Non-linear iterations and the higher Baire spaces

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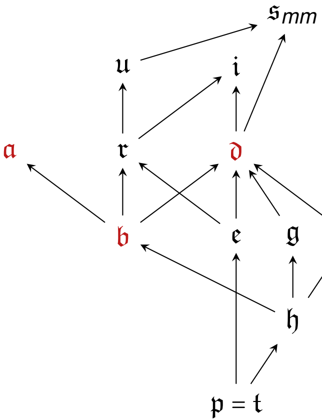

Figure: The van Douwen diagram of combinatorial invariants

Definition (Bounding and Domination)

Let f and g be functions from κ to κ.

1. Then g eventually dominates f, denoted by $f <^* g$, if $\exists n < \kappa$ $\forall m > n\ (f(m) < g(m))$.
2. A family $\mathcal{F} \subseteq {}^\kappa\kappa$, is dominating if $\forall g \in {}^\kappa\kappa\ \exists f \in \mathcal{F}\ (g <^* f)$.
3. A family $\mathcal{F} \subseteq {}^\kappa\kappa$ is unbounded if $\forall g \in {}^\kappa\kappa\ \exists f \in \mathcal{F}\ (f \not<^* g)$.
4. $\mathfrak{b}_\kappa$ and $\mathfrak{d}_\kappa$ denote the least cardinalities of an unbounded and dominating family respectively.
5. Finally, $\mathfrak{c}_\kappa = 2^\kappa$.

Lemma (Cummings, Shelah)

Let κ be a regular uncountable. Then

$$\kappa^+ \leq \mathrm{cf}(\mathfrak{b}(\kappa)) = \mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq \mathfrak{c}(\kappa) = 2^\kappa.$$

Realizing a cardinal constellation

Theorem (Cummings, Shelah)

Assume $\kappa^{<\kappa} = \kappa$, GCH above κ and (β, δ, μ) such that

$$\kappa^+ \leq \beta = \mathrm{cf}(\beta) \leq \mathrm{cf}(\delta) \leq \mu \text{ and } \kappa < \mathrm{cf}(\mu).$$

Then there is a cardinal preserving generic extension in which

$$\mathfrak{b}(\kappa) = \beta, \mathfrak{d}(\kappa) = \delta \text{ and } 2^\kappa = \mu.$$

Definition (Hechler and restricted Hechler poset)

- The Hechler poset $\mathbb{H}$ consists of all (s, f) where $s \in \kappa^{<\kappa}, f \in {}^{\kappa}\kappa\}$.
- The extension relation is defined as follows $(t, g) \leq_{\mathbb{H}} (s, f)$ iff $s \subseteq t$, $\forall n \in \kappa\ (g(n) \geq f(n))$ and $\forall i \in \operatorname{dom}(t) \backslash \operatorname{dom}(s)\ (t(i) > f(i))$.
- If $A \subseteq {}^{\kappa}\kappa$, then $\mathbb{H}(A) = \{(s, f) : s \in \kappa^{<\kappa}, f \in A\}$ equipped with the same extension relation is known as restricted Hechler forcing.

$\mathbb{H}(A)$ adjoins a κ-real eventually dominating the elements in A.

Definition

Let $(P, \leq_P)$ be a partial order.

1. We call $U \subseteq P$ unbounded if $\forall p \in P\ \exists q \in U\ (q \not\leq_P p)$.
2. $\mathfrak{b}(P) = \min\{|U| : U \subseteq P \text{ is unbounded}\}$.
3. A subset $D \subseteq P$ is dominating if $\forall p \in P\ \exists q \in D\ (p \leq_P q)$.
4. $\mathfrak{d}(P) = \min\{|D| : D \subseteq P \text{ is dominating}\}$.

Lemma

Any poset P has a well-founded and dominating subposet P' of P with

$$\mathfrak{d}(P) = \mathfrak{d}(P') \text{ and } \mathfrak{b}(P) = \mathfrak{b}(P').$$

Theorem (Cummings, Shelah)

Let $\kappa = \kappa^{<\kappa}$ and let Q be a well-founded poset with $\mathfrak{b}(Q) \geq \kappa^+$. Then there is a forcing $\mathbb{H}(\kappa, Q)$, which is κ-closed and κ^+-cc and such that

$$V^{\mathbb{H}(\kappa,Q)} \models Q \text{ can be cofinally embedded into } ({}^{\kappa}\kappa, <^*)$$

and thus

$$V^{\mathbb{H}(\kappa,Q)} \models \mathfrak{b}(\kappa) = \mathfrak{b}(Q) \leq \mathfrak{d}(\kappa) = \mathfrak{d}(Q).$$

Definition (Almost Disjointness)

Let $x, y \in [\kappa]^\kappa$.

