2^κ, κ regular, $\kappa^{<\kappa} = \kappa$

Lipschitz game $A, B \subseteq 2^\kappa$ $G_L(A,B)$

I i_0 . . . x

II i_1 y

0/1 length κ

II $x \in A \iff y \in B$

SLO_L $A \leq_L B$ or $B \leq_L \neg A$

Wadge game

the same, except player II can pass

SLO_W $A \leq_W B$ $B \leq_W A$

continuous reductions.

$AD \Rightarrow SLO_L(2^\omega) \Rightarrow SLO_W(2^\omega)$

Thm (Martin Monk) SLO implies that

the Wadge hierarchy on Borel sets is wellfounded.

Q Is the Wadge hierarchy on the κ-Borel subsets of 2^κ well-founded?

SLO_L implies that every non-self-dual Wadge class has a universal set

Q Does every nonselfdual Wadge class of Borel sets admit a universal set?

dense in Y

$$Y_0 = \{x \in Y \mid x(i)=1 \text{ for an even number of } i\text{'s}\}$$

$$Y_1 = \{$$

$$Y = \{x \in 2^\kappa \mid x(i)=1 \text{ for only finitely many } i\text{'s}\}$$

closed set of size κ

κ^+

Diff(closed)

$$N_t = \{x \supseteq t\}$$

$$t \in 2^{<\kappa}$$

Prop. $Y_0 \notin \mathrm{Diff}(\text{closed})$

$Y_0 \neq C \setminus D$ $\quad C, D \subseteq Y$

clod clod

Prop. If $A \in D_\xi(\text{clod})$, then $A \leq_W Y_0$.

need a strategy for I

clods

Proof sketch:

$\xi = 1$ $\quad$ A is a closed.

$$T_A = \{ t \in 2^{<\omega} \mid \exists x \in A \; t \subseteq x \}.$$

As long as I plays in T_A, play $y \upharpoonright \alpha$ for a fixed $y \in Y_0$.

When first I plays outside of T_A, choose some $z \in A$ with $z \supseteq y \upharpoonright \alpha$

Prop. If $A \leq Y_0$, then either $A \in \mathrm{Diff}$ (clopen) or $Y_0 \leq A$.

Proof idea

Kuratowski-Hausdorff derivation relative to a set A:

$C_0 = \partial A$ $\qquad$ $C_0 \supseteq C_1 \supseteq C_2$ closed

$C_1 = \partial(A \cap C_0)^{C_0}$

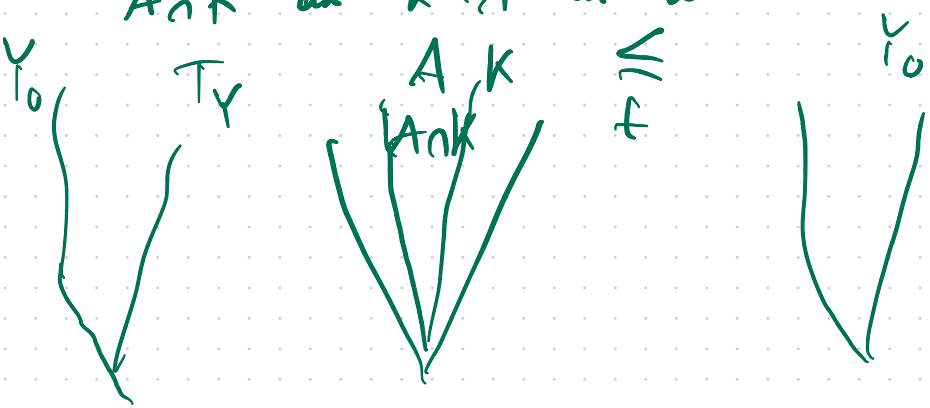
intersection at limits.

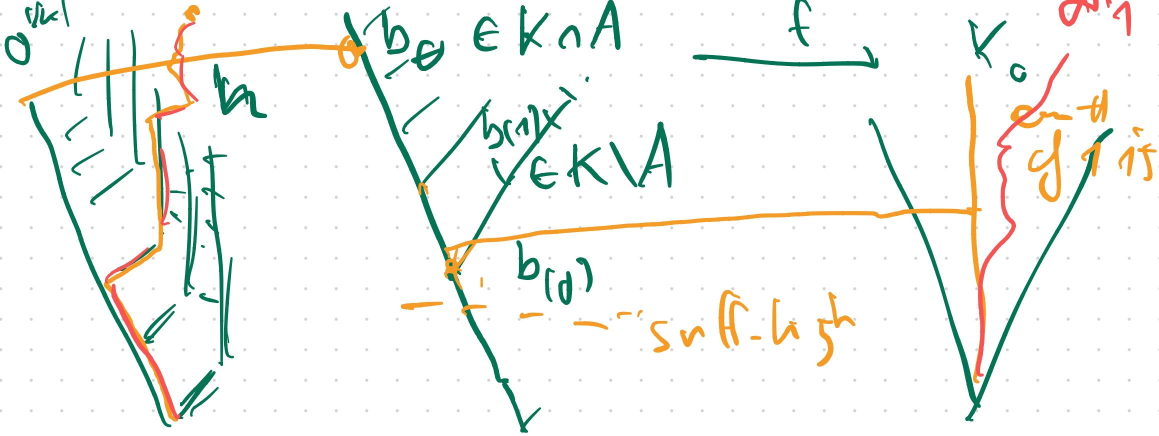
stabilize at a closed set K.

case $K = \emptyset$ scattered $\to$ A in the diff hier.

case $K \neq \emptyset$.

