Open Hypergraphs, Covering with Closed Sets and Games

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Joint work in progress with Philipp Schlicht

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Very general dichotomies have emerged for graphs and hypergraphs which imply several old and new theorems in descriptive set theory.

General Aims:

- Versions of these dichotomies for generalized Baire spaces.
- Lift known applications to the uncountable setting.
- New applications.

Carroy, Miller and Soukup (2020) found a generalization of Feng's open graph dichotomy to infinite dimensional directed hypergraphs on analytic sets of reals, which we have lifted to definable subsets of generalized Baire spaces.

The setup:

κ always denotes an infinite cardinal with $\kappa^{<\kappa} = \kappa$.

${}^{\kappa}d$ always has the bounded topology τ_b for any discrete topological space d, with basic open sets $N_t := \{x \in {}^{\kappa}d : t \subseteq x\}$, where $t \in {}^{<\kappa}d$.

A d-dihypergraph on a set $X \subseteq {}^{\kappa}\kappa$ is a set of nonconstant sequences in ${}^{d}X$.

Fix the box topology on ${}^{d}X$ with basic open sets $\prod_{i \in d} U_i$, where each U_i is open in X.

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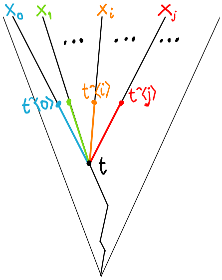
The open graph dichotomy:

$\mathrm{OGD}_{\kappa}(X)$ states that for any open graph G on X, either

- G admits a κ-coloring (i.e., X is the union of κ many G-independent sets),
- or G has a κ-perfect complete subgraph (i.e., there is a continuous embedding $f : {}^{\kappa}2 \to X$ of the complete graph $\mathbb{K}_{{}^{\kappa}2}$ into G.)

The open dihypergraph dichotomy

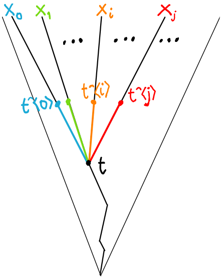


$\mathsf{ODD}^d_\kappa(X)$: For all box-open d-dihypergraphs H on X, either H admits a κ-coloring, or there is a continuous homomorphism $f : {}^\kappa d \to X$ from $\mathbb{H}_{{}^\kappa d} := \bigcup_{t \in {}^{<\kappa} d} \prod_{i \in d} N_{t^\frown \langle i \rangle}$ to H.

$\mathsf{ODD}^d_\kappa(X, \mathsf{Def}_\kappa)$ denotes the restriction to definable box-open dihypergraphs.

By "definable", we always mean "definable from a κ-sequence of ordinals".

The open dihypergraph dichotomy



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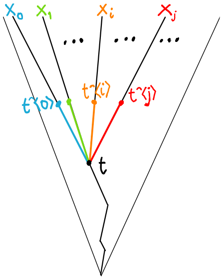

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By "definable", we always mean "definable from a κ-sequence of ordinals".

Theorem (Schlicht, Sz, 2023)

After a Lévy collapse of λ to κ^+, the following hold for all definable subsets X of ${}^\kappa\kappa$:

- $\mathsf{ODD}^\kappa_\kappa(X)$ *if λ is Mahlo.*
- $\mathsf{ODD}^\kappa_\kappa(X, \mathsf{Def}_\kappa)$ *if λ is inaccessible.*

Some Applications

Let $X \subseteq {}^\kappa\kappa$. $\mathsf{ODD}^\kappa_\kappa(X)$ implies each of the following:[1]

- Versions of the Hurewicz dichotomy:
  - either X is covered by κ-many κ-compact sets, or X contains a closed subset of ${}^\kappa\kappa$ which is homeomorphic to ${}^\kappa\kappa$.
  - Either $X \subseteq \bigcup_{\alpha<\kappa}[T_\alpha]$ for $<\kappa$-splitting trees T_α or X contains a superperfect subset.
- The Kechris-Louveau-Woodin dichotomy characterizing when X can be separated from $Y \subseteq {}^\kappa\kappa \setminus \{X\}$ by a $\mathbf{\Sigma}^0_2(\kappa)$ set.
- The determinacy of Väänänen's perfect set game of length κ for all subsets of ${}^\kappa\kappa$.
- The asymmetric κ-Baire property.
- The Jayne-Rogers theorem any $f : X \to {}^\kappa\kappa$ is $\mathbf{\Delta}^0_2(\kappa)$-measurable if and only if it is a union of κ many continuous functions on relatively closed subsets of X.

[1] The first two and last one were obtained for $\kappa = \omega$ by Carroy-Miller-Soukup (2020). For $\kappa > \omega$, all of these were obtained by Schlicht and I (2023).

The Closed-Sets Covering Property

$\mathcal{F}$ always denotes a family of closed subsets of ${}^{\kappa}\kappa$. $\mathcal{I}_{\mathcal{F}}$ is the κ-ideal generated by $\mathcal{F}$ (i.e., the closure of $\mathcal{F}$ under taking unions of size κ and subsets).