1. The sets x and y are almost disjoint if $|x \cap y| < \kappa$.
2. A family $\mathcal{A} \subseteq [\kappa]^\kappa$ is κ-almost disjoint if any two pairwise distinct elements in $\mathcal{A}$ are almost disjoint.
3. An almost disjoint family is κ-maximal almost disjoint (κ-mad) if it is maximal with respect to inclusion.
4. The almost disjointness number $\mathfrak{a}_\kappa$ is the minimal size of a κ-maximal almost disjoint family of cardinality at least κ and is denoted $\mathfrak{a}_\kappa$.

Strong Witnesses

Definition (Hechler poset for adding a mad family)

Let λ be an ordinal. Then $\mathbb{H}_\lambda$ consists of all partial functions

$$p : \lambda \times \kappa \to 2,$$

with $\mathrm{dom}(p) = F_p \times n_p$ where $F_p \in [\lambda]^{<\kappa},\ n_p \in \kappa$ and extension relation:

$$q \leq p \text{ iff } p \subseteq q \text{ and } \forall i \in n_q \backslash n_p\ |q^{-1} \cap F_p \times \{i\}| \leq 1.$$

Properties

1. If G is a $\mathbb{H}_\lambda$-generic, then for each $\alpha \in \gamma$ let $A_\alpha = \{i : \exists p \in G\ p(\alpha, i) = 1\}$.
2. Then $\mathcal{A}_\lambda = \{A_\alpha : \alpha < \lambda\}$ is κ-almost disjoint.
3. Moreover, if $\lambda \geq \kappa^+$ then $\mathcal{A}_\lambda$ is κ-maximal almost disjoint.
4. If $\alpha \leq \beta$, then $\mathbb{H}_\beta$ decomposes to $\mathbb{H}_\beta \simeq \mathbb{H}_\alpha * \dot{\mathbb{H}}_{[\alpha,\beta)}$ as follows:
   - Let G be a $\mathbb{H}_\alpha$-generic In $V[G]$.
   - Let $\mathbb{H}_{[\alpha,\beta)}$ be the poset of all (p, H), where
   - $p : (\beta \backslash \alpha) \times \kappa \to 2$ has domain $\mathrm{dom}(p) = F_p \times n_p$, $H \in [\alpha]^{<\kappa}$;
   - $(p, H) \leq (q, K)$ iff $p \leq_{\mathbb{H}_\beta} q$, $K \subseteq H$, $n_p \geq n_q$ and for every $j \in F_q$, $k \in n_p \backslash n_q$ and $i \in K$, if $k \in A_i$, then $p(j, k) = 0$ holds.

Lemma (Brendle, F.)

Let $\mathbb{P}$ and $\mathbb{Q}$ be posets with $\mathbb{P} \lessdot \mathbb{Q}$. Suppose $\dot{\mathbb{A}}$ (resp. $\dot{\mathbb{B}}$) is a $\mathbb{P}$-name (resp. $\mathbb{Q}$-name) for a poset and $i: \mathbb{A} \to \mathbb{B}$ is an embedding in $V^{\mathbb{Q}}$ such that

- $i(\mathbb{1}_{\mathbb{A}}) = \mathbb{1}_{\mathbb{B}}$,
- $\forall p, p' \in \mathbb{A} \left(p \leq p' \to i(p) \leq i(p')\right)$,
- $\forall p, p' \in \mathbb{A} \left(p \perp p' \leftrightarrow i(p) \perp i(p')\right)$ and
- Every max antichain of $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$ is mapped to a max antichain of $\dot{\mathbb{B}}$ in $V^{\mathbb{Q}}$.

Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$.

We will make use of the following notation: If $(A, <_A)$ is a poset and $y \in A$, then

$$A_y = \{x \in A : x <_A y\} \text{ and } y \uparrow_A = \{x \in A : y <_A x\}.$$

Theorem (GCH)

Let κ be a regular infinite cardinal, β and δ cardinals such that

$$\kappa^+ \leq \beta = \mathrm{cof}(\beta) \leq \mathrm{cof}(\delta).$$

There is a well-founded (index) partial order $(W, <_W)$ of cardinality δ, which has a least and largest elements, denoted c and m respectively, and such that for $Q = W \backslash \{m, c\}$, $<_Q = (Q \times Q) \cap <_W$ the following holds:

$$\mathfrak{b}(Q) = \beta, \mathfrak{d}(Q) = \delta, \forall b \in Q(|b_{\uparrow Q}| \geq \delta).$$

Fix $(W, <_W)$, $(Q, <_Q)$ and let $Q' = Q \cup \{m\}$, $<_{Q'} = (Q' \times Q') \cap <_W$.

