

Descriptive aspects of large cardinals beyond HOD

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Introduction

Following Woodin, seminal results of Jensen and Silver can be formulated in the following way:

Theorem (The L-Dichotomy, Jensen & Silver)

Exactly one of the following statements holds:

- If λ is a singular cardinal, then λ is singular in L and $(\lambda^+)^L = \lambda^+$ holds (" L is close to V ").
- Every regular cardinal is inaccessible in L (" L is far from V ").

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- Every regular cardinal is inaccessible in L (" L is far from V ").

A surprising result of Woodin shows that strong large cardinal assumptions imply an analog of this dichotomy for the inner model HOD .

Theorem (Simplified HOD Dichotomy, Woodin)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\text{HOD}} = \lambda^+$ holds ("*HOD is close to V*").
- Every regular cardinal $\kappa \geq \delta$ is measurable in HOD ("*HOD is far from V*").

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Are there canonical extensions of ZFC that prove that HOD is far from V?

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Questions

Are there canonical extensions of ZFC that prove that HOD is far from V? Are there such axioms that imply $V \neq \text{HOD}$?

All standard large cardinal axioms are compatible with the assumption that $V = \text{HOD}$ and therefore do not provide affirmative answers to these questions.

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In the following, we will observe that more interesting things can be said about the relationship between V and HOD in this setting.

Definition (Goldberg & Schlutzenberg, ZF)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $j : V_\beta \longrightarrow V_\beta$ with the property that $\alpha < \text{crit}(j) < \lambda$ and λ is the first non-trivial fixed point of j .

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If $j : V \longrightarrow V$ is an elementary embedding, then the first non-trivial fixed point of j is a rank-Berkeley cardinal.

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Proposition (ZF)

Rank-Berkeley cardinals are cardinals of countable cofinality that are regular in HOD.

Proof.

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$$c(\alpha) < j(c(\alpha)) = j(c)(j(\alpha)) = c(\alpha),$$

a contradiction.



Exacting cardinals

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Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $V_\lambda \cup \{\lambda\} \subseteq X$, and
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Note that, in order to prove that a cardinal λ is exacting, it suffices to find a single embedding $j : X \rightarrow V_\beta$ satisfying $V_\lambda \cup \{\lambda\} \subseteq X \prec V_\beta \prec_{\Sigma_2} V$, $j(\lambda) = \lambda$ and $j \upharpoonright \lambda \neq \text{id}_\lambda$.

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Theorem (Aguilera–Bagaria–L.)

If λ is exacting, then λ is a singular cardinal that is regular in HOD_{V_λ} .

Corollary

If there is an exacting cardinal above an extendible cardinal, then eventually all regular cardinals are measurable in HOD .

Let λ be an exacting cardinal. Then there is a non-trivial elementary embedding $j : V_\lambda \longrightarrow V_\lambda$ and results of Kunen imply $\text{cof}(\lambda) = \omega$.

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If we pick $\alpha < \text{cof}(\lambda)^{\text{HOD}_{\{z\}}}$ with $c(\alpha) > \text{crit}(j)$, then we have

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For this purpose, remember that a cardinal λ is Jónsson if every structure in a countable first-order language whose domain has cardinality λ has a proper elementary substructure of cardinality λ .

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The next result shows that exactingness is equivalent to a strengthening of this property that incorporates external features of the given structure.

Theorem (Aguilera–Bagaria–L.)

The following are equivalent for each cardinal λ with $|V_\lambda| = \lambda$:

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- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ isomorphic to a structure in $\mathcal{C}$.

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In another direction, exactingness can also be represented as a natural strengthening of the existence of $\mathcal{I}_3$ -embeddings:

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Theorem (Aguilera–Bagaria–Goldberg–L.)

The following are equivalent for every cardinal λ :

- λ is an exacting cardinal.
- For every non-empty, ordinal definable subset A of $V_{\lambda+1}$, there exist $x, y \in A$ and an elementary embedding

$$j : (V_\lambda, \in, x) \longrightarrow (V_\lambda, \in, y).$$

Moreover, the consistency strength of exacting cardinal is surprisingly low:

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Definition

- An I3-embedding is a non-trivial elementary embedding $j : V_\lambda \longrightarrow V_\lambda$ for some limit ordinal λ .
- An I2-embedding is a non-trivial elementary embedding $j : V \longrightarrow M$ with $V_\lambda \subseteq M$, where λ is the first non-trivial fixed point of j .

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Theorem (Aguilera–Bagaria–Goldberg–L.)

Let $j : V \longrightarrow M$ be an I2-embedding with critical point κ and let

$$U = \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

If G is generic over V for Prikry forcing with U , then κ is an exacting cardinal in $V[G]$.

Finally, the consistency of the existence of an exacting cardinal above an extendible cardinal can be established from *large cardinals beyond Choice*.