Definition. Suppose $X \subseteq {}^{\kappa}\kappa$, C is a class.

$\mathsf{CCP}_{\kappa}^{\mathsf{C}}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^{\kappa}\kappa$, either $X \in \mathcal{I}_{\mathcal{F}}$ or X has an $\mathcal{I}_{\mathcal{F}}$-positive subset $Y \in \mathsf{C}$.

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Theorem (Louveau)

$\mathsf{CCP}^{\mathbf{\Sigma}^1_1}_\omega(X)$ *holds for all subsets of* ${}^\omega\omega$ *in Solovay's model.*

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Theorem (Louveau)

$\mathsf{CCP}_{\omega}^{\mathbf{\Sigma}^1_1}(X)$ *holds for all subsets of* ${}^{\omega}\omega$ *in Solovay's model.*

Theorem (Solecki)

$\mathsf{CCP}_{\omega}^{\mathbf{\Pi}^0_2}(X)$ *holds for all analytic subsets of* ${}^{\omega}\omega$.

By Solecki's result, $\mathsf{CCP}_{\omega}^{\mathbf{\Sigma}^1_1}(X) \iff \mathsf{CCP}_{\omega}^{\mathbf{\Pi}^0_2}(X)$ for all $X \subseteq {}^{\omega}\omega$.

The Closed-Sets Covering Property

Definition. Suppose $X \subseteq {}^{\kappa}\kappa$.

$\mathrm{CCP}_{\kappa}(X)$: For any family $\mathcal{F}$ of closed subsets of ${}^{\kappa}\kappa$, either $X \in \mathcal{I}_{\mathcal{F}}$ or there is a continuous function $f : {}^{\kappa}\kappa \to X$ with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.

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$\mathsf{CCP}_\kappa(X, \mathsf{Def}_\kappa)$ is the restriction to definable families $\mathcal{F}$ of closed sets.

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$\mathsf{CCP}_\kappa(X, \mathsf{Def}_\kappa)$ is the restriction to definable families $\mathcal{F}$ of closed sets.

Theorem

For any $X \subseteq {}^\kappa\kappa$:

- $\mathsf{CCP}_\kappa(X) \iff \mathsf{ODD}^\kappa_\kappa(X)$.
- $\mathsf{CCP}_\kappa(X, \mathsf{Def}_\kappa) \iff \mathsf{ODD}^\kappa_\kappa(X, \mathsf{Def}_\kappa)$.

Hence $\mathsf{CCP}_\kappa(X)$ holds for all definable sets X after a Lévy-collapse of a Mahlo cardinal to κ^+, and an inaccessible suffices for $\mathsf{CCP}_\kappa(X, \mathsf{Def}_\kappa)$.

Form CCP to ODD

Proof sketch.

Suppose H is a box-open κ-dihypergraph on X. Let $\mathcal{F}$ be the family of all closed H-independent subsets of ${}^{\kappa}\kappa$.

$Y \in \mathcal{I}_{\mathcal{F}} \iff H \restriction Y$ has a κ-coloring, for all $Y \subseteq X$.

Lemma

The existence of the following objects is equivalent:

- *a continuous homomorphism from* $\mathbb{H}_{\kappa_\kappa}$ *to* H,
- *a continuous map* $f : {}^{\kappa}\kappa \to X$ *with* $f(N_t) \notin \mathcal{I}_{\mathcal{F}}$ *for all* $t \in {}^{<\kappa}\kappa$.

Hence $\mathsf{CCP}^{\mathcal{F}}_{\kappa}(X) \iff \mathsf{ODD}^{H\restriction X}_{\kappa}$.

Proof of the Lemma.

$\Downarrow$: Suppose $f : {}^{\kappa}\kappa \to X$ is a continuous homomorphism from $\mathbb{H}_{{}^{\kappa}\kappa}$ to X.

Claim. $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$.

Proof. Suppose $f(N_t) \subseteq \bigcup_{\alpha<\kappa} X_\alpha$ where each $X_\alpha \in \mathcal{F}$. Construct a continuous increasing sequence $\langle t_\alpha : \alpha < \kappa \rangle$ such that $t_0 = t$ and for each $\alpha < \kappa$,

- $t_{\alpha+1}$ is an immediate successor of t_α
- $f(N_{t_{\alpha+1}}) \cap X_\alpha = \emptyset$.

This is possible since otherwise, there exists x_i in $f(N_{t_\alpha \frown \langle i \rangle}) \cap X_\alpha$ for each $i < \kappa$. Since f is a homomorphism, $\langle x_i : i < \kappa \rangle \in H \restriction X_\alpha$. So $X_\alpha \notin \mathcal{F}$. □

$\Uparrow$: Suppose $f : {}^{\kappa}\kappa \to X$ is continuous with $f(N_t) \in \mathcal{I}_{\mathcal{F}}^{+}$ for all $t \in {}^{<\kappa}\kappa$. Construct continuous strict order preserving maps $\phi, \iota : {}^{<\kappa}\kappa \to {}^{<\kappa}\kappa$ such that for all $t \in {}^{<\kappa}\kappa$,

- $\prod_{i<\kappa} N_{\phi(t \frown \langle i \rangle)} \cap X \subseteq H$,
- $f(N_{\iota(t)}) \subseteq N_{\phi(t)}$.

