Stochastic control and EQFT

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Statistical Mechanics/EQFT

Want to consider measures of the form

$$\mathrm{d}\nu = \exp(-S(\varphi))\mathrm{d}\varphi$$

- $\mathrm{d}\varphi$ is the Lebeque measure on some space of configuartions $\mathcal{S}'(\Lambda)$ and e.g $\Lambda = \varepsilon\mathbb{Z}^d, \mathbb{R}^d, \mathbb{T}^d$.
- S is an action, typically

$$S(\varphi) = \int \lambda V(\varphi) + m^2\varphi^2 + |\nabla\varphi|^2 \mathrm{d}x$$

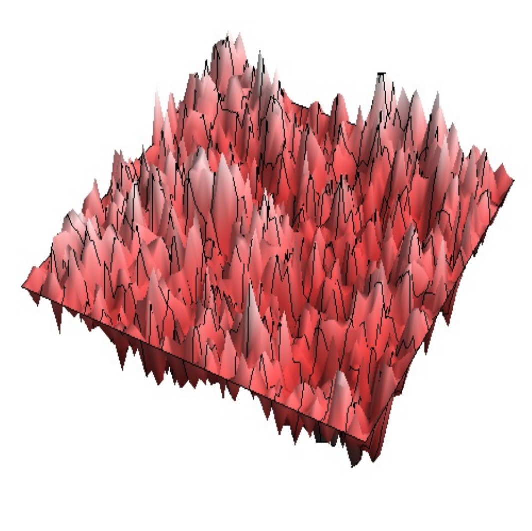
$$V(\varphi) = \cos(\beta\varphi), \exp(\beta\varphi), \varphi^4$$

$\mathrm{d}\varphi$ does not make sense if the configuartions space is infinite dimesional $\Rightarrow$ use the quadratic term of the action to pass to a gaussian measure.

$$d\mu = \exp\left(-\int m^2\varphi^2 + |\nabla\varphi|^2 dx\right) \quad \text{Gaussian Free Field}$$

◇ Gaussian measure with covariance $(m^2 - \Delta)^{-1}$.

◇ μ probability measure supported on distributions of regularity $-\frac{d-2}{2} - \delta$ for any $\delta > 0$



$\Rightarrow$ Cannot define $V(\varphi)$ on the support on μ in a straightforward fashion.

Wick ordering

Now consider $d=2$. Consider an approximation of μ_T of μ with covariance

$$\mathcal{C}_T = \rho_T(D)(m^2-\Delta)^{-1} = \int_0^T J_t^2 \mathrm{d}t \qquad J_t = (m^2-\Delta)^{-1/2}\sigma_t(D)$$

$$\rho_T = 1 \, \text{for} \, |x| \leqslant T \, \text{compactly supported} \, \sigma_t = \sqrt{\frac{\mathrm{d}}{\mathrm{d}t}\rho_t}$$

Then with $\phi_T = \rho_T(D)\phi$

$$[\![\phi_T^4]\!] = \phi_T^4 - \alpha_T \phi^2 + \beta_T \rightarrow [\![\phi_\infty^4]\!] \in \mathcal{C}_{\mathrm{loc}}^{-\delta}(\Lambda)$$

and this limit exists μ almost surely. Similarly we can consider

$$[\![\sin(\beta\phi_T)]\!] = T^{\beta^2/4\pi} \sin(\beta\phi_T) \in \mathcal{C}_{\mathrm{loc}}^{-\beta^2/4\pi-\delta}(\Lambda)$$

and this limit also exists almost surely. (Complex GMC).

Questions

- ♢ Existence of measures in the continuum/infinite volume limit
- ♢ Uniqueness, Decay of correlations, OS Axioms
- ♢ Description of the measure in some sense
- ♢ Pathwise properties
- ♢ Large deviations in Semiclassical limt

Renormalization

- We are interested in an "effective theory", i.e what we observe at "low" (finite) freqencies.
- Consider functional $f\colon \mathcal{S}'(\Lambda) \to \mathbb{R}$ and

$$\mathcal{L}(f) = \lim_{T\to\infty} \int \exp(-f(\varphi))\exp(-V_T(\varphi))\mathrm{d}\mu(\varphi)$$

and assume that $f(\varphi) = f(P_t\varphi)$ where P_T is a projector on frequencies $\leqslant t$.

