Wadge classes in Generalized Descriptive Set Theory

at the second level of the Borel hierarchy

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Our setup

In this talk, we work in ZFC.

Our setup

Let κ be an uncountable cardinal that satisfies the condition $\kappa^{<\kappa} = \kappa$.

We consider the **Generalized Cantor space** $({}^{\kappa}2, \tau_b)$, where τ_b is the *bounded topology*, i.e. the topology generated by the sets

$$N_s({}^{\kappa}2) := \{ x \in {}^{\kappa}2 \mid s \subseteq x \}, \qquad s \in {}^{<\kappa}2.$$

We denote by $\mathbf{Bor}(\kappa^+)$ the κ^+-algebra generated by τ_b.

We recall that the $\mathbf{Bor}(\kappa^+)$-sets admit a stratification into $\boldsymbol{\Sigma}^0_\xi(\kappa^+), \boldsymbol{\Pi}^0_\xi(\kappa^+)$ and $\boldsymbol{\Delta}^0_\xi(\kappa^+)$ sets, $1 \leq \xi < \kappa^+$.

Wadge (or continuous) reducibility

Definition

Let X, Y be topological spaces. Given $A \subseteq X$, $B \subseteq Y$

$$A \leq_{\mathrm{W}}^{X,Y} B$$

if there exists a continuous $f : X \to Y$ such that $f^{-1}(B) = A$.

- Continuous reducibility is a transitive and reflexive relation, that is, a preorder (or quasi-order).
- We set:
  - $A <_{\mathrm{W}}^{X} B$ iff $A \leq_{\mathrm{W}}^{X} B$ and $B \not\leq_{\mathrm{W}}^{X} A$.
  - $A \equiv_{\mathrm{W}}^{X} B$ iff $A \leq_{\mathrm{W}}^{X} B$ and $B \leq_{\mathrm{W}}^{X} A$.

  The equivalence classes induced by $\leq_{\mathrm{W}}^{X}$ are called Wadge degrees:

  $$[A]_{\mathrm{W}}^{X} = \{B \mid A \equiv_{\mathrm{W}}^{X} B\}$$

- The preorder $\leq_{\mathrm{W}}^{X}$ induces a partial order on $\mathcal{P}(X)/_{\equiv_{\mathrm{W}}^{X}}$ we call this partial order the **Wadge hierarchy** on X.

Wadge Hierarchy on ${}^{\omega}2$

Wadge's Lemma

For all $A, B \in \mathbf{Bor}({}^{\omega}2)$,

$$A \leq_{\mathrm{W}} B \quad \text{or} \quad \neg B = {}^{\omega}2 \setminus B \leq_{\mathrm{W}} A.$$

We call this the **Wadge Semi-Linear Ordering principle** for $\leq_W$ (SLO). Given $\boldsymbol{\Gamma}$, we write SLO($\boldsymbol{\Gamma}$) if SLO holds for any $A, B \in \boldsymbol{\Gamma}$.

By SLO, for every $A \subseteq {}^{\omega}2$:

- either $A \leq_{\mathrm{W}} \neg A$, then A (or $[A]_{\mathrm{W}}$) is **selfdual**,
- or A and $\neg A$ are $\leq_{\mathrm{W}}$-incomparable, then A is called **non-selfdual** and $\{[A]_{\mathrm{W}}, [\neg A]_{\mathrm{W}}\}$ is a maximal antichain, called a non-selfdual pair.

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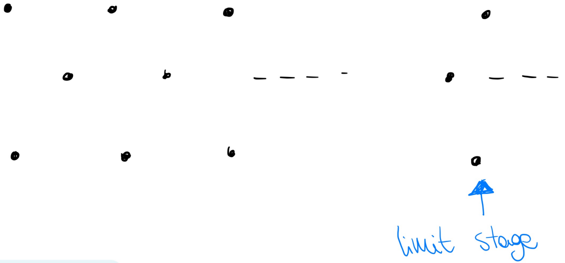
Recall: there is no κ^+-Borel determinacy when $\kappa > \omega$!

