Topology and Cardinal Invariants on Singular Higher Baire Spaces

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Notation

We let μ denote a singular cardinal and $\kappa = \mathrm{cf}(\mu)$.

We let $\langle \mu_\xi \mid \xi \in \kappa \rangle$ denote a continuous strictly increasing sequence cofinal in μ.

Topology

Given a space of functions ${}^{\delta}\rho$ and a (partial) function $s : D \to \rho$ with $D \subseteq \delta$, we write $[s] = \{f \in {}^{\delta}\rho \mid s \subseteq f\}$.

Let ν be a cardinal, then the *$<\nu$-box topology* on ${}^{\delta}\rho$ is the topology generated by basic clopens $[s]$ such that $s : D \to \rho$ has $|D| < \nu$.

The *bounded topology* on ${}^{\delta}\rho$ is the topology generated by basic clopens $[s]$ such that $s : D \to \rho$ and D is bounded in δ.

We will discuss the $<\kappa$-box, bounded, and $<\mu$-box topology, abbreviated as "κ", "bd", and "μ", e.g. in subscripts.

There are several closely related sets of functions that serve as generalisation of the classical Baire (and Cantor) space:

1. ${}^{\mu}\mu$
2. ${}^{\mu}2$
3. ${}^{\mu}\kappa$
4. ${}^{\kappa}\mu$
5. $\prod_{\xi \in \kappa} \mu_\xi = \mathcal{K}$

Each of these sets of functions may be given the $<\kappa$-box topology, the $<\mu$-box topology or the bounded topology.

Definition

Let (X, τ) be a topological space. A *local basis* for $x \in X$ is a set $B \subseteq \tau$ such that $x \in V$ for all $V \in B$ and each $U \in \tau$ with $x \in U$ has $V \in B$ with $V \subseteq U$.

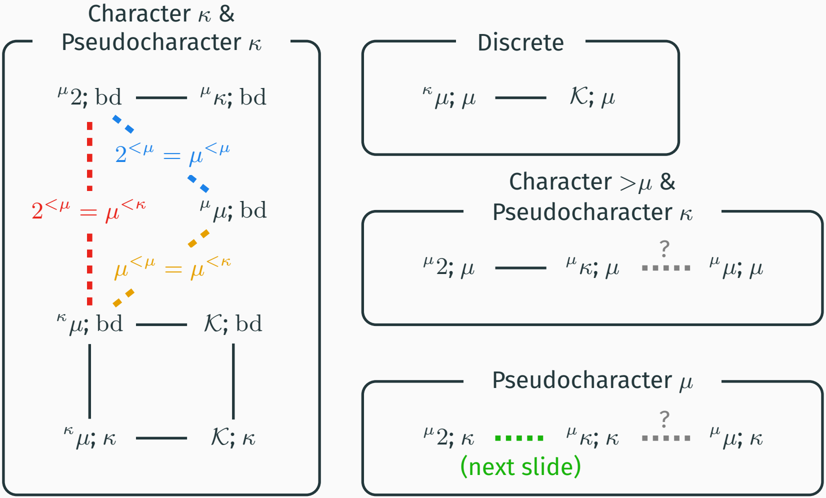
(X, τ) has *character* ν if all $x \in X$ have a local basis of cardinality (at most) ν.

(X, τ) has *pseudocharacter* ν if each singleton $\{x\}$ with $x \in X$ is the intersection of (at most) ν-many open sets.

(X, τ) is *discrete* if $\{x\}$ is open for each $x \in X$.

Note: character, pseudocharacter and discreteness are topological invariants.

15 Different Spaces — Or Are They?



A space (X, τ) is $<\kappa$-*compact* if every open cover of X has a subcover of size $<\kappa$. The cardinal κ is *weakly compact* if and only if ${}^{\kappa}2$ with the $<\kappa$-box topology is $<\kappa$-compact. The cardinal κ is *strongly compact* if and only if ${}^{\theta}2$ with the $<\kappa$-box topology is $<\kappa$-compact for every θ.