- Decompose $\mu = \mu_t * \mu_{t,T}$ where μ_t has covariance $\mathcal{C}_t$ and $\mu_{t,T}$ has covariance $\mathcal{C}_T - \mathcal{C}_t$.

$$\begin{aligned} & \int \exp(-f(\tilde\varphi))\exp(-V_T(\tilde\varphi))\mathrm{d}\mu(\tilde\varphi) \\ = & \int \exp(-f(\varphi))\exp(-V(\varphi+\psi))\mathrm{d}\mu_t(\varphi)\mathrm{d}\mu_{t,T}(\psi) \\ = & \int \exp(-f(\varphi))\exp(-V_{t,T}(\varphi))\mathrm{d}\mu_t(\varphi) \end{aligned}$$

with

$$V_{t,T}(\varphi) = -\log \int \exp(-V(\varphi+\psi))\mathrm{d}\mu_{t,T}(\psi)$$

♢ Want to show that the limit $T \to \infty$ exists if we keep $t < \infty$ fixed.

♢ Can derive a PDE for the effective potential.

Proposition 1. *Assume that* $V_T \in C^2(L^2(\mathbb{R}^2))$. *Then* $V_{t,T}$ *satisfies*

$$\frac{\partial}{\partial t} V_{t,T}(\varphi) + \frac{1}{2}\mathrm{Tr}(\dot{\mathcal{C}}_t \,\mathrm{Hess}\, V_{t,T}(\varphi)) - \frac{1}{2}\|J_t \nabla V_{t,T}(\varphi)\|^2_{L^2(\mathbb{R}^2)} = 0$$

$$V_{T,T}(\varphi) = V_T(\varphi).$$

Furthermore if $V_T \in C^2(L^2(\mathbb{R}^2))$ *then* $V_{t,T} \in C([0,T], C^2(L^2(\mathbb{R}^2))) \cap C^1([0,T], C(L^2(\mathbb{R}^2)))$.

◊ Want to study

$$\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V(Y_T)+\int_0^T l_s(Y_s,u_s)\mathrm{d}s\right]$$

with $\mathcal{H}$ hilbert space (e.g $\mathbb{R}^n$), $V:\mathcal{H}\to\mathbb{R}$, $V\in C^2(\mathcal{H})$ and $l:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$,

$$\mathrm{d}Y_s=\beta(s,u_s)\mathrm{d}s+\sigma_s\mathrm{d}X_s \qquad Y_0=0.$$

$$\sigma:\mathcal{H}\to\mathcal{H}\,\text{linear} \qquad \beta:\mathbb{R}\times\mathcal{H}\to\mathbb{R}.$$

$$\mathbb{H}_a=\{\text{space of processes } [0,T]\to\mathcal{H}\,\text{adapted to } X\}.$$

Introduce the value function

$$V_{t,T}(\varphi)=\mathbb{E}\left[V(Y_{t,T})+\int_t^T l_s(Y_{t,s},u_s)\mathrm{d}s\right]$$

where now

$$\mathrm{d}Y_{t,s}=\beta(s,u_s)\mathrm{d}s+\sigma_s\mathrm{d}X_s \qquad Y_t=\varphi.$$

Proposition 2. (Bellmann)

$$\inf_u \mathbb{E}\Big[V(Y_T)+\int_0^T l_s(Y_s,u_s)\mathrm{d}s\Big]=\inf_u \mathbb{E}\Big[V_{t,T}(Y_T)+\int_0^t l_s(Y_s,u_s)\mathrm{d}s\Big]$$

Furthermore if u is a minimizer of the l.h.s, then $u|_{[0,t]}$ is a minimizer of the r.h.s.