Wadge Hierarchy on ${}^{\omega}2$

The Wadge hierachy on Borel subsets of ${}^{\omega}2$ has the following properties:

- it is semi-well-ordered;
- selfdual degrees and nonselfdual pairs alternate, starting with the nonselfdual pair $\{[{}^{\omega}2]_W, [\emptyset]_W\}$ at the bottom, followed by the selfdual degree of all nontrivial clopen subsets;
- at limit levels there is a nonselfdual pair.



Boldface and Wadge pointclasses

Let X topological space. A **boldface pointclass** $\boldsymbol{\Gamma}$ in is a subset of $\mathcal{P}(X)$ downward closed under $\leq_W^X$. The dual of $\boldsymbol{\Gamma}$ is $\check{\boldsymbol{\Gamma}} = \{\neg A \mid A \in \boldsymbol{\Gamma}\}$.
We say that $\boldsymbol{\Gamma}$ is **nonselfdual** if $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$, and **selfdual** otherwise.

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A boldface pointclass $\boldsymbol{\Gamma}(X)$ is a **Wadge class** if it is of the form

$$A \downarrow_X = \{B \subseteq X \mid B \leq_{\mathrm{W}} A\}$$

for some $A \in \boldsymbol{\Gamma}(X)$; When $\boldsymbol{\Gamma}(X) = A \downarrow_X$, A is called **complete** (for $\boldsymbol{\Gamma}$), and we say that $\boldsymbol{\Gamma}$ is generated by A.

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By SLO, every non-selfdual boldface pointclass $\boldsymbol{\Gamma}(X)$ is a Wadge class:

Let $\boldsymbol{\Gamma}$ be a non-selfdual boldface pointclass. If SLO($\boldsymbol{\Gamma}$) holds, then

$$A \text{ is } \boldsymbol{\Gamma}\text{-complete} \iff A \in \boldsymbol{\Gamma}(X) \setminus \check{\boldsymbol{\Gamma}}(X).$$

N.B not all selfdual boldface pointclasses in X are Wadge classes.

Wadge pointclasses

The Wadge hierarchy is isomorphic to the structure of all Wadge classes ordered by inclusion: $[A]_{\mathrm{W}} \mapsto A\downarrow$.
In particular:

- If $\boldsymbol{\Gamma} = A\downarrow_X$ then $\boldsymbol{\Gamma}$ is selfdual if and only if A (or $[A]_{\mathrm{W}}$) is selfdual.
- A nonselfdual pair $\{[A]_{\mathrm{W}}, [\neg A]_{\mathrm{W}}\}$ corresponds to the pair of distinct nonselfdual Wadge classes $(\boldsymbol{\Gamma}, \check{\boldsymbol{\Gamma}})$, with $\boldsymbol{\Gamma} = A\downarrow_X$.
- If SLO($\boldsymbol{\Gamma}$) holds, then for every $\boldsymbol{\Gamma}', \boldsymbol{\Gamma}'' \subseteq \boldsymbol{\Gamma}$,

$$\boldsymbol{\Gamma}' \subseteq \boldsymbol{\Gamma}'' \text{ or } \check{\boldsymbol{\Gamma}}'' \subseteq \boldsymbol{\Gamma}'$$

- For every Wadge class $\boldsymbol{\Gamma}$, we can consider the coarse Wadge class $\boldsymbol{\Gamma}^* = \boldsymbol{\Gamma} \cup \check{\boldsymbol{\Gamma}}$. If SLO holds and $\leq_{\mathrm{W}}^X$ is well founded, then $\subseteq$ on coarse Wadge classes is a well-order.

