The forcability problem

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The 8th Workshop on Generalized Descriptive Set Theory
Helsinki, August 21 2025

Origin and motivation

This is joint work with Obrad Kasum.

Problem (Entscheidungsproblem)

Is there a general procedure (algorithm) for determining whether a given formula ϕ of first order logic is universally valid or equivalently satisfiable?

The Entscheidungsproblem was regarded as "the main problem of mathematical logic" and "the most general problem of mathematics".

Origin and motivation

In his 1900 Paris lecture, Hilbert says:

> *When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science ... No statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps.*

We known that there is no general decision procedure for the Entscheidungsproblem.

Origin and motivation

Definition

Let ϕ be an infinitary propositional formula. Then ϕ is consistent iff there exists a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there exists a satisfying assignment for ϕ.

Definition

Let ϕ be an infinitary propositional formula. Then ϕ is proper-consistent (resp. semiproper-consistent, ssp-consistent[1]) iff there exists a proper (resp. semiproper, ssp) poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there exists a model for ϕ.

[1] The abbreviation "ssp" stands for "stationary set preserving".

Origin and motivation

Instead of adding a satisfying assignment to an infinitary formula ϕ, we can ask if a poset in $\mathscr{K}$ can:

- add a witness for a given Σ_1-formula $\exists x \theta(x, a)$, for some parameter a,
- add a model for a given $\mathscr{L}_{\infty,\omega}$-sentence ϕ, or
- given a poset $\mathbb{Q}$ and a family of dense sets $\mathcal{D}$ in $\mathbb{Q}$, add a $\mathcal{D}$-generic filter in $\mathbb{Q}$.

Origin and motivation

We want to know when forcing axioms such as PFA or MM imply a certain statement ϕ.

The motivation comes for the theorem of Asperó-Schindler stating that MM^{++} implies Woodin's axiom $(*)$. The proof seems very different from previous applications of forcing axioms.

We want a general method that will give us all known applications of forcing axioms, including the Asperó-Schindler result.

Origin and motivation

Let ϕ be an infinitary propositional formula. We can assume that ϕ is in Conjunctive Normal Form, i.e. ϕ is just a set of clauses, and each clause $C \in \phi$ is a set of literals (a variable or a negated variable). Let $\mathcal{V}$ be the set of variables. We are looking for an assignment $s : \mathcal{V} \to \{0, 1\}$ such that, for every $C \in \phi$, there is $l \in C$, such that $s(l) = 1$.

Origin and motivation

We consider the following game $\Game^{\mathrm{meg}}(\phi)$ of length ω:

I	$C_0 \in \phi$		$C_1 \in \phi$		$\cdots$
II		w_0		w_1	$\cdots$

At stage i, Player I chooses a clause $C_i \in \phi$ and Player II chooses a finite partial assignment w_i, such that $w_{i-1} \subseteq w_i$, and there is a literal $l \in C_i$ with $w_i(l) = 1$. Player II wins the game if she can continue playing infinitely many moves. Otherwise Player I wins.

$\Game^{\mathrm{meg}}(\phi)$ is closed for Player II so one of the players has a winning strategy.

Origin and motivation

We want to imitate the following characterization of consistency.

Theorem

Let ϕ be an infinitary CNF formula. Then ϕ is consistent if and only if Player II has a winning strategy in $\Game^{\mathrm{meg}}(\phi)$.

If Player II has a winning strategy, we let $\mathbb{H}$ denote the set of all finite partial assignments w such that Player II has a winning strategy compatible with w. Then $\mathbb{H}$ ordered by reverse inclusion adds a satisfying assignment for ϕ.

Main result

For a beth-fixed point κ and a formula ϕ we describe versions of the above game that correspond to proper, semi-proper, and ssp-consistency. They are denoted by $\Game^{\mathrm{p}}_{\kappa}(\phi)$, $\Game^{\mathrm{sp}}_{\kappa}(\phi)$, and $\Game^{\mathrm{ssp}}_{\kappa}(\phi)$.