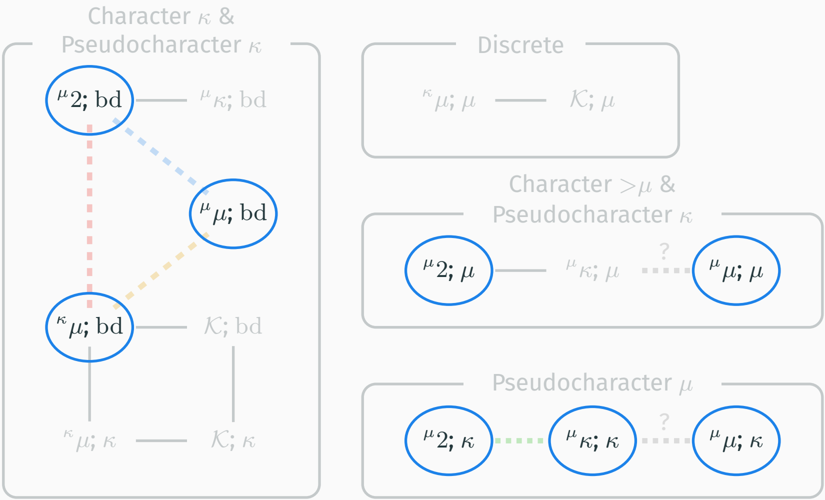
Theorem

If κ is strongly compact, then $({}^{\mu}2, \kappa)$ and $({}^{\mu}\kappa, \kappa)$ **are not** homeomorphic.

Theorem

If κ is not weakly compact, then $({}^{\mu}2, \kappa)$ and $({}^{\mu}\kappa, \kappa)$ **are** homeomorphic.

At Most 8 Spaces!



Meagre Sets

A subset A of a topological space is *nowhere dense (nwd)* if every nonempty open contains a nonempty open disjoint from A. For ν a cardinal, a subset of a topological space is ν*-meagre* if it is the union of ν-many nwd sets.

For each of our 8 spaces, every ν-meagre set is nwd if $\nu < \kappa$. Moreover, some κ-meagre set is not nwd. Finally, at least 6 of our spaces are κ^+-meagre in themselves (so, $\mathrm{cov}(\mathcal{M}_\tau^X) = \kappa^+$).

Notation

For a space (X, τ), we write $\mathcal{M}_\tau^X$ for the κ-meagre ideal of X. E.g.: $\mathcal{M}_{\mathrm{bd}}^{\kappa_\mu}$, $\mathcal{M}_\kappa^{\mu_\mu}$, etc.

For a space (X, τ), we may consider the forcing $\mathbb{C}^X_\tau$ consisting of nonempty open sets ordered by inclusion. For instance, $\mathbb{C}^{\omega_\omega}_{\text{product}}$ is just Cohen forcing.

Lemma *Cf. Landver 1992, Lemma 1.3*

If $\mathbb{C}^X_\tau$ collapses κ^+, then $\operatorname{cov}(\mathcal{M}^X_\tau) = \kappa^+$.

Proof. Let $\dot{f}$ name an injection from $(\kappa^+)^{\mathbf{V}}$ to κ and for each $\alpha \in \kappa^+$ let D_α be the set of conditions deciding $\dot{f}(\alpha)$. Then $X \setminus \bigcup D_\alpha$ is nwd in X, and $\{X \setminus \bigcup D_\alpha \mid \alpha \in \kappa^+\}$ covers X: otherwise there would be $p \in \bigcap_{\alpha\in\kappa^+} D_\alpha$, which is absurd. □

Theorem

$$\begin{aligned}\kappa^+ &= \operatorname{cov}(\mathcal{M}^{\kappa_\mu}_{\text{bd}}) = \operatorname{cov}(\mathcal{M}^{\mu_\mu}_{\text{bd}}) = \operatorname{cov}(\mathcal{M}^{\mu_2}_{\text{bd}}) \\ &= \operatorname{cov}(\mathcal{M}^{\mu_\mu}_{\mu}) = \operatorname{cov}(\mathcal{M}^{\mu_2}_{\mu}) = \operatorname{cov}(\mathcal{M}^{\mu_\mu}_{\kappa}).\end{aligned}$$

Note that $\mathbb{C}_\kappa^{\mu 2}$ is forcing equivalent to the κ-support product of κ-Cohen forcing $\mathbb{C}_\kappa^{\kappa 2}$ of length μ. Hence, Landver's Lemma is not usable to determine whether $\operatorname{cov}(\mathcal{M}_\kappa^{\mu 2}) = \kappa^+$.

Theorem

$\kappa^+ \leq \operatorname{cov}(\mathcal{M}_\kappa^{\mu 2}) \leq \operatorname{cov}(\mathcal{M}_\kappa^{\kappa 2})$.

Conjecture

$\operatorname{cov}(\mathcal{M}_\kappa^{\mu 2}) = \operatorname{cov}(\mathcal{M}_\kappa^{\kappa 2})$.