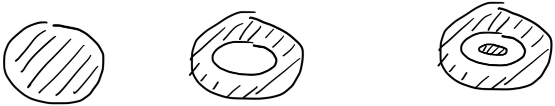
Difference Hierarchy

Definition

Let $\theta \geq 1$ be an ordinal. If $(C_\eta)_{\eta<\theta}$ is a decreasing sequence of subsets of a set X, we define $C = D_\theta\left((C_\eta)_{\eta<\theta}\right) \subseteq X$ by

$$x \in C \iff \begin{cases} x \in \bigcap_{\eta<\theta} C_\eta \vee \min\{\eta < \theta \mid x \notin C_\eta\} \text{ is odd} & \text{for } \theta \text{ odd} \\ x \notin \bigcap_{\eta<\theta} C_\eta \wedge \min\{\eta < \theta \mid x \notin C_\eta\} \text{ is odd} & \text{for } \theta \text{ even} \end{cases}$$

$D_1(C_0) = C_0$ $\quad$ $D_2(C_0, C_1) = C_0 \setminus C_1$ $\quad$ $D_3(C_0, C_1, C_2) = C_0 \setminus C_1 \cup C_2$ $\quad$ $D_\omega((C_i)_{i<\omega}) = \bigcup_{i<\omega} (C_{2i} \setminus C_{2i+1})$



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For $\theta \geq 1$, X a topological space and $\boldsymbol{\Gamma}(X)$ boldface pointclass, we let:

$$\mathbf{Dif}_\theta\left(\boldsymbol{\Gamma}(X)\right) = \left\{D_\theta\left((C_\eta)_{\eta<\theta}\right) \mid C_\eta \in \boldsymbol{\Gamma}(X), \eta < \theta\right\}.$$

We also define $\check{\mathbf{Dif}}_\theta\left(\boldsymbol{\Gamma}(X)\right)$ to be the dual class of $\mathbf{Dif}_\theta\left(\boldsymbol{\Gamma}(X)\right)$.

- $\mathbf{Dif}_\theta\left(\boldsymbol{\Gamma}\right)$ is a boldface pointclass.
- $\mathbf{Dif}_\theta\left(\boldsymbol{\Gamma}\right) \cup \check{\mathbf{Dif}}_\theta\left(\boldsymbol{\Gamma}\right) \subseteq \mathbf{Dif}_{\theta+1}\left(\boldsymbol{\Gamma}\right) \cap \check{\mathbf{Dif}}_{\theta+1}\left(\boldsymbol{\Gamma}\right)$
- if $Y \subseteq X$, then $\mathbf{Dif}_\theta\left(\boldsymbol{\Gamma}(Y)\right) = \{A \cap Y \mid A \in \mathbf{Dif}_\theta\left(\boldsymbol{\Gamma}(X)\right)\}$.

Difference hierarchy on ${}^{\omega}2$

1. For every $\alpha < \omega_1$, the difference hierarchy over $\mathbf{\Pi}^0_\alpha({}^{\omega}2)$ has length ω_1.
2. The first ω_1-many nonselfdual levels of the Wadge hierarchy on ${}^{\omega}2$ are occupied by the difference classes $\mathbf{Dif}_\theta\left(\mathbf{\Pi}^0_1({}^{\omega}2)\right)$ and their duals.
3. The Hausdorff-Kuratowski theorem holds:

Theorem (Hausdorff, Kuratowski)

In every polish space X and for any $1 \leq \alpha < \omega_1$,

$$\mathbf{\Delta}^0_{\alpha+1}(X) = \bigcup_{1 \leq \theta < \omega_1} \mathbf{Dif}_\theta\left(\mathbf{\Pi}^0_\alpha(X)\right).$$

Difference hierarchy on ${}^{\kappa}2$

Recall: Given X topological space, if $\mathbf{\Gamma}(X)$ is selfdual, then there is no X-universal set for $\mathbf{\Gamma}(X)$.

Proposition

Let X be a topological space of weight $\leq \kappa$, $\mathbf{\Gamma}$ a boldface pointclass.
If $\mathbf{\Gamma}(X)$ has a ${}^{\kappa}2$-universal set, then $\mathbf{Dif}_\theta(\mathbf{\Gamma}(X))$ has a ${}^{\kappa}2$-universal set for every $1 < \theta < \kappa^+$.
In particular, $\mathbf{Dif}_\theta\left(\mathbf{\Pi}^0_\alpha(\kappa^+)\right) \neq \check{\mathbf{Dif}}_\theta\left(\mathbf{\Pi}^0_\alpha(\kappa^+)\right)$ is nonselfdual for every $1 < \theta < \kappa^+$ and $\alpha < \kappa^+$.

