\; C_{sin} - C_0 \approx = -(D^2/8)\,(h-h_c)^2$$

Strong coupling: gaussian case

$$h \gg h_c = 2$$

Polarization and costs match at leading order:
Fully Gaussian scenario



Polarization

$$m_{sin} \approx m_{opt} \approx 1 - 1/\sqrt{4\,h}$$

Costs

$$C_{opt} - C_0 \approx C_{sin} - C_0 \approx D^2\,[-h/2 + (3/4)\sqrt{h}]$$

Controls do ***not*** match in general

but

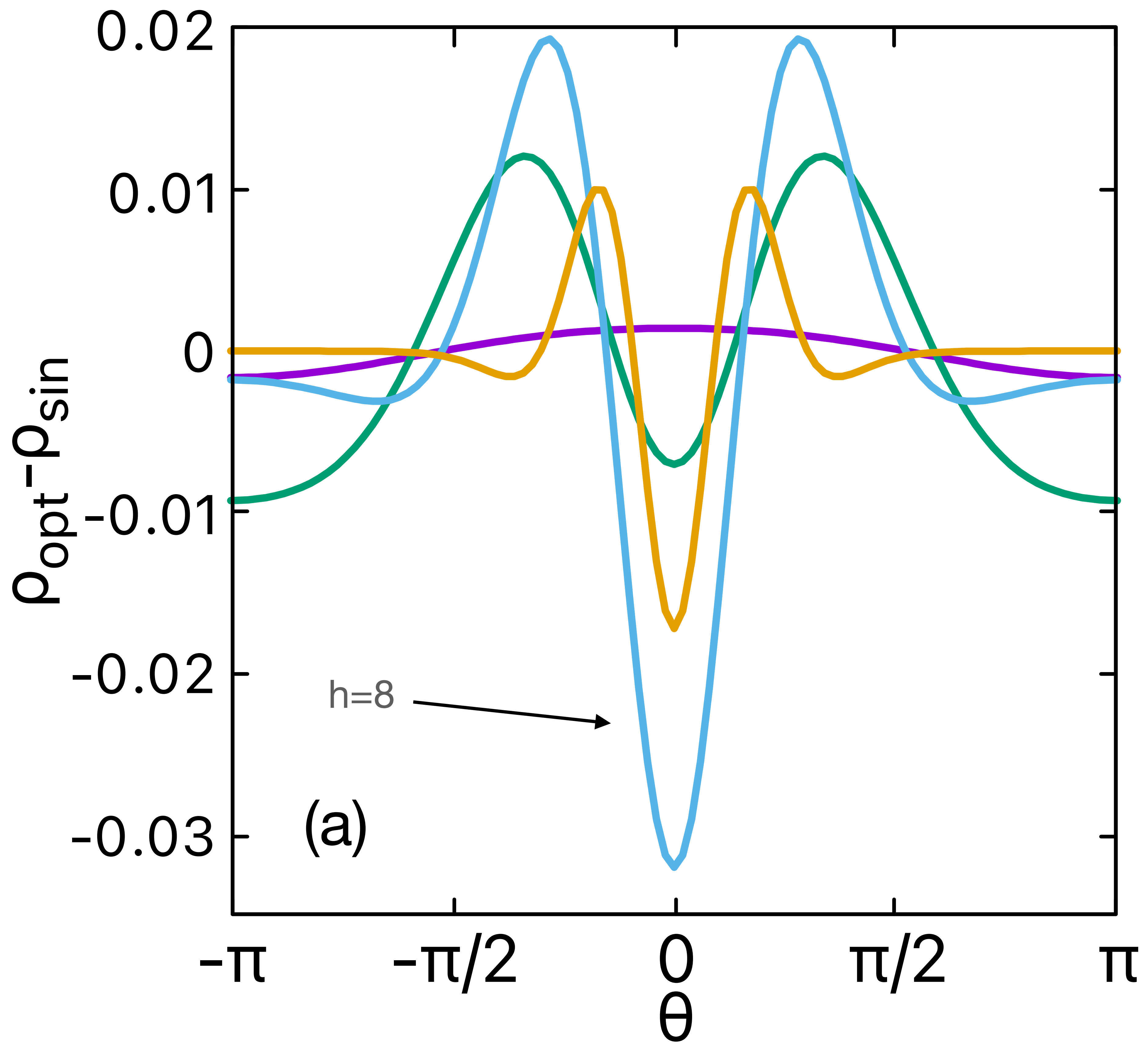
They do for small angles

$$f_{sin} \approx f_{opt} \approx -D\sqrt{h}\,sin(\theta) \quad \text{if} \quad |\theta| \ll 1$$