Theorem

Let ϕ be an infinitary CNF propositional formula. Suppose there exists a proper class of inaccessible cardinals. TFAE.

- *ϕ is proper (resp. semiproper, ssp) consistent.*
- *For some inaccessible κ, Player II has a winning strategy in $\Game^{\mathrm{p}}_{\kappa}(\phi)$ (resp. $\Game^{\mathrm{sp}}_{\kappa}(\phi)$, $\Game^{\mathrm{ssp}}_{\kappa}(\phi)$).*

Main result

This theorem essentially characterizes when a model to a $\mathbf{\Sigma}_1$ formula can be added by forcing of a given kind.

Moreover, the set of winning positions for Player II gives the desired poset.

What is not obvious is to decide who has a winning strategy, but we give some sufficient conditions for Player II to win these games. This covers all known applications of forcing axioms.

The set-up

For a set X, we let $\widehat{X}$ denote the transitive closure of X.

Definition

A set M is a virtual model iff $M \prec \widehat{M} \models \mathrm{ZFC}^-$. □

We have the following hull operation.

Definition

Let M be a virtual model and X an arbitrary set. Then

$$\mathrm{Hull}(M, X) = \{f(x) : f \in M \text{ is a function and } x \in X \cap \mathrm{dom}(f)\}$$

The set-up

$\mathrm{Hull}(M, X)$ is elementary in $\widehat{M}$ and it contains $X \cap \widehat{M}$. In fact, it is the smallest virtual model with respect to these two properties.

Definition

Suppose that M is a virtual model and that λ is an infinite ordinal. We define $M{\downarrow}\lambda$ to be the image of M under the transitive collapse of $\mathrm{Hull}(M, V_\lambda)$.

The anti-collapse of a virtual model is essentially a partial extender.

The set-up

We let $\beth_{\text{fix}}$ denotes the class of all beth-fixed points.

Definition

Suppose that M is a virtual model. Then λ_M is defined as the largest $\lambda \in \beth_{\text{fix}}$ such that $V_\lambda \in M$, when it exists. □

Definition

Suppose that $\lambda \in \beth_{\text{fix}}$. Then $\mathcal{C}_\lambda$ denotes the set of all virtual model M such that $\lambda_M = \lambda$ and $\widehat{M} = \text{Hull}(M, V_\lambda)$.

Definition

The class $\mathcal{C}$ is defined as $\bigcup_{\lambda \in \beth_{\text{fix}}} \mathcal{C}_\lambda$.

When using virtual models as side conditions, we will require them to be $\in$-chains. In fact, we will require a bit more, as explained in the following definition.

If M is a virtual model we let $\delta_M = M \cap \omega_1$.

Definition

Suppose that $\mathcal{M} \subseteq \mathcal{C}$. Then $\mathcal{M}$ is a vm-chain iff for all $M, N \in \mathcal{M}$,

- if $\delta_M = \delta_N$, then $M = N$,
- if $\delta_M < \delta_N$, then $M \in N$ and $\lambda_M < \lambda_N$. □

The set-up

When passing from a virtual model M to its projection $M{\downarrow}\lambda$, we lose information. We will use the following notion in order to talk about this possibility of increasing the information content of a virtual model.

Definition

Suppose that M and N are virtual models, $M \in \mathcal{C}$, and $\pi : M \longrightarrow N$. Then π is a lifting iff, letting

$$\rho : \widehat{N{\downarrow}\lambda_M} \xrightarrow{\simeq} \mathrm{Hull}(N, V_{\lambda_M}) \prec \widehat{N}$$

be the anti-collapse, we have that $M = N{\downarrow}\lambda_M$ and $\pi = \rho \restriction M$.