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A cardinal κ is *super Reinhardt* if for every ordinal α , there is an elementary embedding $j : V \longrightarrow V$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

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Theorem (Aguilera–Bagaria–L., BG)

If there is a super Reinhardt cardinal, then there is a model of ZFC with an exacting cardinal above an extendible cardinal.

Ultraextracting cardinals

We now consider the possibility of further strengthening the notion of exacting cardinals.

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The proof of the following result uses ideas from Woodin's proof of the Kunen Inconsistency:

Proposition

If λ is a cardinal, $\zeta > \lambda$ is an ordinal with $V_\zeta \prec_{\Sigma_2} V$, X is an elementary submodel of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$ and $j : X \rightarrow V_\zeta$ is an elementary embedding with $j(\lambda) = \lambda$ and $j \upharpoonright \lambda \neq \text{id}_\lambda$, then $\lambda^+ \notin X$ and $[\lambda]^\omega \notin X$.

The above proposition shows that we can strengthen the notion of exacting cardinals by demanding that certain sets are contained in the domains of the elementary embeddings witnessing the given property.

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This motivates the following definition:

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \beta$, there exist

- an elementary submodel X of V_β with $V_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow V_\beta$ with $\alpha < \text{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \upharpoonright V_\lambda \in X$.

As before, this notion is equivalent to a natural strengthening of a rank-into-rank axioms:

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Theorem (Aguilera–Bagaria–Goldberg–L.)

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- For every ordinal definable subset A of $V_{\lambda+1}$, there exists an elementary embedding

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This equivalence can be used to show that ultraextendingness is equivalent to an assumption considered by Woodin in his *Suitable Extender Models I*. paper as a candidate for an axiom that, together with the existence of an extendible cardinal, causes HOD to be far from V .

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The following statements are equiconsistent over ZFC:

- There is an ultraexacting cardinal.
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Ultraexacting cardinals and ordinal definability

We now discuss results showing that analogs of results about ordinal definable sets in determinacy models can be proven for ordinal definable sets in the presence of ultraexacting cardinals.

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We start with analog of the classical result that the ω_1 of a determinacy model is a measurable cardinal in its HOD.

Definition (Woodin)

An uncountable regular cardinal κ is ω -*strongly measurable in HOD* if there is a cardinal $\delta < \kappa$ such that $(2^\delta)^{\text{HOD}} < \kappa$ and HOD contains no partition of the set E_ω^κ of all elements of κ with cofinality ω into δ -many sets that are all stationary in V .

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Lemma (Woodin)

If a cardinal κ is ω -strongly measurable in HOD , then κ is a measurable cardinal in HOD .

Theorem (Aguilera–Bagaria–L.)

If λ is an ultraexacting cardinal, then λ^+ is ω -strongly measurable in HOD .

The proof of this result relies on arguments showing that, given an elementary embedding $j : X \longrightarrow V_\zeta$ witnessing the ultraexactness of a cardinal $\lambda < \zeta$, for certain ordinals

$$\lambda < \alpha \in X \cap \zeta,$$

we can find functions

$$f : L_\alpha(V_{\lambda+1}) \longrightarrow L_{j(\alpha)}(V_{\lambda+1})$$

in X with the property that

$$f \restriction (X \cap L_\alpha(V_{\lambda+1})) = j \restriction (X \cap L_\alpha(V_{\lambda+1})).$$

In the following, fix

- a cardinal λ ,
- an ordinal $\zeta > \lambda$ such that V_ζ is sufficiently elementary in V ,
- an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow V_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda \neq \text{id}_\lambda$ and $j \restriction V_\lambda \in X$.

Extensions to $V_{\lambda+1}$

Remember that, if ξ is an ordinal of countable cofinality and $k : V_\xi \longrightarrow V_\xi$ is an elementary embedding, then the map

$$k_+ : V_{\xi+1} \longrightarrow V_{\xi+1}; \quad A \longmapsto \bigcup \{k(A \cap V_\alpha) \mid \alpha < \xi\}$$

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If $i : V \longrightarrow M$ is an I_2 -embedding with least non-trivial fixed point λ , then

$$(i \upharpoonright V_\lambda)_+ = i \upharpoonright V_{\lambda+1} : V_{\lambda+1} \longrightarrow V_{\lambda+1}$$

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is Σ_1 -elementary. Conversely, every non-trivial Σ_1 -elementary function from $V_{\lambda+1}$ to itself can be extended to an I2-embedding.