Then $[\phi] = f \circ [\iota]$ will be a continuous homomorphism from $\mathbb{H}_{\kappa_d}$ to H. □

Form ODD to CCP

Let $\mathcal{F}$ be a family of closed subsets of ${}^\kappa\kappa$. We may assume $\mathcal{I}_\mathcal{F} \cap \mathbf{\Pi}^0_1 = \mathcal{F}$.

Let H consist of all κ-sequences $\langle x_\alpha : \alpha < \kappa \rangle \in {}^\kappa\kappa$ with $\overline{\{x_\alpha : \alpha < \kappa\}} \notin \mathcal{F}$.

Lemma

A closed subset C of ${}^\kappa\kappa$ is H-independent if and only if $C \in \mathcal{F}$.

Proof. $\Rightarrow$: Take a κ-sequence $\langle x_\alpha : \alpha < \kappa \rangle \in {}^\kappa\kappa$ whose range is dense in C. Then $C = \overline{\{x_\alpha : \alpha < \kappa\}}$ is not in $\mathcal{F}$, so it is $\mathcal{I}_\mathcal{F}$-positive.

$\Leftarrow$: If $H \restriction C$ has a hyperedge $\langle x_\alpha : \alpha < \kappa \rangle \in {}^\kappa\kappa$ then $C \notin \mathcal{F}$ since C is a superset of the $\mathcal{I}_\mathcal{F}$-positive set $\overline{\{x_\alpha : \alpha < \kappa\}}$. □

Hence $\mathsf{CCP}^\mathcal{F}_\kappa(X) \iff \mathsf{ODD}^{H\restriction X}_\kappa$ by the previous slide. □

It's all the same in the countable setting

Lemma

$\mathsf{CCP}_\omega(X) \iff \mathsf{CCP}_\omega^{\boldsymbol{\Sigma}_1^1}(X)$ *for all* $X \subseteq {}^\omega\omega$.

Proof.

It suffices to show $\mathsf{CCP}_\omega(\boldsymbol{\Sigma}_1^1)$. Let X be an $\mathcal{I}_\mathcal{F}$-positive analytic set, and let $f : {}^\omega\omega \to X$ be a continuous surjection. For all $t \in {}^{<\omega}\omega$, take an infinite maximal antichain A_t of nodes u in ${}^{<\omega}\omega$ with $t \subseteq u$ and $f(N_u) \in \mathcal{I}_\mathcal{F}^+$.

Construct a strict order preserving map $\phi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $\langle \phi(t^\frown\langle i\rangle) : i < \omega\rangle$ enumerates $A_{\phi(t)}$ without repetitions for each $t \in {}^{<\omega}\omega$.

$[\phi](x) := \bigcup_{t \subsetneq x} \phi(t)$ for all $x \in {}^\omega\omega$. Then $g := f \circ [\phi]$ is a continuous map from ${}^\omega\omega$ to X with $g(N_t) \in \mathcal{I}_\mathcal{F}^+$ for all $t \in {}^{<\omega}\omega$. □

Example: the κ-perfect set property

Let $\mathsf{CCP}_\kappa(X,\mathcal{F})$ and $\mathsf{CCP}^{\mathsf{C}}_\kappa(X,\mathcal{F})$ denote the versions for a single family $\mathcal{F}$ of closed sets.

If $\mathcal{F}$ is the family of singletons, $\mathsf{CCP}_\kappa(X,\mathcal{F})$ is equivalent to the κ-perfect set property.

If $V = L$, then

- $\mathrm{PSP}_\kappa(\mathbf{\Sigma}^1_1)$ fails for all $\kappa = \kappa^{<\kappa} > \omega$ (Friedman, Hyttinen, Kulikov).
- $\mathrm{PSP}_\kappa(\mathsf{C}_\kappa)$ fails for $\kappa = \omega_2$ and the class C_κ of continuous images of ${}^\kappa\kappa$ (Lücke, Schlicht).

So $\mathsf{CCP}_\kappa(X)$ does not follow from either $\mathsf{CCP}^{\mathbf{\Sigma}^1_1}_\kappa(X)$ or $\mathsf{CCP}^{\mathsf{C}_\kappa}_\kappa(X)$.

Example: the asymmetric κ-Baire property

$X \subseteq {}^\kappa\kappa$ has the κ-Baire property if there is an open set $U \subseteq {}^\kappa\kappa$ such that $X \triangle U$ is κ-meager (i.e. the union of κ-many nowhere dense sets).