From this we can derive a PDE for $V_{t,T}$ which looks like

$$\frac{\partial}{\partial t}v(t,\varphi)+\frac{1}{2}\inf_{a\in\mathcal{H}}[\,\mathrm{Tr}(\sigma^2\mathrm{Hess}\,v(t,\varphi))+\langle\nabla v,\beta(t,a)\rangle_{\mathcal{H}}+l(t,\varphi,a)]=0 \qquad (1)$$

Proposition 3. (Verification) *Assume that $v\in C([0,T],C^{2,\mathrm{loc}}(\mathcal{H}))\cap C^{1,\mathrm{loc}}([0,T],C(\mathcal{H}))$ and v solves (1) with $v(T,\varphi)=V_T(\varphi)$. Furthermore assume that there exists $u\in\mathbb{H}_a$ and Y such that u,Y satisfy the state equation and*

$$u_t\in\mathrm{argmin}_{a\in\Lambda}[\,\mathrm{Tr}(\sigma^2\,\mathrm{Hess}\,v(t,Y_t))+\langle\nabla v(t,Y_t),\beta(t,a)\rangle_H+l(t,Y_t,a)]. \qquad (2)$$

Then $v(t,\varphi)=V_{t,T}(\varphi)$ and the pair u,Y is optimal.

Concrete situation

$\mathcal{H} = L^2(\mathbb{R}^2)$ and

$$\begin{aligned} \beta(t,a) &= J_t a \\ \sigma_t &= J_t \\ l(t,Y_t,a) &= \frac{1}{2}\|a\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Then (2) becomes a minimization problem for a quadratic functional and reduces to

$$u_t = -J_t \nabla v(t, Y_{s,t}).$$

This means if we can solve the equation

$$\mathrm{d}Y_{s,t} = -J_t^2 \nabla v(t, Y_{s,t})\mathrm{d}t + J_t \mathrm{d}X_t, \tag{3}$$

we can apply the verification theorem.

Furthermore in this case (1) takes the form

$$\frac{\partial}{\partial t} v(t,\varphi) + \frac{1}{2}\mathrm{Tr}(J_t^2 \mathrm{Hess}\, v(t,\varphi)) - \frac{1}{2}\|J_t \nabla v(t,\varphi)\|_{L^2(\mathbb{R}^2)}^2 = 0, \tag{4}$$

which is precisely the Polchinski equation.

Corollary 4.

$$-\log \mathbb{E}[e^{-V_T(\varphi + W_{t,T})}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[V_T(Y_{s,T}(u,\varphi)) + \frac{1}{2}\int_s^T \|u_t\|_{L^2}^2 \mathrm{d}t \right]$$

where $\mathbb{H}_a$ *is the space of processes adapted to* X_t *such that* $\mathbb{E}[\int_0^\infty \|u_t\|_{L^2}^2 \mathrm{d}t] < \infty$ *and* $Y_t(u,\varphi)$ *satisfies*

$$\mathrm{d}Y_{s,t}(u,\varphi) = -J_t^2 u_t \mathrm{d}t + J_t \mathrm{d}X_t$$

$$Y_{s,s}(u,\varphi) = \varphi.$$

Construction of Φ^4_2 in finite volume

Take $\Lambda = \mathbb{T}^2$ and denote

$$I_T(u) = \int_0^T J_t u_t \mathrm{d}t \qquad W_T = \int_0^T J_t \mathrm{d}X_t.$$

From previous slide we have with $V_T(\varphi_T) = \int [\![\varphi_T]\!] \mathrm{d}x$

$$\begin{aligned} & -\log \int \exp(-f(\varphi) - V_T(\varphi)) \\ = & \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\int_\Lambda [\![(W_T + I_T(u))^4]\!] \mathrm{d}x + \frac{1}{2}\int_0^T \|u\|^2_{L^2} \mathrm{d}t\right] \end{aligned}$$

From this we immidiatly see (can also be done by Jensen)

$$-\log \int \exp(-f(\varphi) - V_T(\varphi)) \leqslant \mathbb{E}\left[f(W_T) + \int_\Lambda [\![(W_T)^4]\!] \mathrm{d}x\right] = \mathbb{E}[f(W_T)]$$

It is not hard do thow that

$$\|I_T(u)\|_{H^1} \leqslant \left(\int_0^T \|u\|^2_{L^2} \mathrm{d}t\right)^{1/2}.$$