The bookkeeping

1. Fix a surjective bookkeeping function $F : Q \to \beta$ such that for all $\alpha \in \beta$, $F^{-1}(\alpha)$ is cofinal in Q. That is

$$\forall \alpha < \beta \forall b \in Q(b_{\uparrow_Q} \cap F^{-1}(\alpha) \neq \varnothing).$$

   Note F exists, since $|Q| = \delta \geq \beta$ and $\forall b \in Q(|b_{\uparrow_Q}| \geq \delta)$.

2. Let $<$ denote the product order on $(\beta + 1) \times W$. That is $(\alpha_0, a_0) < (\alpha_1, a_1)$ iff $\alpha_0 \in \alpha_1$ and $a_0 <_W a_1$, or $\alpha_0 = \alpha_1$ and $a_0 <_W a_1$.

Definition

For each (α, a) in $(\beta+1) \times W$ define recursively a partial order $P_{\alpha,a}$ and take $V_{\alpha,a} = V^{P_{\alpha,a}}$. For each $\alpha \leq \beta$ let $P_{\alpha,c} = \mathbb{H}_\alpha$. Let $(\alpha, a) \in (\beta+1) \times Q'$. Suppose

1. For each $(\gamma, b) <_{lex} (\alpha, a)$ the poset $P_{\gamma,b}$ has been defined.
2. If $b \neq c$, also a $P_{\gamma,c}$-name $\dot{T}_{\gamma,b}$ for a forcing notion is given, so that

$$P_{\gamma,b} = P_{\gamma,c} * \dot{T}_{\gamma,b}.$$

   and whenever $(\alpha_0, a_0) < (\alpha_1, a_1) < (\alpha, a)$, $c \neq a_0$ then

$$\Vdash_{P_{\alpha_1,c}} \dot{T}_{\alpha_0,a_0} \lessdot \dot{T}_{\alpha_1,a_1}.$$

Then in particular, for each $(\alpha_0, a_0) < (\alpha_1, a_1) \leq (\alpha, a)$ we have $P_{\alpha_0,a_0} \lessdot P_{\alpha_1,a_1}$.

We proceed to define $P_{\alpha,a}$. In $V_{\alpha,c}$ let $T_{\alpha,a}$ be the poset of all functions p with

1. $\mathrm{dom}(p) = Q'_a$ and
2. for each $b \in Q'_a$ with $F(b) \geq \alpha$, $\Vdash_{T_{\alpha,b}} p(b) \in \{\emptyset\}$
3. for each $b \in Q'_a$ with $F(b) < \alpha$, $\Vdash_{T_{\alpha,b}} p(b) \in \mathbb{H}(\dot{H}^\alpha_b)$, where

$$\dot{H}^\alpha_b \text{ is a } T_{\alpha,b} \text{ name for } V^{F(b),b} \cap {}^\kappa\kappa.$$

4. $\mathrm{supp}(p) = \{b \in Q'_a, F(b) < \alpha : \Vdash_{T_{\alpha,b}} p(b) \neq 1_{\mathbb{H}(\dot{H}^\alpha_b)}\}$ is of size $< \kappa$.

The $T_{\alpha,a}$ extension relation is defined as follows: $p \leq_{T_{\alpha,a}} q$ iff

$$\operatorname{supp}(q) \subseteq \operatorname{supp}(p) \text{ and}$$

for each $b \in \operatorname{supp}(q)$, if $b \in Q'_a, F(b) < \alpha$ then

$$p \restriction b \Vdash_{T_{\alpha,b}} p(b) \leq_{\mathbb{H}(\dot{H}^\alpha_b)} q(b),$$

where $p \restriction b := p \restriction Q'_b$ and w.l.o.g. we can assume

$$p(b) = (s^p_b, \dot{f}^p_b)$$

where the stem s^p_b is in the ground model and $\dot{f}^p_b$ is a nice $T_{\alpha,b}$-name for a κ-real in $V^{P_{F(b),b}} \cap {}^\kappa\kappa$. Finally, define

$$P_{\alpha,a} = P_{\alpha,c} * \dot{T}_{\alpha,a}.$$

Properties of the poset

1. If $\alpha \leq \alpha' \leq \beta$ and $a \in Q'$, then $V_{\alpha',c} \vDash T_{\alpha,a} \lessdot T_{\alpha',a}$.
2. $\forall b \in W \; \forall \alpha < \alpha' \leq \beta \; (P_{\alpha,b} \lessdot P_{\alpha',b})$.