Theorem (Halko, Shelah)

The κ-Baire property holds for κ-Borel sets, but it fails for κ-analytic sets (for the club filter) when $\kappa > \omega$.

Definition (Schlicht). X has the asymmetric κ-Baire property if the the Banach-Mazur game of length κ for X is determined.

In this game, players $\mathbf{I}$ and $\mathbf{II}$ play a strictly increasing sequence $\langle s_\alpha : \alpha < \kappa \rangle$ in ${}^{<\kappa}\kappa$. $\mathbf{I}$ plays in all even rounds (including limit rounds). $\mathbf{I}$ wins if $\bigcup_{\alpha<\kappa} s_\alpha \in X$.

Example: the asymmetric κ-Baire property

Proposition

If $\mathcal{F}$ is the family of nowhere dense sets, then

- $\mathsf{CCP}_\kappa(X, \mathcal{F})$ *is equivalent to the asymmetric κ-Baire property.*
- $\mathsf{CCP}_\kappa^{\mathsf{Borel}_\kappa}(X, \mathcal{F})$ *implies the κ-Baire property.*

So $\mathsf{CCP}_\kappa(X)$ does not imply $\mathsf{CCP}_\kappa^{\mathbf{\Pi}^0_2}(X)$ or even $\mathsf{CCP}_\kappa^{\mathsf{Borel}_\kappa}(X)$.

The Transitive Closed Hypergraph Dichotomy

- $\mathbb{K}^d_X := {}^dX \setminus \{\text{constant sequences}\}$ is the complete d-hypergraph on X.
- H is box-closed if its complement $H^{\mathsf{c}} := \mathbb{K}^d_X \setminus H$ is box-open.
- A d-hypergraph H is a d-dihypergraph which is closed under permutations of hyperedges (i.e. $\langle x_{\pi(i)} : i < d\rangle \in H$ for all $\pi \in Sym(d)$ and $\langle x_i : i < d\rangle \in H$).
- H is transitive if all of its vertical sections
$$H_{\langle x_1,\dots x_i,\dots\rangle} := \{x \in X : \langle x, x_1, \dots x_i, \dots\rangle \in H\} \text{ are } H\text{-cliques.}$$
- H is weakly transitive if all of its vertical sections are unions of κ-many H-cliques.

$\mathsf{TCHD}^d_\kappa(X)$ states that for any box-closed weakly transitive d-hypergraph H on X, either

- X is a union of κ-many H-cliques,
- or there exists a κ-perfect H-independent set.

The Transitive Closed Hypergraph Dichotomy

Theorem (He)

$\mathsf{TCHD}^d_\omega(X)$ *holds for all analytic subsets* X *of* ${}^\omega\omega$ *and all* $d < \omega$*.*

Theorem

Let $d < \kappa$*. Suppose* $\diamondsuit_\kappa$ *or* κ *is inaccessible or* $\kappa = \omega$*. For any* $X \subseteq {}^\kappa\kappa$*:*

- $\mathsf{ODD}^d_\kappa(X) \Longrightarrow \mathsf{TCHD}^d_\kappa(X)$,
- $\mathsf{ODD}^d_\kappa(X, \mathsf{Def}_\kappa) \Longrightarrow \mathsf{TCHD}^d_\kappa(X, \mathsf{Def}_\kappa)$.

Hence $\mathsf{TCHD}^d_\kappa(X)$ holds for all definable sets X after a Lévy-collapse of a Mahlo cardinal to κ^+, and an inaccessible suffices for the restriction $\mathsf{TCHD}^d_\kappa(X, \mathsf{Def}_\kappa)$ to definable dihypergraphs.

The Transitive Closed Hypergraph Dichotomy

Theorem (He)

$\mathsf{TCHD}^d_\omega(X)$ *holds for all analytic subsets* X *of* ${}^\omega\omega$ *and all* $d < \omega$.

Theorem

Let $d < \kappa$. *Suppose* $\diamondsuit^i_{\kappa,d}$. *For any* $X \subseteq {}^\kappa\kappa$:

- $\mathsf{ODD}^d_\kappa(X) \Longrightarrow \mathsf{TCHD}^d_\kappa(X)$,
- $\mathsf{ODD}^d_\kappa(X, \mathsf{Def}_\kappa) \Longrightarrow \mathsf{TCHD}^d_\kappa(X, \mathsf{Def}_\kappa)$.

$\diamondsuit^i_{\kappa,d}$: There exists a sequence $\langle A_\alpha \subseteq {}^\alpha d : \alpha < \kappa \rangle$ such that $|A_\alpha| < \kappa$ for all $\alpha < \kappa$ and $\{\alpha < \kappa : x{\restriction}\alpha \in A_\alpha\}$ is cofinal in κ for all $x \in {}^\kappa d$.

From ODD to TCHD

Proof (for inaccessible cardinals and ω).

Suppose H is a box-closed weakly transitive d-hypergraph on X.