Expanding we have

$$\begin{aligned} & \mathbb{E}\Big[f(W_T + I_T(u)) + \int_\Lambda [\![(W_T + I_T(u))^4]\!] \mathrm{d}x + \frac{1}{2}\int_0^T \|u\|^2_{L^2}\mathrm{d}t \Big] \\ = \; & \mathbb{E}\Big[f(W_T + I_T(u)) + \int_\Lambda [\![W_T^3]\!] I_T(u)\mathrm{d}x + 4\int_\Lambda [\![W_T^2]\!] I_T^2(u)\mathrm{d}x + 6\int_\Lambda W_T I_T^3(u)\mathrm{d}x \\ & + \int I_T^4(u)\mathrm{d}x + \frac{1}{2}\int_0^T \|u\|^2_{L^2}\mathrm{d}t \Big] \end{aligned}$$

Now to get the corresponding lower bound to our upper bound we need

$$\mathbb{E}|\mathrm{red}| \leqslant C + \delta \mathbb{E}[\mathrm{green}].$$

For example

$$\begin{aligned} & \mathbb{E}\int_\Lambda [\![W_T^3]\!] I_T(u)\mathrm{d}x \\ \leqslant \; & C\mathbb{E}\| [\![W_T^3]\!] \|^2_{H^{-1}(\Lambda)} + \varepsilon \mathbb{E}\|I_T(u)\|^2_{H^1(\Lambda)} \\ \leqslant \; & C + \varepsilon \mathbb{E}\|I_T(u)\|^2_{H^1(\Lambda)}. \end{aligned}$$

Similar for the other terms $\Rightarrow$ Uniform upper and lower bounds on the Laplace tranform.

Now partition function diverges so we have to consider

$$\lim_{\rho \to 1} \mathcal{W}^{\rho}(f) - \mathcal{W}^{\rho}(0)$$

where $\rho \in C_c^{\infty}(\mathbb{R}^2)$

$$\mathcal{W}^{\rho}(f) = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[f(W_\infty + I_\infty(u)) + \int \rho V_\infty(W_\infty + I_\infty(u)) + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2 \mathrm{d}t \right]$$

$\Rightarrow$ Have to study the optimizer on the r.h.s and control the depencede on f. E.g. want something like

$$\int_0^\infty \int \exp(\gamma|x|)|u_t^{f,\rho} - u_t^{0,\rho}|^2 \mathrm{d}x\mathrm{d}t.$$

where $u^{f,\rho}$ is the optimizer on the r.h.s. Then we can pass to the limit in

$$\lim_{\rho \to 1} \mathcal{W}^{\rho}(f) - \mathcal{W}^{\rho}(0)$$

and obtain an expression for the laplace transform. Proving decay of correlations is also possible.

Euler Lagrange equations

We can study the optimizer via it's EL equations. For $h \in \mathbb{H}_a$

$$
\begin{aligned}
& \mathbb{E}[\nabla f(W_\infty + I_\infty(u^{f,\rho}))I_\infty(h)] \\
= \ & \mathbb{E}\Big[\int \rho \nabla V(W_\infty + I_\infty(u^{f,\rho}))I_\infty(h)\mathrm{d}x\Big] \\
& + \mathbb{E}\Big[\int_0^\infty \int u_t^{f,\rho} h_t \mathrm{d}x\mathrm{d}t\Big]
\end{aligned}
$$

So taking difference

$$
\begin{aligned}
& \mathbb{E}[\nabla f(W_\infty + I_\infty(u^{f,\rho}))I_\infty(h)] \\
= \ & \mathbb{E}\Big[\int \rho(\nabla V(W_\infty + I_\infty(u^{f,\rho})) - \nabla V(W_\infty + I_\infty(u^{0,\rho})))I_\infty(h)\mathrm{d}x\Big] \\
& + \mathbb{E}\Big[\int_0^\infty \int (u_t^{f,\rho} - u_t^{\rho})\, h_t \mathrm{d}x\mathrm{d}t\Big]
\end{aligned}
$$