Thus, altogether we have

$$\forall \alpha, \alpha' \leq \beta \; \forall a, b \in W \; (\alpha \leq \alpha' \wedge a <_W b \rightarrow P_{\alpha,a} \lessdot P_{\alpha',b}).$$

Remark

In $V_{\alpha,c}$ the following holds: Let $p, q \in T_{\alpha,a}$ for some $a \in Q'$ be such that

$$\text{for each } b \in \operatorname{supp}(q) \cap \operatorname{supp}(p), s_b^p \subseteq s_b^q \vee s_b^p \supseteq s_b^q.$$

Then p, q are compatible, with a common extension $r \in T_{\alpha,a}$ where:

1. $\operatorname{supp}(r) = \operatorname{supp}(p) \cup \operatorname{supp}(q)$ and
2. $\Vdash_{T_{\alpha,b}} r(b) = p(b)$ if $b \in \operatorname{supp}(p) \setminus \operatorname{supp}(q)$
3. $\Vdash_{T_{\alpha,b}} r(b) = q(b)$ if $b \in \operatorname{supp}(q) \setminus \operatorname{supp}(p)$
4. $\Vdash_{T_{\alpha,b}} r(b) = (s_b^r, \dot{f}_b^r)$ if $b \in \operatorname{supp}(p) \cap \operatorname{supp}(q)$, where

$$s_b^r = s_b^p \cup s_b^q$$

and $\dot{f}_b^r$ is a $T_{\alpha,b}$-name for the pointwise maximum of $\dot{f}_b^q$ and $\dot{f}_b^p$.

Lemma

Let $\alpha \leq \beta$, $a \in W$. Then $P_{\alpha,a}$ is κ^+-c.c. and is κ-closed.

Lemma

Suppose $b \in W$, then the following hold:

1. If $p \in P_{\beta,b}$ then $p \in P_{\alpha,b}$ for some $\alpha < \beta$.
2. If $\dot{f}$ is a $P_{\beta,b}$-name for a κ-real then it is a $P_{\alpha,b}$-name for some $\alpha < \beta$.

Definition (Strong diagonalization)

Let $M \subseteq N$ be models of ZFC, $\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\kappa]^\kappa$ and $A \in N \cap [\kappa]^\kappa$. Then we say

$$\star(M, N, \mathcal{B}, A)$$

holds, if for every $h \in M \cap {}^{\kappa \times [\gamma]^{<\kappa}}\kappa$ and $m \in \kappa$ we can find $n \geq m$, $F \in [\gamma]^{<\kappa}$ satisfying

$$[n, h(n, F)) \setminus \bigcup_{\alpha \in F} B_\alpha \subseteq A.$$

Lemma (Brendle,F.)

If $\star(M, N, \mathcal{B}, A)$ holds, then for every $B \in M \cap ([\kappa]^\kappa \setminus I(\mathcal{B}))$ we have $|A \cap B| = \kappa$.

Lemma (Brendle, F.)

If $G_{\gamma+1}$ is $\mathbb{H}_{\gamma+1}$-generic, $G_\gamma = G_{\gamma+1} \cap \mathbb{H}_\gamma$ and $\mathcal{A}_\gamma = \{A_\alpha\}_{\alpha<\gamma}$ where

$$A_\alpha = \{i : \exists p \in G_{\gamma+1}\ p(\alpha, i) = 1\}$$

for each $\alpha \leq \gamma$, then

$$\star(V[G_\gamma], V[G_{\gamma+1}], \mathcal{A}_\gamma, A_\gamma).$$

Lemma (Brendle, F.)

Let $M \subseteq N$ be models of ZFC, $P \in M$ a forcing poset such that $P \subseteq M$, G a P-generic filter over N (hence also P-generic over M). Then if

$$\mathcal{B} = \{B_\alpha\}_{\alpha<\gamma} \subseteq M \cap [\kappa]^\kappa, A \in N \cap [\kappa]^\kappa \text{ and } \largestar(M, N, \mathcal{B}, A)$$

holds, then

$$\largestar(M[G], N[G], \mathcal{B}, A).$$

Lemma

$\forall b \in W \ \forall \alpha < \beta \ (\largestar(V_{\alpha,b}, V_{\alpha+1,b}, \mathcal{A}_\alpha, A_\alpha)).$

Theorem (Bag, F.)

$$V_{\beta,m} \vDash \mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta \leq \mathfrak{d}_\kappa = \delta.$$

$\mathfrak{b}_\kappa \geq \beta$: Let $B \subseteq V_{\beta,m} \cap {}^\kappa\kappa$ be such that $|B| < \beta$.