SSP consistency

We consider the case of SSP consistency.
Fix an infinitary CNF formula ϕ. We think of ϕ as a set of clauses.
We may assume that ϕ is consistent.
Let $\mathbb{H}$ be the poset of finite partial assignments which are winning positions for Player II in the consistency game for ϕ.

proposition

In $V^{\mathbb{H}}$, there exists a satisfying assignment for ϕ.

However, in general, $\mathbb{H}$ collapses ω_1. We want an SSP poset adding a satisfying assignment for ϕ.

SSP consistency

Let $\lambda \in \beth_{\mathrm{fix}}$ such that $\phi \in V_\lambda$.

Definition (The poset of preconditions $\mathbb{P}^*_\lambda$)

- Let $p \in \mathbb{P}^*_\lambda$ iff $p = (w_p, \mathcal{M}_p)$ where: $w_p \in \mathbb{H}$, $\mathcal{M}_p$ is a finite subset of $\mathcal{C}_{<\lambda}$ and is a vm-chain.
- Let $p \leq q$ iff $w_p \supseteq w_q$, and for all $N \in \mathcal{M}_q$ there is $M \in \mathcal{M}_p$ such that $\delta_M = \delta_N$ $\lambda_M = \lambda_N$ and $M \supseteq N$.

Let

$$\mathbb{P}^* = \bigcup_{\lambda \in \beth_{\mathrm{fix}}} \mathbb{P}^*_\lambda.$$

Suppose that $\kappa \in \beth_{\mathrm{fix}}$ and $p \in \mathbb{P}^* \cap V_\kappa$. Then the game $\Game^{\mathrm{ssp}}_\kappa(p, \phi)$ is defined as the length ω two Player game of the form

I		Q_0		Q_1		$\cdots$
II	p_{-1}		p_0		p_1	$\cdots$

where $p_{-1} \in \mathbb{P}^* \cap V_\kappa$ is such that $p_{-1} \leq p$ and for all $n < \omega$, the following is satisfied.

SSP consistency

Player I must ensure that either

- $Q_n = C$ for some clause C of ϕ, or
- $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary,
- $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, such that there exists $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

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SSP consistency

Player II must ensure that

- $p_n \in \mathbb{P}^* \cap V_\kappa$ and $p_n \leq p_{n-1}$.
- if $Q_n = C$ for some clause C of ϕ, then there exists $l \in C$ such that $w_{p_n}(l) = 1$.
- if $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary, then there exist

  $$M \prec (H((2^\kappa)^+), \in, \kappa, p_{n-1}, U)$$

  and $\lambda \in \beth_{\mathrm{fix}} \cap \kappa$ such that

  $$\kappa \cap \mathrm{Hull}(M, V_\lambda) \subseteq \lambda,$$

  $M{\downarrow}\lambda \in \mathcal{M}_{p_n}$, and $\delta_M \in S$.

SSP consistency

- if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exists $q \in E$ such that $\delta(\mathrm{Hull}(M, q)) = \delta(M)$ and $p_n \leq q$.

The infinite plays with no rules broken are won by Player II.

Definition

Suppose that $\kappa \in \beth_{\mathrm{fix}}$. We let $\mathbb{C}_\kappa^{\mathrm{ssp}}(\phi)$ consist of all $p \in \mathbb{P}^* \cap V_\kappa$ such that Player II wins $\Game_\kappa^{\mathrm{ssp}}(p, \phi)$.

SSP consistency

Proposition

Suppose that $\kappa \in \beth_{\text{fix}}$ and that $\mathbb{C}_\kappa^{\text{ssp}}(\phi)$ is non-empty. Then $\mathbb{C}_\kappa^{\text{ssp}}(\phi)$ is stationary set preserving and adds a satisfying assignment for ϕ.

Theorem

Suppose there exists a proper class of inaccessible cardinals. Then an infinitary formula ϕ is ssp-consistent iff there is an inaccessible κ such that Player II has a winning strategy in $\eth_\kappa^{\text{ssp}}(\emptyset, \phi)$

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The AS-condition

While the above theorem gives a characterization when a given infinitary formula is ssp-consistent, it is not easy to use.