As a consequence:

- The difference hierarchy on $\mathbf{\Pi}^0_1(\kappa^+)$ has length κ^+.
- $\mathbf{Dif}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right) \cap \check{\mathbf{Dif}}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right) \subsetneq \mathbf{Dif}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right) \subsetneq \mathbf{Dif}_{\theta+1}\left(\mathbf{\Pi}^0_1(\kappa^+)\right) \cap \check{\mathbf{Dif}}_{\theta+1}\left(\mathbf{\Pi}^0_1(\kappa^+)\right)$

Difference hierarchy on ${}^{\kappa}2$

Are the difference classes $\mathbf{Dif}_\theta\left(\mathbf{\Pi}^0_1(\kappa^+)\right)$ (and their duals) Wadge classes ?

Complete canonical sets for $\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$

Let $\mathcal{C}^{(1)}$ be any proper $\mathbf{\Pi}^0_1(\kappa^+)$-set. Recursively, for every $1 \leq \alpha < \kappa^+$:

$$\mathcal{C}^{(\alpha+1)} = \begin{cases} \bigcup_{\alpha<\kappa} 0^{(\alpha)} {}^\frown \langle 1 \rangle {}^\frown \mathcal{C}^{(\alpha)} & \text{for } \alpha \text{ odd;} \\ \bigcup_{\alpha<\kappa} 0^{(\alpha)} {}^\frown \langle 1 \rangle {}^\frown \mathcal{C}^{(\alpha)} \cup \{0^\kappa\} & \text{for } \alpha \text{ even.} \end{cases}$$

and for $\alpha < \kappa^+$ limit ordinal we define

$$\mathcal{C}^{(\alpha)} = \begin{cases} \bigcup_{i<\mathrm{cof}(\gamma)} \bigcup_{\beta<\kappa} 0^{(\mathrm{cof}(\gamma)\cdot\beta+i)} {}^\frown \langle 1 \rangle {}^\frown \mathcal{C}^{(\alpha_i)} & \text{if } \mathrm{cof}(\alpha) < \kappa; \\ \bigcup_{i<\mathrm{cof}(\gamma)} 0^{(i)} {}^\frown \langle 1 \rangle {}^\frown \mathcal{C}^{(\alpha_i)} & \text{if } \mathrm{cof}(\alpha) = \kappa. \end{cases}$$

where $\langle \alpha_i \mid i < \mathrm{cof}(\alpha) \rangle$ is cofinal in α and for every $i < \mathrm{cof}(\alpha)$, $\mathcal{C}^{(\alpha_i)}$ is complete for $\mathbf{Dif}_{\alpha_i}(\mathbf{\Pi}^0_1(\kappa^+))$.
We also define $\mathcal{U}^{(\alpha)} = {}^\kappa 2 \setminus \mathcal{C}^{(\alpha)}$ for every $1 \leq \alpha < \kappa^+$.

Proposition

For every $1 \leq \alpha < \kappa^+$, $\mathcal{C}^{(\alpha)}$ is complete for the $\mathbf{Dif}_\alpha(\mathbf{\Pi}^0_1(\kappa^+))$, hence $\mathcal{U}^{(\alpha)}$ is complete for $\check{\mathbf{Dif}}_\alpha(\mathbf{\Pi}^0_1(\kappa^+))$.

SLO along the difference hierarchy

Theorem

For every $\theta < \kappa^+$, $\mathsf{SLO}(\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+)))$.

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For every $\theta < \kappa^+$, $\mathsf{SLO}(\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+)))$.