By $\mathsf{ODD}^d_\kappa(X)$, we can assume there exists a continuous homomorphism f from $\mathbb{H}_{^\kappa d}$ to H^c. Construct a continuous order preserving map $\phi : {}^{<\kappa}d \to {}^{<\kappa}\kappa$ with

$\prod_{i<d} N_{\phi(t_i)} \subseteq H^\mathsf{c}$ for all $\alpha < \kappa$ and all non-constant sequences $\langle t_i : i < d\rangle$ in ${}^\alpha d$.

At successor stages, use box-openness and the following lemma repeatedly.

Lemma. H^c is a dense subset of ${}^d(\mathrm{ran}(f))$.

Proof. Let U_i be an open subset of $\mathrm{ran}(f)$ for all $i < d$. Take any $\overline{x} = \langle x_1, \ldots, x_i, \ldots\rangle$ in $\prod_{1\le i<d} U_i$. It suffices to show $H^\mathsf{c}_{\overline{x}} \cap U_0 \neq \emptyset$. Otherwise $U_0 \subseteq H_{\overline{x}}$ and hence $H^\mathsf{c}\restriction U_0$ is κ-colorable by weak transitivity. But such a coloring can be pulled back to a κ-coloring of $\mathbb{H}_{^\kappa d}$, which cannot exist. □

From ODD to TCHD

Proof (for inaccessible cardinals and ω).

Suppose H is a box-closed weakly transitive d-hypergraph on X.

By $\mathsf{ODD}^d_\kappa(X)$, we can assume there exists a continuous homomorphism f from $\mathbb{H}_{\kappa_d}$ to H^{c}. Construct a continuous order preserving map $\phi : {}^{<\kappa}d \to {}^{<\kappa}\kappa$ with

$\prod_{i<d} N_{\phi(t_i)} \subseteq H^{\mathsf{c}}$ for all $\alpha < \kappa$ and all non-constant sequences $\langle t_i : i < d \rangle$ in ${}^{\alpha}d$.

At successor stages, use box-openness and the following lemma repeatedly.

Lemma. H^{c} is a dense subset of ${}^{d}(\mathrm{ran}(f))$.

The set of all $[\phi(x)] := \bigcup_{\alpha<\kappa} \phi(x{\restriction}\alpha)$, where $x \in {}^{\alpha}d$, forms a κ-perfect H-independent set. □

From ODD to TCHD

Proof (from $\lozenge^i_{\kappa,d}$).

$\lozenge^i_{\kappa,d}$ is equivalent to the following d-dimensional version:

There exists a sequence $\langle B_\alpha \subseteq \mathbb{K}^d_{\alpha\, d} : \alpha < \kappa \rangle$ such that

- $|B_\alpha| < \kappa$ for all $\alpha < \kappa$,
- for all $\langle x_0, \dots, x_i, \dots \rangle \in \mathbb{K}^d_{\kappa\, d}$ the set $\{\alpha < \kappa : \langle x_0 \restriction \alpha, \dots, x_i \restriction \alpha, \dots \rangle \in A_\alpha\}$ is cofinal in κ.

Construct a continuous order preserving map $\phi : {}^{<\kappa}d \to {}^{<\kappa}\kappa$ with $\prod_{i<d} N_{\phi(t_i)} \subseteq H^{\mathsf{c}}$ for all $\alpha < \kappa$ and all $\langle t_i : i < d \rangle \in B_\alpha$.

□

Aims:

- Characterize ODD via games of length κ.
- Determinacy of very general classes of games of length κ.

Feng's games

Feng characterized the open graph dichotomy for sets of reals via a game of length ω. We lift this for $<\kappa$-dimensional dihypergraphs on ${}^{\kappa}\kappa$.

Feng's games

Suppose H is a box-open d-dihypergraph on ${}^{\kappa}\kappa$, where $2 \leq d \leq \kappa$, and $X \subseteq {}^{\kappa}\kappa$.

$\mathcal{F}_{\kappa}(X, H)$ is the following game of length κ:

I	$\langle t_i^0 : i < d \rangle$		$\dots$		$\langle t_i^\alpha : i < d \rangle$		$\dots$
II		i_0		$\dots$		i_α		$\dots$

where $t_i^\alpha \in {}^{<\kappa}\kappa$, $\prod_{i<d} N_{t_i^\alpha} \subseteq H$, $i_\alpha < d$ and $t_{i_\beta}^\beta \subseteq t_i^\alpha$ for all $\beta < \alpha, i < d$.
I wins if $\bigcup_{\alpha<\kappa} t_{i_\alpha}^\alpha \in X$.

ODD_κ^I denotes the restriction of $\mathsf{ODD}_\kappa^d(X)$ to a single d-dihypergraph I on X.

Theorem

- $\mathsf{ODD}_\kappa^{H \restriction X} \implies \mathcal{F}_\kappa(X, H)$ *is determined.*
- *If* $d < \kappa$, *then* $\mathsf{ODD}_\kappa^{H \restriction X} \iff \mathcal{F}_\kappa(X, H)$ *is determined.*

Proof sketch. Winning strategies for $\mathbf{I}$ correspond in a straightforward way to continuous homomorphisms from $\mathbb{H}_{\kappa_\kappa}$ to $H \restriction X$.