Lemma

The map

$$(j \restriction V_\lambda)_+ : V_{\lambda+1} \longrightarrow V_{\lambda+1}$$

is an elementary embedding that is contained in X and satisfies

$$(j \restriction V_\lambda)_+ \restriction (X \cap V_{\lambda+1}) = j \restriction (X \cap V_{\lambda+1}).$$

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The elementarity of j and the above equality then imply that $(j \upharpoonright V_\lambda)_+$ is an elementary embedding of $V_{\lambda+1}$ into itself in X . Finally, the correctness properties of X ensure that $(j \upharpoonright V_\lambda)_+$ also has this property in V . $\square$

Lemma

Let γ be an ordinal in X such that there is a surjection $s : V_{\lambda+1} \rightarrow \gamma$ in X with $j(s) \in X$. Then there exists a unique function $j_\gamma : \gamma \rightarrow j(\gamma)$ that is an element of X and satisfies

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Define $j_\gamma : \gamma \rightarrow j(\gamma)$ to be the unique function satisfying

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$$j_\gamma(s(x)) = j(s)((j \upharpoonright V_\lambda)_+(x))$$

for all $x \in V_{\lambda+1}$. Then j_γ possesses all of the listed properties. $\square$

Remember that, given $E \subseteq V_{\lambda+1}$, we let $\Theta^{L(V_{\lambda+1}, E)}$ denote the least ordinal γ such that $L(V_{\lambda+1}, E)$ does not contain a surjection from $V_{\lambda+1}$ onto γ .

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The basic structure theory of $L(V_{\lambda+1}, E)$ now yields the following statement:

Lemma

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{L(V_{\lambda+1}, E)}$ with $j(\gamma) \in X$, then there is a surjection $s : V_{\lambda+1} \rightarrow \gamma$ in X with $j(s) \in X$.

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Corollary

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{L(V_{\lambda+1}, E)}$ with $j(\gamma) = \gamma$, then there is a function $j_\gamma : \gamma \longrightarrow \gamma$ in X with

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We now use the above results to prove that successors of ultraexacting cardinals are measurable in HOD.

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Definition (Woodin)

An uncountable regular cardinal κ is ω -*strongly measurable in HOD* if there is a cardinal $\delta < \kappa$ such that $(2^\delta)^{\text{HOD}} < \kappa$ and HOD contains no partition of the set E_ω^κ of all elements of κ with cofinality ω into δ -many sets that are all stationary in V .

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Theorem (Aguilera–Bagaria–L.)

If λ is an ultraextending cardinal, then λ^+ is ω -strongly measurable in HOD.

Let λ be an ultraexacting cardinal. Fix an ordinal $\zeta > \lambda$ with $V_\zeta \prec_{\Sigma_3} V$, an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and an elementary embedding $j : X \longrightarrow V_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda \neq \text{id}_\lambda$ and $j \restriction V_\lambda \in X$.

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Assume, towards a contradiction, that there is a partition of $E_\omega^{\lambda+}$ into $\text{crit}(j)$ -many stationary sets that is an element of HOD. Let

$$\vec{S} = \langle S_\alpha \mid \alpha < \text{crit}(j) \rangle$$

denote the least such partition in the canonical well-ordering of HOD. Set

$$j(\vec{S}) = \langle S'_\beta \mid \beta < j(\text{crit}(j)) \rangle.$$

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denote the least such partition in the canonical well-ordering of HOD. Set

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Then $j(\vec{S})$ is the minimal partition of $E_\omega^{\lambda+}$ into $j(\text{crit}(j))$ -many stationary sets in HOD. In particular, the sequence $j(\vec{S})$ is an element of X .

By an earlier lemma, there is a function $f : \lambda^+ \longrightarrow \lambda^+$ in X with

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Elementarity then implies that f is strictly increasing and continuous at limit ordinals of countable cofinality.

This implies that

$$C = \{\gamma < \lambda^+ \mid f(\gamma) = \gamma\}$$

is an ω -club in λ^+ that is an element of X and we can find

$$\gamma \in C \cap S'_{\text{crit}(j)} \cap X.$$

Then there exists $\alpha < \text{crit}(j)$ with $\gamma \in S_\alpha$ and the fact that $j(\gamma) = f(\gamma) = \gamma$ implies that

$$\gamma \in S'_\alpha \cap S'_{\text{crit}(j)} = \emptyset,$$

a contradiction. □

Using the above techniques, it is possible to derive other properties of HOD in the presence of ultraexacting cardinals that resemble results from the determinacy setting. For example:

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Definition

Given an inner model N and ordinals $\lambda < \vartheta$, we say that the ordinal λ is N - ϑ -Berkeley if for every $\alpha < \lambda$ and every transitive set M in N that contains λ as an element and has cardinality less than ϑ in N , there exists a non-trivial elementary embedding $j : M \rightarrow M$ with $\alpha < \text{crit}(j) < \lambda$.

Using the above techniques, it is possible to derive other properties of HOD in the presence of ultraexacting cardinals that resemble results from the determinacy setting. For example:

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Theorem (Aguilera–Bagaria–L.)

Every ultraexacting cardinal λ is $\text{HOD-}\Theta^{L(V_{\lambda+1})}$ -Berkeley.

Thank you for listening!