We now describe a much more manageable condition which is sufficient for the Asperó-Schindler theorem. We fix an infinitary formula ψ which is consistent. We then define inductively posets P_λ, for $\lambda \in \beth_{\text{fix}}$. They all add a satisfying assignment for ϕ, but they may not be ssp. Once we reach an inaccessible κ satisfying the AS-condition we have that $\mathbb{P}_\kappa$ is ssp.
For $p \in \mathbb{P}^*_\lambda$ and for $\bar{\lambda} \in \lambda \cap \beth_{\text{fix}}$, we define

$$p \restriction \lambda := (w_p, \{M \in \mathcal{M}_p : \lambda_M < \lambda\}).$$

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The AS-condition

Suppose that $p \in \mathbb{P}^*_\lambda$. The game $\mathcal{G}_\lambda(p)$ is played as follows.

- Set first $p_{-1} := p$.
- In round $n < \omega$, Player I plays Q_n and Player II answers by $p_n \in \mathbb{P}^*_\lambda$ satisfying $p_n \leq p_{n-1}$.
- If $Q_n = C$ is a clause in ϕ then there is $l \in C$ such that $w_{p_n}(l) = 1$,
- If $Q_n = (M, D)$ where $M \in \mathcal{M}_{p_{n-1}}$ and $D \in M$ is dense in $\mathbb{P}_{\lambda_M}$, then there exists $q \in D$ such that $\delta(\text{Hull}(M, q)) = \delta(M)$ and $p_n \leq p_{n-1}, q$.
- Player II wins this run of the game if she can play infinitely many moves. Otherwise Player I wins.

The AS-condition

Definition

$\mathbb{P}_\lambda$ is the suborder of $(\mathbb{P}^*_\lambda, \leq)$ consisting of those $p \in \mathbb{P}^*_\lambda$ for which Player II wins $\mathcal{G}_\lambda(p)$.

Definition

Suppose that $\theta \gg \kappa$ is regular and $M \prec (H_\theta, \in, \kappa, \phi)$ is countable. Then M is good iff for all $p \in \mathbb{P}_\kappa \cap M$, there exist $q \in \mathbb{P}_\kappa$ and $\lambda \in \kappa \cap \beth_{\text{fix}}$ such that

- $\kappa \cap M \subseteq \lambda$,
- $q \leq p$,
- $M{\downarrow}\lambda \in \mathcal{M}_q$.

The AS-condition

Proposition

Suppose that $\theta \gg \kappa$ is regular and $M \prec H_\theta$ is good. Then $\mathbb{P}_\kappa$ is semiproper for M.

Corollary

If there exists a local club of good models $M \prec H_\theta$, then $\mathbb{P}_\kappa$ is stationary set preserving.

Local club

$C \subseteq [X]^\omega$ is a *local club* iff for weak-club many $\bar{X} \in [X]^{\omega_1}$, the set $C \cap [\bar{X}]^\omega$ contains a club.

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Definition

ϕ is *AS good* at κ iff

$\forall S$ which are stationary subsets of ω_1,

$V^{\mathrm{Col}(\omega,<\kappa)}$ satisfies that:

$\forall \mu$ which are models for ϕ,

$\exists \tau$ which is an elementary embedding from V to some W with critical point ω_1^V and which satisfies that $\omega_1^V \in \tau(S)$, and

$\exists \hat{\mu}$ which is a model for $\tau(\phi)$ satisfying that

for all $\psi \in \phi\downarrow$, $\hat{\mu}(\tau(\psi)) = \mu(\psi)$.

The AS-condition

Theorem

If ϕ is AS-good at κ, then $\mathbb{P}_\kappa$ is stationary set preserving.

The main point is that this allows us to prove the Asperó-Schindler theorem.

Theorem (Asperó-Schindler)

Assume MM^{++} implies Woodin's axiom $(*)$.

Thank You!

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