If $H \restriction X$ has a κ-coloring $X := \bigcup_{\alpha<\kappa} X_\alpha$, then $\mathbf{II}$ wins by making sure that the α^{th} color is avoided in round α (i.e., $N_{t^\alpha_{i_\alpha}} \cap X_\alpha = \emptyset$).

Now, suppose σ is a winning strategy for $\mathbf{II}$. Let Run_σ denote the set of those positions $p := \langle t_\alpha, r_\alpha : \alpha \leq \beta \rangle$ which follow σ.

A position $p \in Run_\sigma$ is good for $x \in X$ if $\bigcup_{\alpha<\beta} t^\alpha_{i_\alpha} \subseteq x$.

$$X_p := \{x \in X : p \text{ is maximal good for } x\}.$$

Claim. $X = \bigcup_{p \in Run_\sigma} X_p$ and each X_p is H-independent.

If $d < \kappa$, then $|Run_\sigma| = \kappa$, so $H \restriction X$ has a κ-coloring.

□

$\mathrm{ODD}_\kappa^\kappa$ via games

Carroy-Miller-Soukup characterized $\mathrm{ODD}_\omega^\omega(X)$ for subsets X of ${}^\omega\omega$ via a slowed down version of Feng's games. We lift this to the uncountable setting.

Suppose H is a κ-dihypergraph on ${}^\kappa\kappa$ and $X \subseteq {}^\kappa\kappa$.

$\mathcal{G}_\kappa(X, H)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		i_0		i_1		$\dots$		i_α		$\dots$

where $t_\alpha \in {}^{<\kappa}\kappa$, $i_\alpha < 2$, and $t_\beta \subseteq t_\alpha$ for all $\beta < \alpha$ with $i_\beta = 1$.

Let $\mathrm{supp}_\kappa := \{\alpha < \kappa : i_\alpha = 1\}$.

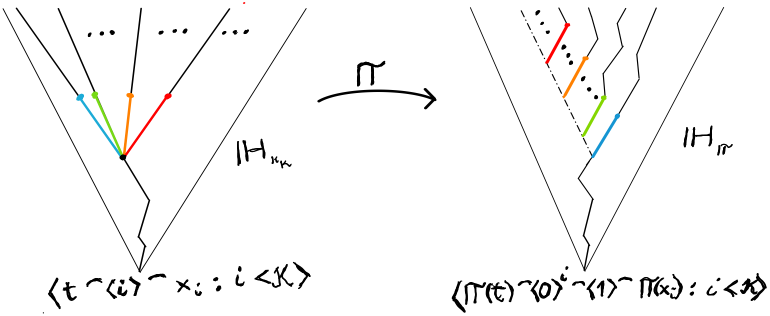
- If $|\mathrm{supp}_\kappa| = \kappa$, then **I** wins if $x := \bigcup_{\alpha \in \mathrm{supp}_\kappa} t_\alpha$ is in X
- If $|\mathrm{supp}_\kappa| < \kappa$, then **I** wins if $\prod_{\alpha<\kappa} N_{t_{m+\alpha}} \subseteq H$ where m is the least ordinal with $i_\beta = 0$ for all $\beta \geq m$.

$\mathsf{ODD}_\kappa^\kappa$ via games

Theorem. For all box-open κ-dihypergraphs H on ${}^\kappa\kappa$ and all $X \subseteq {}^\kappa\kappa$,

$$\mathsf{ODD}_\kappa^{H\restriction X} \iff \mathcal{G}_\kappa(X, H) \text{ is determined.}$$

Proof idea. Let $\pi(x) := \bigoplus_{\alpha<\kappa} \langle 0\rangle^{x(\alpha)} {}^\frown \langle 1\rangle$ for all $x \in {}^\kappa\kappa$. Let $\mathbb{H}_\pi := \pi^d(\mathbb{H}_{{}^\kappa\kappa})$.



Winning strategies for $\mathbf{I}$ correspond to continuous homomorphisms from $\mathbb{H}_\pi$ to $H\restriction X$. The proof for $\mathbf{II}$ uses a similar idea as the previous proof. $\square$

Kechris's games

Kechris introduced a general class of games of length ω which encompasses many of the classical games characterizing dichotomies for subsets of ${}^{\omega}\omega$. We consider the versions of length κ for subsets of the κ-Baire space.

Kechris's games

Let upw$({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $F : R \to \text{upw}({}^{<\kappa}d)$.
$\mathcal{K}_\kappa(X, F)$ is the following game of length κ:

$$\begin{array}{lcccccccccc} \mathbf{I} & t_0 & & t_1 & & \dots & & t_\alpha & & \dots & \\ \mathbf{II} & & r_0 & & r_1 & & \dots & & r_\alpha & & \dots \end{array}$$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in F(r_\alpha)$ for all $\alpha < \kappa$.