The proof is by induction on $\theta < \kappa^+$. We will use the following fact:

Fact

Let $\mathbf{\Gamma}$ be a non selfdual boldface pointclass. If:

1. $\mathsf{SLO}(\mathbf{\Gamma} \cap \check{\mathbf{\Gamma}})$
2. A is $\mathbf{\Gamma}$-complete $\iff A \in \mathbf{\Gamma} \setminus \check{\mathbf{\Gamma}}$

then, $\mathsf{SLO}(\mathbf{\Gamma})$.

SLO along the difference hierarchy

First, assume $\theta = 1$.

- SLO($\mathbf{\Delta}^0_1(\kappa^+)$).

let $A, B \in \Delta^0_1(\kappa^+)$. Assume $A, B \notin \{\emptyset, {}^\kappa 2\} \Rightarrow \exists\, b \in B$
$\exists\, c \in {}^\kappa 2 \setminus B$

Set $f : {}^\kappa 2 \to {}^\kappa 2$

$$x \longmapsto \begin{cases} b & \text{if } x \in A \\ c & \text{otherwise} \end{cases}$$

SLO along the difference hierarchy

First, assume $\theta = 1$.

- SLO($\mathbf{\Delta}^0_1(\kappa^+)$).
- Let $C \subseteq {}^{\kappa}2$. If $C \in \mathbf{\Pi}^0_1(\kappa^+) \setminus \mathbf{\Sigma}^0_1(\kappa^+)$, then C is $\mathbf{\Pi}^0_1(\kappa^+)$-complete.

Let $C \subseteq {}^{\kappa}2$, $C \in \Pi^0_1(\kappa^+) \setminus \Sigma^0_1(\kappa^+)$ and $D \in \Pi^0_1(\kappa^+)$. Let $x \in \partial C$.
We want $D \leq_W C$. Consider $G_W(D, C)$:

- As long as I plays nodes of T_D,
  II plays initial segments of x.
- If I ever reaches $t \in \partial T_D$,
  II picks an extension y of its previous play s.t. $y \notin C$
  and plays initial segments of y.

SLO along the difference hierarchy

First, assume $\theta = 1$.

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Fact

Let $\boldsymbol{\Gamma}$ be a non selfdual boldface pointclass. If:

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then, SLO($\boldsymbol{\Gamma}$).

By the Fact: SLO($\boldsymbol{\Sigma}^0_1(\kappa^+)$) and SLO($\boldsymbol{\Pi}^0_1(\kappa^+)$).

SLO along the difference hierarchy

Now, assume $\theta > 1$. If $\theta = \beta + 1$ successor ordinal:

- SLO($\boldsymbol{\Gamma}$) for $\boldsymbol{\Gamma} = \mathbf{Dif}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{Dif}}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+))$.
- Every proper $\mathbf{Dif}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+))$-subset $C \subseteq {}^\kappa 2$ is $\mathbf{Dif}_\theta(\boldsymbol{\Pi}^0_1(\kappa^+))$-complete.

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Theorem

For any $\beta < \kappa^+$ and for any decreasing sequence $(C_i)_{i \leq \beta}$ of closed subsets of ${}^\kappa 2$, TFAE:

1. *there exists $x \in \partial C_\beta$ for every $\alpha < \kappa$ $D_{\beta+1}((C_i)_{i\leq\beta}) \cap N_{x\restriction\alpha} \notin \check{\mathbf{Dif}}_\beta$.*
2. *$D_{\beta+1}((C_i)_{i\leq\beta})$ is $\mathbf{Dif}_{\beta+1}$-complete.*
3. *$D_{\beta+1}((C_i)_{i\leq\beta})$ is a proper $\mathbf{Dif}_{\beta+1}$-set*

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By the Fact: SLO($\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$) and SLO($\check{\mathbf{Dif}}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$).

SLO along the difference hierarchy

Now, assume $\theta > 1$. If θ is a limit ordinal:

- SLO($\mathbf{\Gamma}$) for $\mathbf{\Gamma} = \mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{Dif}}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$.
- Every proper $\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$-subset $C \subseteq {}^\kappa 2$ is $\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$-complete.