- By $\mathfrak{b}(Q) = \beta$ and counting nice names

$$\exists b \in Q \, \exists \alpha < \beta \, (B \subseteq V_{\alpha,b} \cap {}^\kappa\kappa).$$

- As $\forall \gamma < \beta \; \forall c \in Q \, (c \uparrow_Q \cap F^{-1}(\gamma) \neq \emptyset)$

  there is $b' \in Q$ with $b < b'$ and $F(b') = \alpha$.

- Then $P_{\alpha+1,b'}$ adds a dominating κ-real over

$$V_{\alpha,b'} \cap {}^\kappa\kappa \supseteq V_{\alpha,b} \cap {}^\kappa\kappa,$$

  hence B is not unbounded.

$\mathfrak{a}_\kappa \leq \beta$: The family $\mathcal{A}_\beta = \{A_\alpha : \alpha < \beta\}$ added in the first column is κ-mad in $V_{\beta,m}$.

Suppose note:

- Then $\exists x \in V_{\beta,m} \cap [\kappa]^\kappa \; \forall A_\alpha \in \mathcal{A}_\beta \; (|x \cap A_\alpha| < \kappa)$.
- But then $\exists \alpha < \beta \; (x \in V_{\alpha,m} \cap [\kappa]^\kappa)$.
- However $\star(V_{\alpha,m}, V_{\alpha+1,m}, \mathcal{A}_\alpha, A_\alpha)$ holds and so

$$|A_\alpha \cap x| = \kappa,$$

  which is a contradiction.

Thus

$$V_{\beta,m} \vDash \beta \leq \mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa \leq \beta.$$

$\delta \geq \mathfrak{d}_\kappa$: Let $\dot{f}$ be a $P_{\beta,m}$-name for a κ-real.

- Since $\mathfrak{b}(Q) = \beta \geq \kappa^+$ and β is regular,

$$\text{there are } b \in Q, \alpha < \beta \text{ such that } f \in V_{\alpha,b} \cap {}^\kappa\kappa.$$

- Let $D \subseteq Q$ be dominating of size δ and let $d \in D$ be such that $b <_Q d$.
- As $\forall \gamma < \beta\ \forall c \in Q\ (c\uparrow_Q \cap F^{-1}(\gamma) \neq \emptyset)$,

$$\exists d_{\alpha,b} \in Q \text{ such that } d_{\alpha,b} > d, F(d_{\alpha,b}) = \alpha.$$

- Then $P_{\alpha+1,d_{\alpha,b}}$ adds a dominating real $g^{d_{\alpha,b}}$ over $V_{\alpha,d_{\alpha,b}} \supseteq V_{\alpha,b}$.
- Thus $\{g^{d_{\alpha,b}} : d \in D, \alpha \in \beta\}$ is dominating.
- It remains to observe that $\{g^{d_{\alpha,b}} : d \in D, \alpha \in \beta\}$ is of size $\delta \cdot \beta = \delta$.

$\delta \leq \mathfrak{d}_\kappa$:

- For each $a \in Q$ and $P_{\beta,m}$-generic filter G, let

$$f_G^a = \bigcup\{t_a : \exists p \in G\ (p(a) = (t_a, \dot{f}_a))\}.$$

- (Claim) If $g \in V_{F(a),a}$ and $b \nleq_Q a$, then $V_{\beta,m} \models f_G^b \nleq^* g$.
- Now, let $F \subseteq V_{\beta,m} \cap {}^\kappa\kappa$ be of size less than δ.
- Note that for each $f \in F$ there is $a_f \in Q$ such that $f \in V_{F(a_f),a_f} \cap {}^\kappa\kappa$.
- Now $|\{a_f : f \in F\}| < \delta$, so $\{a_f : f \in F\}$ is not dominating in Q.
- Hence $\exists u \in Q\ \forall f \in F\ (u \nleq_Q a_f)$.
- By the above Claim we have $\forall f \in F\ (f_G^u \nleq^* f)$.
- Hence F is not dominating. □

Theorem (Bag, F.)