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Example

- $d := 2$, $R := 2$ and $F(r) := \{t \in {}^{<\kappa}2 : t(0) = r\}$ characterizes the κ-perfect set property.

Kechris's games

Let upw$({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $F : R \to \text{upw}({}^{<\kappa}d)$.
$\mathcal{K}_\kappa(X, F)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in F(r_\alpha)$ for all $\alpha < \kappa$.

Example

- $d := \kappa$, $R := {}^{<\kappa}\kappa$ and $F(r) := \{t \in {}^{<\kappa}\kappa : r \subseteq t\}$ is the Banach-Mazur game (for the assymmetric κ-Baire property).

Kechris's games

Let upw$({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $F : R \to \text{upw}({}^{<\kappa}d)$.
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where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in F(r_\alpha)$ for all $\alpha < \kappa$.

Example

- $d := \kappa$, $R := \kappa$ and $F(r) := \{t \in {}^{<\kappa}\kappa \mid t(0) \geq r\}$ characterizes a variant of the Hurewicz dichotomy.

Kechris's games

Let $\mathrm{upw}({}^{<\kappa}d)$ denote the set of upwards closed subsets of ${}^{<\kappa}d$. Let $X \subseteq {}^{\kappa}d$.

Let R be a nonempty set (requirements) and $F : R \to \mathrm{upw}({}^{<\kappa}d)$.
$\mathcal{K}_\kappa(X, F)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}d$ and $r_\alpha \in R$. **I** wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$ and $t_{\alpha+1} \in F(r_\alpha)$ for all $\alpha < \kappa$.

Theorem

$\mathsf{ODD}^\kappa_\kappa(X, \mathsf{Def}_\kappa)$ *implies that* $\mathcal{K}_\kappa(X, F)$ *is determined for all nonempty sets* R *of size* $\leq \kappa$ *and all nontrivial* $F : R \to \mathrm{upw}({}^{<\kappa}d)$.

F is nontrivial if for all $i \in d$, there exists $r \in R$ such that $t(0) \neq i$ for all $t \in F(r)$.

Generalizing Kechris's and Feng's games

Suppose $R \subseteq \mathcal{P}(\kappa)$, F is a function with domain $R \times {}^{<\kappa}\kappa$ with $F(r,t) \in \mathrm{upw}({}^{r}({}^{<\kappa}\kappa))$ for all r, t. $\mathcal{F}_\kappa(X, F)$ is the following game of length κ:

I	t_0		$\langle t^1_i : i \in r_0\rangle$		$\dots$	t_ω		$\langle t^{\omega+1}_i : i \in r_\omega\rangle$		$\dots$
II		r_0		i_1, r_1	$\dots$		r_ω		$i_{\omega+1}, r_{\omega+1}$	$\dots$

In successor rounds $\alpha+1$, **I** plays $t^{\alpha+1}_i \in {}^{<\kappa}\kappa$ for all $i \in r_\alpha$ which extend $t_\beta := t^{\beta}_{i_\beta}$ for all successor ordinals $\beta \leq \alpha$, so that $\langle t^{\alpha+1}_i : i < r_\alpha\rangle \in F(r_\alpha, \bigoplus_{\beta<\alpha} t_\beta)$. **II** plays $i_{\alpha+1} < r_\alpha$ and $r_{\alpha+1} \in R$. In round 0 and limit rounds, **I** plays $t_\alpha \in {}^{<\kappa}\kappa$ extending t_β for all $\beta < \alpha$ and **II** plays $r_\alpha \in R$.

I wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$.

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I	t_0		$\langle t_i^1 : i \in r_0 \rangle$		$\dots$	t_ω		$\langle t_i^{\omega+1} : i \in r_\omega \rangle$	$\dots$
II		r_0		i_1, r_1	$\dots$		r_ω	$i_{\omega+1}, r_{\omega+1}$	$\dots$

In successor rounds $\alpha+1$, **I** plays $t_i^{\alpha+1} \in {}^{<\kappa}\kappa$ for all $i \in r_\alpha$ which extend $t_\beta := t_{i_\beta}^{\beta}$ for all successor ordinals $\beta \leq \alpha$, so that $\langle t_i^{\alpha+1} : i < r_\alpha \rangle \in F(r_\alpha, \bigoplus_{\beta<\alpha} t_\beta)$. **II** plays $i_{\alpha+1} < r_\alpha$ and $r_{\alpha+1} \in R$. In round 0 and limit rounds, **I** plays $t_\alpha \in {}^{<\kappa}\kappa$ extending t_β for all $\beta < \alpha$ and **II** plays $r_\alpha \in R$.

I wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$.