Theorem

For $\theta < \kappa^+$ limit and for any decreasing sequence $(C_i)_{i<\theta}$ of closed subsets of ${}^\kappa 2$, TFAE:

1. *there exists $x \in \partial(\bigcap_{i<\theta} C_i)$ for every $\alpha < \kappa$ and $\beta < \gamma$, $D_\theta((C_i)_{i<\theta}) \cap N_{x \restriction \alpha} \notin \mathbf{Dif}_\beta$.*
2. *$D_\theta((C_i)_{i<\theta})$ is $\mathbf{Dif}_\theta$-complete.*
3. *$D_\theta((C_i)_{i<\theta})$ is a proper $\mathbf{Dif}_\theta$-set.*

By the Fact: SLO($\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$) and SLO($\check{\mathbf{Dif}}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$).

Difference hierarchy on ${}^{\kappa}2$

Are the selfdual boldface pointclasses $\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{Dif}}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$ Wadge classes?

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Are the selfdual boldface pointclasses $\mathbf{Dif}_{\theta}(\mathbf{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{Dif}}_{\theta}(\mathbf{\Pi}^0_1(\kappa^+))$ Wadge classes?

- θ is a successor ordinal: yes. This implies that: $\mathbf{Dif}_{\theta}(\mathbf{\Pi}^0_1(\kappa^+)) \cup \check{\mathbf{Dif}}_{\theta}(\mathbf{\Pi}^0_1(\kappa^+)) \subsetneq \mathbf{Dif}_{\theta+1}(\mathbf{\Pi}^0_1(\kappa^+)) \cap \check{\mathbf{Dif}}_{\theta+1}(\mathbf{\Pi}^0_1(\kappa^+))$

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- θ is a limit ordinal, $\mathrm{cof}(\theta) < \kappa$: yes.
- θ is a limit ordinal, $\mathrm{cof}(\theta) = \kappa$:
  - If κ is not weakly compact: yes
  - If κ is weakly compact: no, indeed $\bigcup_{\beta<\theta} \mathbf{Dif}_\beta = \mathbf{Dif}_\theta \cap \check{\mathbf{Dif}}_\theta$.

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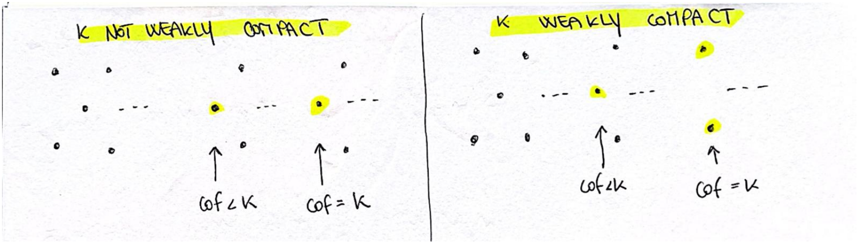

Hausdorff-Kuratwoski theorem

Does Hausdorff-Kuratwoski theorem hold when $\kappa > \omega$?

We recall:

Theorem (Hausdorff, Kuratowski)

In every polish space X and for any $1 \leq \alpha < \omega_1$,

$$\boldsymbol{\Delta}^0_{\alpha+1}(X) = \bigcup_{1\leq\theta<\omega_1} \mathbf{Dif}_\theta \left(\boldsymbol{\Pi}^0_\alpha(X)\right)$$

A counterexample to Hausdorff-Kuratowski in GDST

Theorem

Let $X, Y, Z \subseteq {}^{\kappa}2$ nonempty. If $Y \cap Z$ is dense and codense in Y, then

1. $Z \notin \mathbf{Dif}_{\theta}(\mathbf{\Pi}^0_1(Y, \kappa^+))$ for any $\theta < \kappa^+$.
2. for all $\theta < \kappa^+$ and all $A \in D_{\theta}(\mathbf{\Pi}^0_1(X, \kappa^+))$, $A \leq_W^{X,Y} Z$.

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2. for all $\theta < \kappa^+$ and all $A \in D_\theta(\mathbf{\Pi}^0_1(X, \kappa^+))$, $A \leq_W^{X,Y} Z$.