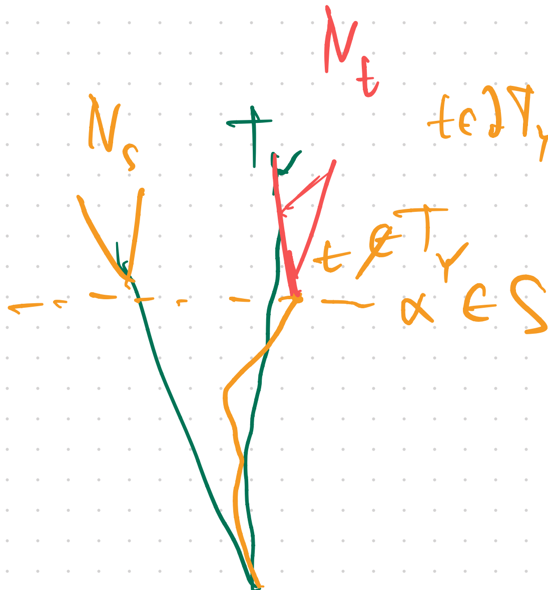
$A \cap K$ and $K \setminus A$ are dense in K.





$Y_0 \quad Y_S \quad Y_{S'}$

$\mathrm{Diff}(c[\omega])$



$Y_S := Y_0 \cup$

$\{ t^\frown x \mid t$ has a cofinal wo seq. of 1's and $|t| \in S \}$

S stationary co-stationary.

$S, S' \subseteq \kappa$

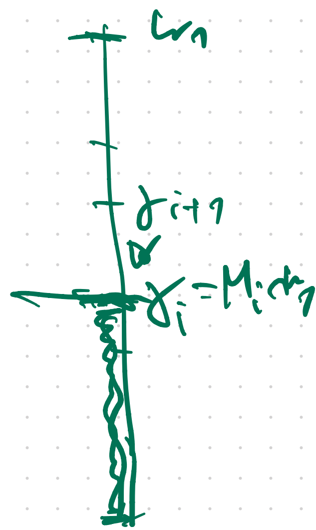
Prop. If $(S \triangle S') \cap \mathrm{cof}^{\kappa}_{\omega}$ stationary, then $Y_S, Y_{S'}$ are $\leq_W$-incomparable.

$\kappa = \omega_1$

Proof sketch Suppose $Y_S \overset{f}{\leq} Y_{S'}$

$f, S, S' \in M_0 \prec M_1 \prec M_2$ countable

$(M_i)_{i<\kappa}$



D

Pick some $\gamma_i \in S \setminus S'$.

Put $(\gamma_i^j)_{j\in\omega}$ cofinal in γ_i.

We construct a sequence $(x_n)_{n\in\omega}$

in $2^{\omega_1} \cap M_i$

satisfying other seq. $(\alpha_n), (\beta_n)$

in $M \cap \omega_1$ st.

1. $x_{n+1}(\alpha) = 1$ at arbitrary places in $[\alpha_n, \alpha_{n+1}]$

2. $f(x_{n+1})(\beta) = 1$ for at least one and at most finitely many $\beta \in [\beta_n, \beta_{n+1}]$

3. $x_n \restriction \alpha_n \subseteq x_{n+1}$

4. $f(x) \restriction \beta_n = f(x_n) \restriction \beta$ if $x \supseteq x_n \restriction \alpha_n$

5. $\alpha_n, \beta_n \geq \gamma_i^n$

$SLO_L(\mathrm{clop}(2^{\omega_1})) \Rightarrow \underline{SLO(2^{\omega})}$

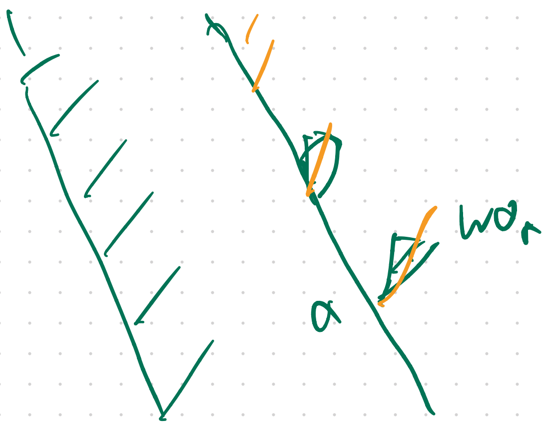
with ZF

$\underline{Q}$ (Coloring) is $SLO_L(2^{\omega_1})$ consist ?

If $SLO_L(2^{\omega_1})$ holds, then $SLO(2^{\omega})$

$\Rightarrow \underline{PSP(2^{\omega})}$

<u>Lemma</u> $\underline{SLO_W}(\mathrm{clopen}(2^{\omega_1})) \Rightarrow$ exists an $\underline{\omega_1\text{-seq. of reals.}}$



<u>Prop.</u> Suppose ZF holds and $\kappa = \mu^+$ is regular. Then $SLO_W(2^{\kappa})$ fails for Δ^0_2 subsets of 2^{κ}.