Example

- To obtain Kechris's game for $F' : \kappa \to \mathrm{upw}({}^{<\kappa}\kappa)$, let $R := \{\{\alpha\} : \alpha < \kappa\}$, $F(\{\alpha\}, t) := F'(\alpha)$ for all $t \in {}^{<\kappa}\kappa$

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II		r_0		i_1, r_1	$\dots$	r_ω		$i_{\omega+1}, r_{\omega+1}$	$\dots$

In successor rounds $\alpha + 1$, **I** plays $t_i^{\alpha+1} \in {}^{<\kappa}\kappa$ for all $i \in r_\alpha$ which extend $t_\beta := t_{i_\beta}^{\beta}$ for all successor ordinals $\beta \leq \alpha$, so that $\langle t_i^{\alpha+1} : i < r_\alpha \rangle \in F(r_\alpha, \bigoplus_{\beta<\alpha} t_\beta)$. **II** plays $i_{\alpha+1} < r_\alpha$ and $r_{\alpha+1} \in R$. In round 0 and limit rounds, **I** plays $t_\alpha \in {}^{<\kappa}\kappa$ extending t_β for all $\beta < \alpha$ and **II** plays $r_\alpha \in R$.

I wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$.

Example

- To obtain Feng's game for a d-dihypergraph H, let $R := \{d\}$ and $F(d,t) := \{\langle t_i : i < d \rangle : \prod_{i<d} N_{t^\frown t_i} \subseteq H\}$.

Generalizing Kechris's and Feng's games

Suppose $R \subseteq \mathcal{P}(\kappa)$, F is a function with domain $R \times {}^{<\kappa}\kappa$ with $F(r,t) \in \mathsf{upw}({}^{r}({}^{<\kappa}\kappa))$ for all r, t. $\mathcal{F}_\kappa(X, F)$ is the following game of length κ:

I	t_0		$\langle t^1_i : i \in r_0 \rangle$		$\dots$	t_ω		$\langle t^{\omega+1}_i : i \in r_\omega \rangle$	$\dots$
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I wins if $\bigoplus_{\alpha<\kappa} t_\alpha \in X$.

Theorem (?). Suppose R consists of pairwise disjoint subsets. Then

$$\mathsf{ODD}^\kappa_\kappa(X) \implies \mathcal{F}_\kappa(X, F) \text{ is determined.}$$

Suppose $s \notin \kappa$, $R \subseteq \mathcal{P}(\kappa)$ and F is a function with domain $R \times {}^{<\kappa}\kappa$ with $F(r,t) \in \mathsf{upw}({}^{r}({}^{<\kappa}\kappa))$ for all r, t. $\mathcal{G}_\kappa(X, F)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}\kappa$ and $r_\alpha \in R \cup \{s\}$. **II** has to play $r_\alpha \in R$ if the order type of $\mathsf{supp}_\alpha := \{\beta < \alpha : i_\beta \in R\}$ is a limit ordinal.

- If $|\mathsf{supp}_\kappa| = \kappa$, then **I** wins if $x := \bigoplus_{\alpha \in \mathsf{supp}_\kappa} t_\alpha$ is in X
- If $|\mathsf{supp}_\kappa| < \kappa$, then **I** wins if $\langle t_{m+\alpha} : \alpha \in r_m \rangle \in F(r_{m-1}, t_{m-1})$ where m is the least ordinal with $r_\beta = s$ for all $\beta \geq m$.

Generalizing Kechris's and the CMS games

Suppose $s \notin \kappa$, $R \subseteq \mathcal{P}(\kappa)$ and F is a function with domain $R \times {}^{<\kappa}\kappa$ with $F(r,t) \in \text{upw}({}^{r}({}^{<\kappa}\kappa))$ for all r, t. $\mathcal{G}_\kappa(X, F)$ is the following game of length κ:

I	t_0		t_1		$\dots$		t_α		$\dots$	
II		r_0		r_1		$\dots$		r_α		$\dots$

where $t_\alpha \in {}^{<\kappa}\kappa$ and $r_\alpha \in R \cup \{s\}$. **II** has to play $r_\alpha \in R$ if the order type of $\text{supp}_\alpha := \{\beta < \alpha : i_\beta \in R\}$ is a limit ordinal.

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Example

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Generalizing Kechris's and the CMS games

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Example

- To obtain the Carroy-Miller-Soukup game for a d-dihypergraph H, let $R := \{d\}$ and $F(d,t) := \{\langle t_i : i < d \rangle : \prod_{i<d} N_{t^\frown t_i} \subseteq H\}$.

Generalizing Kechris's and the CMS games

Suppose $s \notin \kappa$, $R \subseteq \mathcal{P}(\kappa)$ and F is a function with domain $R \times {}^{<\kappa}\kappa$ with $F(r,t) \in \mathsf{upw}({}^{r}({}^{<\kappa}\kappa))$ for all r, t. $\mathcal{G}_\kappa(X, F)$ is the following game of length κ:

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Conjecture. Suppose R consists of pairwise disjoint subsets. Then

$$\mathsf{ODD}^\kappa_\kappa(X) \implies \mathcal{G}_\kappa(X, F) \text{ is determined.}$$

Thank you!