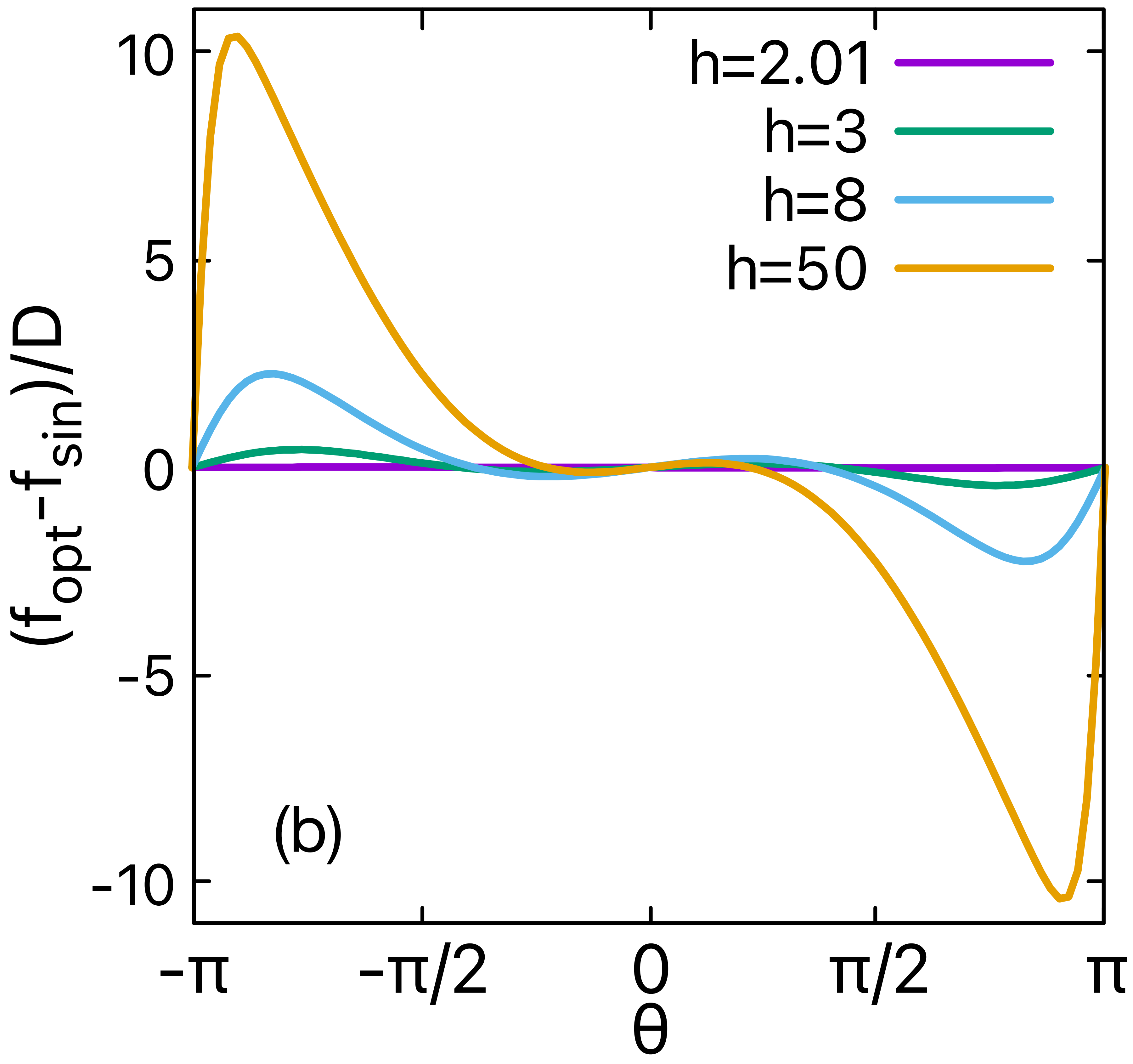
Only small angles count in this regime

General comparison

Differences in probability density peak around $h \approx 8$



Differences in control increase with h

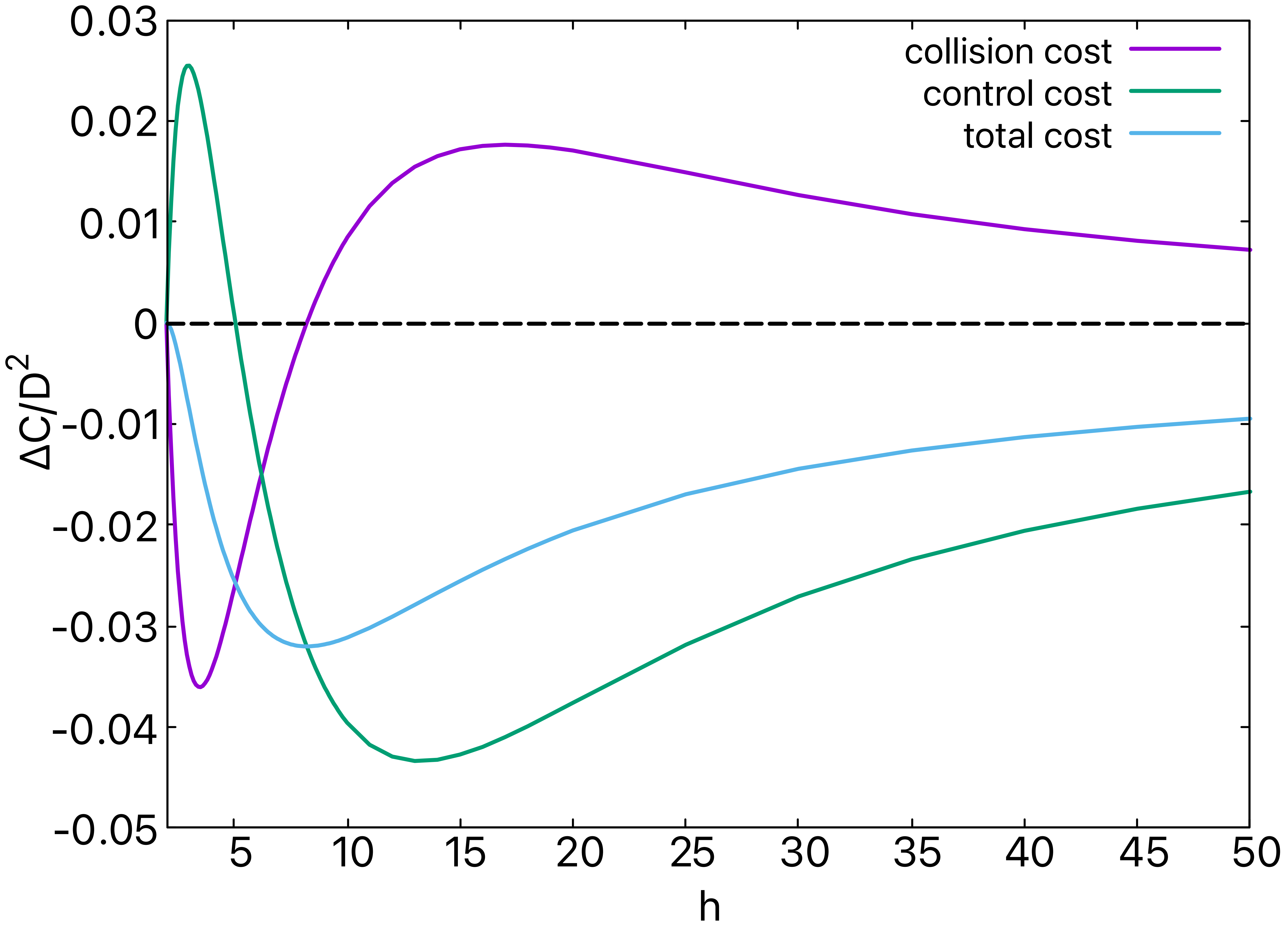


We should look at costs

Cost comparison: optimal vs best sinusoidal control

$$\Delta C(h) = C_{optimal}(h) - C_{sinusoidal}(h)$$

Universal behaviour as a function of h once rescaled by D^2



For any given h, the difference ΔC can become arbitrarily large

but

What about relative cost $\Delta C / C$?

No universal behaviour: $\Delta C / C$

Depends on g_0

$$C_{coll} = (\delta\, D^2/2)\,[g_0 - m^2\, g_1]$$

Two notable cases:

1) pure alignment reward: $g_0 = 0$

$\max(\Delta C/C) = -1/8$ at $h = 0$

2) pure collision: $g_0 = g_1$

$\max(\Delta C/C) \approx -.02$ at $h \approx 4.8$

Optimal solution vs sinusoidal model

Summary

Critical case → Equivalence up to redefinition of m

$$m_{sin} \approx \sqrt{(1/2)\,(h-2)} \qquad m_{opt} \approx \sqrt{(4/7)\,(h-2)}$$

Strong coupling case → Equivalence at leading order

General cost analyis → Relative cost differences are always small in realistic scenarios

Conclusion

1) Optimal control theory or mean field game formalism -> promising framework for collective behaviours
2) Exact mean field solution -> critical behaviour
3) Remarkably Vicsek-like (sinusoidal) interaction close to optimal control, at least in the mean field formulation.

Outlook

1) Beyond mean field
2) Different dynamics
3) Beyond spatial homogeneity

Thank you for your attention