Role of convexity

Imagine if V was convex. Then testing with $h = \exp(\gamma|x|)(u^{f,\rho} - u^{0,\rho})$ we get

$$\begin{aligned}
&\mathbb{E}[\exp(\gamma|x|)\nabla f(W_\infty + I_\infty(u^{f,\rho}))I_\infty(u^{f,\rho} - u^{0,\rho})] \\
= \; &\mathbb{E}\left[\int \rho\exp(\gamma|x|)(\nabla V(W_\infty + I_\infty(u^{f,\rho})) - \nabla V(W_\infty + I_\infty(u^{0,\rho})))I_\infty(u^{f,\rho} - u^{0,\rho})\mathrm{d}x\right] \\
&+\mathbb{E}\left[\int_0^\infty \int \exp(\gamma|x|)(u_t^{f,\rho} - u_t^{\rho})^2 \mathrm{d}x\mathrm{d}t\right]
\end{aligned}$$

If V is convex then

$$\int \rho\exp(\gamma|x|)(\nabla V(W_\infty + I_\infty(u^{f,\rho})) - \nabla V(W_\infty + I_\infty(u^{0,\rho})))I_\infty(u^{f,\rho} - u^{0,\rho})\mathrm{d}x \geqslant 0$$

so

$$\mathbb{E}\left[\int_0^\infty \int \exp(\gamma|x|)(u_t^{f,\rho} - u_t^{\rho})^2 \mathrm{d}x\mathrm{d}t\right] \leqslant |\mathbb{E}[\exp(\gamma|x|)\nabla f(W_\infty + I_\infty(u^{f,\rho}))I_\infty(u^{f,\rho} - u^{0,\rho})]|$$

and with a nice f the r.h.s is bounded by

$$\mathbb{E}\left[\int_0^\infty \int \exp(\gamma|x|)(u_t^{f,\rho} - u_t^{\rho})^2 \mathrm{d}x\mathrm{d}t\right]^{1/2}$$

Now $\Lambda = \mathbb{R}^2$ and

$$V_T(\phi) = \lambda T^{\beta^2/4\pi} \cos(\beta\phi).$$

In this case we can obtain quite strong bounds on the minimizer.

Lemma 5. (Envelope theorem)

$$\nabla V_{t,T}(\varphi) = \mathbb{E}[\nabla V_T(W_{t,T} + I_{t,T}(u^{\varphi}) + \varphi)]$$

where u^{φ} *minimizes*

$$\mathbb{E}\left[\int \rho V_T(W_{t,T} + I_{t,T}(u^{\varphi}) + \varphi) + \frac{1}{2}\int_t^T \|u_s\|_{L^2}^2 \mathrm{d}t\right].$$

$\Rightarrow \|\nabla V_{t,T}\|_{L^\infty} \leqslant \|\nabla V_T\|_{L^\infty}$. So

$$\|u_t^{\varphi}\|_{L^\infty} = \|J_t \nabla V_{t,T}(W_{t,T} + I_{t,T}(u^{\varphi}) + \varphi)\|_{L^\infty} \leqslant t^{-1} T^{\beta^2/4\pi}$$

L^∞ Bound

Now lets take

$$
\begin{aligned}
& \|\nabla V_{t,T}(\varphi)\|_{L^\infty} \\
= \; & \|\mathbb{E}[\nabla V_T(W_{t,T} + I_{t,T}(u^\varphi) + \varphi)]\|_{L^\infty} \\
= \; & \left\| \mathbb{E}\left[\nabla V_T(W_{t,T} + \varphi) + \int \nabla V_T(W_{t,T} + \varphi + \theta I_{t,T}(u^\varphi)) I_{t,T}(u^\varphi) \mathrm{d}\theta \right] \right\|_{L^\infty} \\
\leqslant \; & \|\mathbb{E}[T^{\beta^2/4\pi} \sin(\beta(W_{t,T} + \varphi))]\|_{L^\infty} + \mathbb{E}\left[\int \|\nabla V_T(W_{t,T} + \varphi + \theta I_{t,T}(u^\varphi)) I_{t,T}(u^\varphi)\|_{L^\infty} \mathrm{d}\theta \right] \\
\leqslant \; & \|\mathbb{E}[t^{\beta^2/4\pi} \sin(\beta\varphi)]\|_{L^\infty} + t^{-1} T^{\beta^2/4\pi} \\
\leqslant \; & t^{\beta^2/4\pi} + 2t^{\beta^2/4\pi - 1}
\end{aligned}
$$