A similar situation occurs for $\mathcal{M}_\kappa^{\mu \kappa}$.

Domination

Given $f, g \in {}^{\delta}\rho$ and a cardinal ν (either κ or μ in our case), we define the following orders.

Let $f \leq_\nu g$ if $|\{\alpha \in \delta \mid f(\alpha) > g(\alpha)\}| < \nu$.
Let $f \leq_{\mathrm{bd}} g$ if $\{\alpha \in \delta \mid f(\alpha) > g(\alpha)\} \subseteq \beta$ for some $\beta < \delta$.
Let $f \leq_{\mathrm{all}} g$ if $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \delta$.

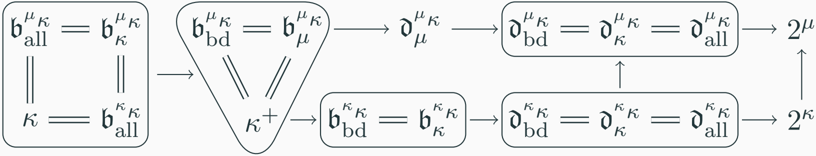
Let $\mathfrak{b}_{(\cdot)}^{{}^{\delta}\rho}$ be the least size of a $\leq_{(\cdot)}$-unbounded subset of ${}^{\delta}\rho$ and let $\mathfrak{d}_{(\cdot)}^{{}^{\delta}\rho}$ be the least size of a $\leq_{(\cdot)}$-dominating subset of ${}^{\delta}\rho$.

Proposition

Assuming ${}^{\delta}\rho$ itself is $\leq_{(\cdot)}$-unbounded, $\mathfrak{b}_{(\cdot)}^{{}^{\delta}\rho} \leq \mathfrak{d}_{(\cdot)}^{{}^{\delta}\rho}$.
$\mathfrak{b}_{\mathrm{all}}^{{}^{\delta}\rho} \leq \mathfrak{b}_{\kappa}^{{}^{\delta}\rho} \leq \mathfrak{b}_{\mathrm{bd}}^{{}^{\delta}\rho} \leq \mathfrak{b}_{\mu}^{{}^{\delta}\rho}$ and $\mathfrak{d}_{\mu}^{{}^{\delta}\rho} \leq \mathfrak{d}_{\mathrm{bd}}^{{}^{\delta}\rho} \leq \mathfrak{d}_{\kappa}^{{}^{\delta}\rho} \leq \mathfrak{d}_{\mathrm{all}}^{{}^{\delta}\rho}$.
If τ is a cofinal subset of ρ, then $\mathfrak{b}_{(\cdot)}^{{}^{\delta}\rho} = \mathfrak{b}_{(\cdot)}^{{}^{\delta}\tau}$ and $\mathfrak{d}_{(\cdot)}^{{}^{\delta}\rho} = \mathfrak{d}_{(\cdot)}^{{}^{\delta}\tau}$.

Domination on ${}^{\kappa}\mu$ is equivalent to domination on ${}^{\kappa}\kappa$; and domination on ${}^{\mu}\mu$ is equivalent to domination on ${}^{\mu}\kappa$.

Theorem *Folklore; as in classical case; Brendle 2022; Hayashi's thesis*



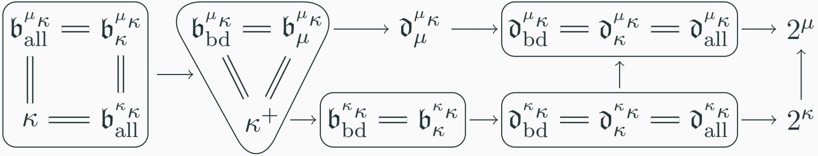
Question

Is $\mathfrak{d}^{{}^\mu\kappa}_{\mu} < \mathfrak{d}^{{}^\mu\kappa}_{\kappa}$ consistent?

Domination and Unbounding on Our (2) Spaces

Domination on ${}^\kappa\mu$ is equivalent to domination on ${}^\kappa\kappa$; and domination on ${}^\mu\mu$ is equivalent to domination on ${}^\mu\kappa$.

Theorem *Folklore; as in classical case; Brendle 2022; Hayashi's thesis*



Theorem *Shelah 2019*

If $\lambda^\kappa < \mu$ for all $\lambda < \mu$, then $\mathfrak{d}^{\mu\kappa}_{\mu} = 2^\mu$.

Theorem *Folklore? Hayashi 2025, § 5*

If κ is uncountable, then $\mathfrak{d}^{\mu\kappa}_{\kappa} < 2^\mu$ is consistent.