Let κ be an infinite regular cardinal, $\kappa^\kappa = \kappa$. Assume GCH at and above κ. If β, δ, μ are infinite cardinals with

$$\kappa^+ \leq \beta = \mathrm{cof}(\beta) \leq \mathrm{cof}(\delta) \leq \delta \leq \mu \text{ and } \mathrm{cof}(\mu) > \kappa,$$

then there is a κ^+-c.c. and κ-closed generic extension in which

$$\mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \beta, \mathfrak{d}_\kappa = \delta \text{ and } \mathfrak{c}_\kappa = \mu.$$

Proof

In the above construction replace $(Q,<_Q)$ by the following poset $(R,<_R)$:

- R consists of pairs (p,i) such that either $i=0 \wedge p \in \mu$ or $i=1 \wedge p \in Q$.
- The order relation is defined as $(p,i) <_R (q,j)$ iff $i=0 \wedge j=1$ or $i=j=1 \wedge p <_Q q$ or $i=j=0 \wedge p<q$ in μ.
- Then $\mathfrak{b}(R)=\mathfrak{b}(Q)=\beta$ and $\mathfrak{d}(R)=\mathfrak{d}(Q)=\delta$ as the map $i: Q \to R$ defined as $b \mapsto (1,b)$ is a cofinal embedding from Q into R.
- The bottom part $(\mu,\in)$ of R ensures that in the final model $\mathfrak{c}_\kappa \geq \mu$ holds.
- Counting nice names implies $\mathfrak{c}_\kappa \leq \mu$ in $V_{\beta,m}$. $\square$

Since $\mathfrak{b}_\kappa = \kappa^+$ implies that $\mathfrak{a}_\kappa = \kappa^+$ for κ regular uncountable, in the Easton extension

$$\mathfrak{b}_\kappa = \mathfrak{a}_\kappa = \kappa^+ < \mathfrak{d}_\kappa = \mathfrak{c}_\kappa$$

holds simultaneously for all $\kappa \in \mathrm{dom}(E)$. Note:

Theorem (Bag, F., Friedman)

In the Easton extension

$$\forall \kappa \in \mathrm{dom}(E)\Big(\mathfrak{sp}(\mathfrak{a}_\kappa) = \{\kappa^+, 2^\kappa\}\Big).$$

Remark

Under some restrictions, one can have a fine global control over $\mathfrak{sp}(\mathfrak{a}_\kappa)$.

Problem

Each of the following constellations in the countable setting is open:

- $\mathfrak{b} < \mathfrak{a} < \mathfrak{d} < \mathfrak{c}$,
- $\mathfrak{b} < \mathfrak{s} < \mathfrak{d} < \mathfrak{c}$,
- $\mathfrak{b} < \mathfrak{r} < \mathfrak{d} < \mathfrak{c}$,
- $\mathfrak{b} < \mathfrak{u} < \mathfrak{d} < \mathfrak{c}$?

Question

Given a set C of regular uncountable cardinals is it consistent that

$$\kappa^+ < \mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa < \mathfrak{d}_\kappa < \mathfrak{c}_\kappa$$

for all $\kappa \in C$ simultaneously?

Let κ be regular uncountable. Then

- (Zapletal) $\mathfrak{s}(\kappa) \geq \kappa$ iff κ is inaccessible, and
- (Suzuki) $\mathfrak{s}(\kappa) > \kappa$ iff κ is weakly compact.

Strong witnesses

Definition

1. A sequence $\langle a_\xi : \xi < \lambda \rangle$, where each a_ξ is in $[\kappa]^\kappa$, is *κ-eventually splitting* if $\forall a \in [\kappa]^\kappa\ \exists \xi < \lambda\ \forall \eta > \xi\ a_\eta$ splits a.
2. A sequence $\langle a_\xi : \xi < \lambda \rangle$, where each a_ξ is in $[\kappa]^\kappa$, is *κ-eventually narrow* if $\forall a \in [\kappa]^\kappa\ \exists \xi < \lambda\ \forall \eta > \xi\ a \not\subseteq^* a_\eta$.

Note that $\tau = \langle a_\xi : \xi < \lambda \rangle$ is κ-eventually splitting iff $\tau' = \langle b_\xi : \xi < \lambda \rangle$, defined as $b_{2\xi} = a_\xi$ and $b_{2\xi+1} = \kappa \setminus a_\xi$, is κ-eventually narrow.