Counterexample 1

Consider the sets:

$$Y = \{x \in {}^{\kappa}2 \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \aleph_0\}$$
$$Y_0 = \{x \in {}^{\kappa}2 \mid \exists n < \omega |\{\alpha < \kappa \mid x(\alpha) = 1\}| = 2n\}.$$

Then $Y_0 \in \mathbf{\Delta}^0_2(\kappa^+)$, but not in $\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$ for any $\theta < \kappa^+$.

A counterexample to Hausdorff-Kuratowski in GDST

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Let $X, Y, Z \subseteq {}^{\kappa}2$ nonempty. If $Y \cap Z$ is dense and codense in Y, then

1. $Z \notin \mathbf{Dif}_{\theta}(\mathbf{\Pi}^0_1(Y, \kappa^+))$ for any $\theta < \kappa^+$.
2. for all $\theta < \kappa^+$ and all $A \in D_{\theta}(\mathbf{\Pi}^0_1(X, \kappa^+))$, $A \leq_W^{X,Y} Z$.

Counterexample 2

Let $\lambda < \kappa$ be a limit ordinal.

$$Y^{\lambda} = \{x \in {}^{\kappa}2 \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \lambda\}$$
$$Y^{\lambda}_0 = \{x \in {}^{\kappa}2 \mid \exists \alpha < \lambda |\{\alpha < \kappa \mid x(\alpha) = 1\}| = 2\alpha\}$$

Then $Y^{\lambda}_0 \in \mathbf{\Delta}^0_2(\kappa^+)$, but not in $\mathbf{Dif}_{\theta}(\mathbf{\Pi}^0_1(\kappa^+))$ for any $\theta < \kappa^+$.

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Proof:

- For every $A \subseteq {}^\kappa 2$, $A \leq_\mathrm{W} Y_0$ if and only if $A \leq_\mathrm{L} Y_0$.
- Let $\mathbf{\Gamma}({}^\kappa 2)$ be non-selfdual and generated by a set A. If $\mathbf{\Gamma}({}^\kappa 2) = \{B \mid B \leq_\mathrm{L} A\}$, then $\mathbf{\Gamma}({}^\kappa 2)$ admits a ${}^\kappa 2$-universal set.
- If $\mathbf{\Gamma}({}^\kappa 2)$ has ${}^\kappa 2$-universal sets, then for every $1 < \theta < \kappa^+$ there is a ${}^\kappa 2$-universal set for $\mathbf{Dif}_\theta(\mathbf{\Gamma}({}^\kappa 2))$.

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Theorem

Let $A \subseteq {}^\kappa 2$ such that $A \leq_W Y_0$. Then, either $A \in \mathbf{Dif}_\theta(\mathbf{\Pi}^1_0(\kappa^+))$ for some $\theta < \kappa^+$ or $Y_0 \equiv_W A$. Therefore, Y_0 is minimal above the difference hierarchy over closed sets.

Y_0 minimal above the difference hierarchy over closed sets.

For any closed set $X \subseteq {}^{\kappa}2$ and $A \subseteq {}^{\kappa}2$, let $\partial_X(A) = cl(X \cap A) \cap cl(X \setminus A)$. The Hausdorff-Kuratowski derivative of X w.r.t. A is defined as:

$$X_0^{(A)} = X$$
$$X_{\alpha+1}^{(A)} = \partial_{X_\alpha^{(A)}}(A)$$
$$X_\gamma^{(A)} = \bigcap_{\alpha<\gamma} X_\alpha^{(A)} \text{ if } \gamma \text{ is limit.}$$

The A-Hausdorff-Kuratowski rank of X is the least $\delta < \kappa^+$ such that $X_\delta^{(A)} = X_{\delta+1}^{(A)}$. If $X_\delta^{(A)} = \emptyset$, we say X is *A-scattered*.

Proposition

Let $A \subseteq {}^{\kappa}2$ such that $A \leq_W Y_0$.