Theorem *Hayashi 2025, § 5*

$\mathrm{cof}([\mu]^\kappa, \subseteq) \leq \mathfrak{d}^{\mu\kappa}_{\mu}$.

More About $\mathcal{M}^{\kappa\mu}_{\mathrm{bd}}$

Theorem *Folklore; as in the classical case*

$\mathfrak{b}_\kappa = \mathfrak{b}_{\mathrm{bd}}^{{}^\kappa\mu} \leq \mathrm{non}(\mathcal{M}_{\mathrm{bd}}^{{}^\kappa\mu}) \leq \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{{}^\kappa\mu})$.

Theorem *Hayashi and vdV.*

$\mu^{<\kappa} \leq \mathrm{non}(\mathcal{M}_{\mathrm{bd}}^{{}^\kappa\mu})$ and $\mathrm{cof}([\mu]^\kappa, \subseteq) \leq \mathrm{non}(\mathcal{M}_{\mathrm{bd}}^{{}^\kappa\mu})$.

Proof. Let $X \subseteq {}^\kappa\mu$ with $|X| < \mu^{<\kappa}$ and $s \in \mu^{<\kappa}$. Then there is some $t \in \mu^{<\kappa}$ extending s with $[t] \cap X = \varnothing$.

Let $X \subseteq {}^\kappa\mu$ with $|X| < \mathrm{cof}([\mu]^\kappa, \subseteq)$. Then there is $y \in [\mu]^\kappa$ with $\forall f \in X (y \nsubseteq \mathrm{ran}(f))$, so $X \subseteq \bigcup_{\alpha\in y} \{f \in {}^\mu\kappa \mid \alpha \notin \mathrm{ran}(f)\}$. □

Question

Is $\mathrm{non}(\mathcal{M}_{\mathrm{bd}}^{{}^\kappa\mu}) < \mu^\kappa$ consistent?

Theorem *Brendle 2022*

If $\tilde{\kappa}$ is regular uncountable and $\tilde{\lambda} = 2^{<\tilde{\kappa}}$, then $\mathfrak{d}_{\mathrm{bd}}^{\tilde{\lambda}\tilde{\kappa}} \leq \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\tilde{\kappa}\tilde{\kappa}})$.

N.B.: Since $\tilde{\lambda}^+ \leq \mathfrak{d}_{\mathrm{bd}}^{\tilde{\lambda}\tilde{\kappa}}$, it follows that $2^{\tilde{\kappa}} < \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\tilde{\kappa}\tilde{\kappa}})$ is consistent, e.g. when $2^{\tilde{\kappa}} = 2^{<\tilde{\kappa}}$.

Theorem *Hayashi and vdV.*

Let $\lambda = \mu^{<\kappa}$, then $\mathfrak{d}_{\mathrm{bd}}^{\lambda\kappa} \leq \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\kappa\mu})$.

Corollary

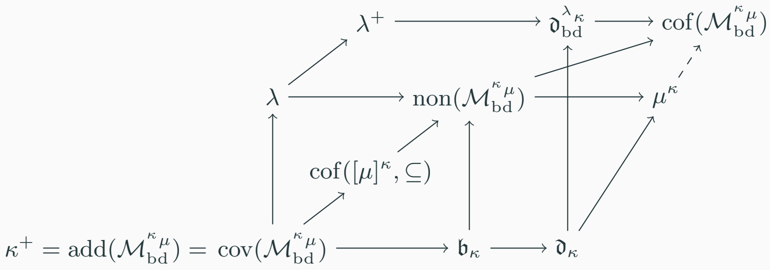
$\lambda^+ \leq \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\kappa\mu})$ and $\mathfrak{d}_\kappa = \mathfrak{d}_{\mathrm{bd}}^{\kappa\mu} \leq \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\kappa\mu})$ and it is consistent that $\mu^\kappa < \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\kappa\mu})$.

Theorem *Hayashi and vdV.*

If $2^\kappa < \mu^\kappa$, then $\mu^\kappa \leq \mathrm{cof}(\mathcal{M}_{\mathrm{bd}}^{\kappa\mu})$.

We don't know if the assumption "$2^\kappa < \mu^\kappa$" is necessary here.

Let $\lambda = \mu^{<\kappa}$.



Apart from the mentioned questions;

What about $\mathrm{non}(\mathcal{M}_\tau^X)$ & $\mathrm{cof}(\mathcal{M}_\tau^X)$ for the 7 other (X, τ)?

References

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