Theorem (F., Bag)

Assume (GCH), let κ be strongly inaccessible and let λ bel such that $\mathrm{cof}(\lambda) > \kappa$. Then every κ-eventually narrow sequence

$$\tau = \langle a_\xi : \xi < \lambda \rangle$$

remains κ-eventually narrow in $V^{\mathbb{H}}$.

Definition

Let κ be regular uncountable.

1. If $\mathcal{A} \subseteq \mathcal{P}(\kappa)$, then $\mathcal{A}$ has the *strong intersection property (SIP)* if $\forall \mathcal{A}' \in [\mathcal{A}]^{<\kappa}\ [|\bigcap \mathcal{A}'| = \kappa]$.
2. $X \subseteq \kappa$ is a *pseudo-intersection of* $\mathcal{A}$ if $X \subseteq^* A$ for any $A \in \mathcal{A}$.

Definition

1. The *generalized pseudo-intersection number* $\mathfrak{p}(\kappa)$ is the minimal size of a family $\mathcal{A} \subseteq [\kappa]^\kappa$ with the SIP but no pseudo-intersection.
2. The invariant $\mathfrak{p}_{cl}(\kappa)$ is the minimal size of a family $\mathcal{A} \subseteq [\kappa]^\kappa$ of clubs (closed and unbounded sets) in κ having no pseudo-intersection.

- Note that $\mathfrak{p}_{cl}(\kappa) = \mathfrak{b}(\kappa)$ regular uncountable κ.
- (F., Montoya, Schilhan and Soukup) Consistently

$$\mathfrak{p}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = (\mathfrak{p}_{cl}(\kappa))$$

Definition

Let $\mathcal{C}$ denote the collection of all clubs in κ.

- $\mathbb{M}(\mathcal{C})$ consists of all pairs (a, C), where $a \in [\kappa]^{<\kappa}$ and $C \in \mathcal{C}$.
- The order is given by $(a', C') \leq (a, C)$ if $C' \subseteq C$ and $a' \setminus a \subseteq C$.

Definition

1. Let λ be an ordinal. A cardinal κ is λ-strongly unfoldable iff
    1. κ is strongly inaccessible
    2. for every κ-model M there is an elementary embedding $j : M \to N$ with critical point κ such that $\lambda < j(\kappa)$ and $V_\kappa \subseteq N$.
2. A cardinal κ is called strongly unfoldable if it is θ-strongly unfoldable for every ordinal θ.

Theorem (Johnstone)

Let κ be strongly unfoldable. Then there is a set forcing extension where the strong unfoldability of κ is indestructible by forcing notions of any size which are $< \kappa$-closed and have the κ^+-c.c..

Theorem (F., Bag)

Let κ be a strongly unfoldable cardinal, $2^\kappa = \kappa^+$ and let $\lambda > \kappa^+$ be a regular uncountable cardinal. Then there is a set forcing generic extension, in which

$$\mathfrak{s}(\kappa) = \kappa^+ < \mathfrak{b}(\kappa) = \mathfrak{c}(\kappa) = \lambda.$$

Let V_0 be the ground model and let V be the $\mathbb{P}^*$-generic extension of V_0, where $\mathbb{P}^*$ is the lottery preparation of κ.

- Then, κ remains strongly unfoldable in any further generic extensions obtained by $< \kappa$-closed, κ^+-cc forcing notions.
- Note that $\mathbb{P}^*$ is κ-cc, of size κ, and so $2^\kappa = \kappa^+$ in V.
- As κ is strongly unfoldable (in particular strongly inaccessible) in V, $V \vDash \kappa^{<\kappa} = \kappa$ holds as well.

- Let $\mathbb{C}_{\kappa^+}$ be the $<\kappa$-support product of κ^+-many copies of the Cohen forcing $2^{<\kappa}$.
- We first add κ^+-many Cohen subsets of κ, $\langle y_\alpha : \alpha < \kappa^+ \rangle$ by forcing with $\mathbb{C}_{\kappa^+}$ and then iteratively diagonalize the club filter for λ-many steps.
- Thus the poset that we are forcing with is $\mathbb{P} = \mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$, where $\dot{\mathbb{M}}(\mathcal{C})_\lambda$ is a $\mathbb{C}_{\kappa^+}$-name for the $<\kappa$-support iteration of $\mathbb{M}(\mathcal{C})$ of length λ.