Now can proceed inductivly to obtain

$$
\sup_\varphi \|\nabla V_{t,T}(\varphi)\|_{L^\infty} \lesssim t^{\beta^2/4\pi}.
$$

from this we get

$$
\|u\|_{L^\infty} \lesssim t^{\beta^2/4\pi - 1} \sup_\varphi \|\nabla V_{t,T}(\varphi)\|_{L^\infty}
$$

We can calculate by Ito's formulate

$$\begin{aligned}
&\int [\![\cos(\beta W_\infty + \beta I_\infty(u))]\!] \mathrm{d}x \\
=\ &\int_0^\infty \int [\![\cos(\beta W_t + \beta I_t(u))]\!] J_t u_t \mathrm{d}x\mathrm{d}t + \text{martingale.} \\
=\ &\int_0^\infty \int J_t [\![\cos(\beta W_t + \beta I_t(u))]\!] u_t \mathrm{d}x + \text{martingale}
\end{aligned}$$

This gives us that

$$\lambda \int_0^\infty \int J_t [\![\cos(\beta W_t + \beta I_t(u))]\!] u_t \mathrm{d}x$$

is semiconvex in u and if λ is sufficiently small

$$\lambda \int_0^\infty \int J_t [\![\cos(\beta W_t + \beta I_t(u))]\!] u_t \mathrm{d}x + \frac{1}{2}\int_0^\infty \|u\|_{L^2}^2 \mathrm{d}t$$

is convex in u.

◊ We can obtain a coupling between the Free Field and the Sine Gordon measure. Set

$$\nu^{\mathrm{SG}} = \frac{1}{Z^\rho}\exp\left(-\int \rho[\![\cos(\beta\phi)]\!]\right)\mathrm{d}\mu \quad Z^\rho = \int \exp\left(-\int \rho[\![\cos(\beta\phi)]\!]\right)\mathrm{d}\mu$$

Proposition 6.

$$\int f(\varphi)\mathrm{d}\nu^\rho_{\mathrm{SG}} = \mathbb{E}[f(W_\infty + I_\infty(u^\rho))]$$

One can show

$$\sup_\rho \|I_\infty(u^\rho)\|_{L^\infty(\mathbb{P}, C^{2-\delta})} < \infty$$

◊Proof uses that

$$\int f(\varphi)\mathrm{d}\nu^\rho_{\mathrm{SG}} = \lim_{s\to 0}\frac{1}{s}\left(\log\int \exp(-s f(\varphi))\mathrm{d}\nu^\rho_{\mathrm{SG}} - \log Z^\rho\right)$$

◊ Bauerschmidt-Hofstetter derive results on the maximum of the Sine-Gordon field.

Want to study semiclassical limit of measures

$$\nu_{\mathrm{SG},\hbar} = \exp\left(-\frac{\lambda}{\hbar}\int_{\mathbb{R}^2} [\![\sin(\beta\phi)]\!] - \frac{1}{\hbar}\int_{\mathbb{R}^2} \phi(m^2-\Delta)\phi \mathrm{d}x \right) = \exp\left(-\frac{\lambda}{\hbar}\int_{\mathbb{R}^2} [\![\sin(\beta\phi)]\!] \mathrm{d}x \right)\mu^{\hbar}$$

where the covariance of $\mu^{\hbar}$ is

$$\hbar(m^2-\Delta)^{-1}.$$

A seqeunce of measures $\nu_{\hbar}$ satisfies a large deviation principle with rate function L if

$$\lim_{\hbar\to 0} -\hbar \log \int \exp\left(-\frac{1}{\hbar} f(\phi) \right) \mathrm{d}\nu_{\hbar} = \inf_{\phi} \{ f(\phi) + L(\phi) \}$$

Proposition 7. *If λ is suffienclty small, $\nu_{\mathrm{SG},\hbar}$ satisfies a large deviations with rate functions*

$$L(\varphi) = \lambda \int_{\mathbb{R}^2} \cos(\beta\phi) \mathrm{d}x + \int_{\mathbb{R}^2} \phi(m^2-\Delta)\phi \mathrm{d}x.$$

Thank you!