- If ${}^{\kappa}2$ is A-scattered, then $A \in \mathbf{Dif}_\theta(\mathbf{\Pi}_0^1(\kappa^+))$ for some $\theta < \kappa^+$.
- If ${}^{\kappa}2$ is not A-scattered. Then, $Y_0 \leq_W A$.

Above the Difference hierarchy

Counterexample 3

Let $Y = \{x \in {}^{\kappa}2 \mid |\{\alpha < \kappa \mid x(\alpha) = 1\}| < \aleph_0\}$ and $S \subseteq \kappa$.

$$Y_S = Y_0 \cup \bigcup_{t \in \partial T_Y, |t| \in S} N_t$$

Then $Y_S \in \mathbf{\Delta}^0_2(\kappa^+)$, but not in $\mathbf{Dif}_\theta(\mathbf{\Pi}^0_1(\kappa^+))$ for any $\theta < \kappa^+$.

Note that $Y_\emptyset = Y_0$.

Above the Difference hierarchy

Theorem

Let $\mathrm{cof}^{\kappa}_{\omega} = \{\gamma < \kappa \mid \mathrm{cof}(\gamma) = \omega\}$. Given $S, S' \subseteq \mathrm{cof}^{\kappa}_{\omega}$:

- If $S \triangle S'$ is stationary in κ, then, $Y_S \not\leq_{\mathsf{W}} Y_{S'}$ and $Y_{S'} \not\leq_{\mathsf{W}} Y_S$.
- If $S \triangle S'$ is non-stationary in κ, then $Y_S \equiv_{\mathsf{W}} Y_{S'}$.

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Therefore, given $S \subseteq \mathrm{cof}^{\kappa}_{\omega}$:

- If $S \neq \emptyset$, Y_S is non-selfdual (because $\neg Y_S \equiv_{\mathrm{W}} Y_{S'}$ with $S' = \kappa \setminus S$).
- If S is stationary in κ, then Y_S and Y_0 are $\leq_{\mathrm{W}}$-incomparable.
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Any stationary $S \subseteq \mathrm{cof}^\kappa_\omega$ is the disjoint union of stationary $(S_i)_{i<\kappa}$.
For any $A \subseteq \kappa$, let $S^A = \bigcup_{i \in A} S_i$.
Given $A, B \subseteq \kappa$, if $A \neq B$ then $S^A \triangle S^B$ is stationary.

Additional results for $\mathbf{\Sigma}^0_2(\kappa^+)$

Theorem

Assume V = L. Then, there are proper $\mathbf{\Sigma}^0_2(\kappa^+)$-sets which are not $\mathbf{\Sigma}^0_2(\kappa^+)$-complete.

Theorem

Let $\lambda > \kappa$ be an inaccessible cardinal. For any $Col(\kappa, < \lambda)$-generic filter G over V, in $V[G]$ every proper $\mathbf{\Sigma}^0_2(\kappa^+)$-set is $\mathbf{\Sigma}^0_2(\kappa^+)$-complete.

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It follows from the an analogue of the *Kechris-Louveau-Woodin dichotomy* by Schlicht, Sziraki: For all disjoint definable subsets $X, Y \subseteq {}^\kappa\kappa$,

either there is a $\mathbf{\Sigma}^0_2(\kappa^+)$-set A separating X from Y,

or

there is a homeomorphism f from ${}^\kappa 2$ onto a closed subset of ${}^\kappa\kappa$ such that $f(\mathbb{Q}_\kappa) \subseteq X$ and $f({}^\kappa 2 \setminus \mathbb{Q}_\kappa) \subseteq Y$,

where $\mathbb{Q}_\kappa = \{x \in {}^\kappa 2 \mid \forall\alpha < \kappa \exists\beta > \alpha(x(\beta) = 0)\}$.

Side note: when κ is singular

Theorem

Let κ be a singular cardinal. Then $(\mathcal{P}(\mathrm{cof}(\kappa)), \subseteq)$ embeds into the Wadge hierarchy on the $\mathbf{\Delta}^0_2(\kappa^+)$ subsets of ${}^{\kappa}2$.

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Thank You!