Properties (F., Montoya, Schilhan, Soukup)

1. This forcing $\mathbb{P}$ has the κ^+-c.c., is κ-closed and forces that $\mathfrak{c}(\kappa) = \lambda$.
2. The set of conditions in $\mathbb{M}(\mathcal{C})_\lambda$ of the form $(\bar{a}, q)$, where
   - i) $\bar{a}$, called the sequence of the stems, is of the form $\langle a_\beta : \beta \in I \rangle$ for some $I \in [\lambda]^{<\kappa}$ and $a_\beta \in [\kappa]^{<\kappa}$ for each $\beta \in I$,
   - ii) q is a function with domain I and for each $\beta \in I$, $q(\beta)$ is a $\mathbb{M}(\mathcal{C})_i$-name for a club,

   is a dense subset of $\mathbb{M}(\mathcal{C})_\lambda$.
3. The set of conditions in $\mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$ of the form $(p, \bar{a}, \dot{q})$, where
   - i) $p \in \mathbb{C}_{\kappa^+}$ and $\bar{a} \in V$,
   - ii) $\dot{q}$ is a $\mathbb{C}_{\kappa^+}$-name for a sequence as in (2, ii).

   is dense in $\mathbb{C}_{\kappa^+} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$.

Remark

1. A nice $\mathbb{M}(\mathcal{C})_\lambda$-name $\dot{x}$ for a subset of κ has the form

$$\bigcup_{\alpha<\kappa} A_\alpha \times \{\check{\alpha}\}$$

   where

   - A_α is an antichain in $\mathbb{M}(\mathcal{C})_\lambda$ and
   - for each $(\bar{a}, q) \in A_\alpha$, $\beta \in \mathrm{dom}(q)$, $q(\beta)$ is a nice $\mathbb{M}(\mathcal{C})_\beta$-name.

2. Thus, nice $\mathbb{M}(\mathcal{C})_\beta$-names are defined by induction on $\beta \in \lambda$.

Claim

If $\dot{x}$ is a nice $\mathbb{M}(\mathcal{C})_\lambda$-name for a subset of κ, then $|\mathrm{trcl}(\dot{x})| \le \kappa$.

Proof

This is seen by induction on the length:

- Suppose the claim holds for nice $\mathbb{M}(\mathcal{C})_\beta$-names for every $\beta < \gamma$ and $\dot{x}$ is a nice $\mathbb{M}(\mathcal{C})_\gamma$-name.
- Then by the definition of nice names, $\dot{x}$ is of the form $\bigcup_{\alpha<\kappa} A_\alpha \times \{\check{\alpha}\}$, where A_α is an antichain in $\mathbb{M}(\mathcal{C})_\gamma$ (thus of size $\le \kappa$).
- For every $(\bar{a}, q) \in A_\alpha$, $|\mathrm{dom}(q)| < \kappa$ and for each $\beta \in \mathrm{dom}(q)$, $q(i)$ is a nice $\mathbb{M}(\mathcal{C})_\beta$-name, which was assumed to have transitive closure of size at most κ. □

1. In the generic extension by $\mathbb{P}$ also $\mathfrak{b}(\kappa) = \mathfrak{p}_{cl}(\kappa) = \lambda$ holds:
   - Let F be a family of clubs in κ of size $< \lambda$.
   - By regularity of λ, F appears at an earlier stage i of the iteration.
   - Thus, the next iterand $\mathbb{M}(\mathcal{C})$ adds a pseudo-intersection of the clubs of V_i, so in particular, a pseudo-intersection of F.
2. $\mathbb{P}$ does not destroy the strongly unfoldability of κ. Thus $V^{\mathbb{P}} \models \mathfrak{s}(\kappa) \geq \kappa^+$.

So it is sufficient to find a splitting family of size κ^+. We will show that the Cohen reals

$$\bar{y} = \langle y_\alpha : \alpha < \kappa^+ \rangle$$

build up such a family.

Claim

$\bar{y} = \{y_\alpha : \alpha < \kappa^+\}$ is splitting in $V^{\mathbb{M}(\mathcal{C})_\lambda}$.

Proof

- Let $\dot{x}$ be a nice $\mathbb{M}(\mathcal{C})_\lambda$-name for a κ-real in $V^{\mathbb{C}_{\kappa^+}} = V[\bar{y}]$.
- There is a $\gamma < \kappa^+$ such that $\dot{x} \in V[\langle y_\alpha : \alpha < \kappa, \alpha \neq \gamma \rangle]$.
- We show that the κ-Cohen real y_γ splits $\dot{x}$.
- W.l.o.g. $\dot{x} \in V$ and we are adding a single Cohen κ-real $y_\gamma = y$ over V (by letting $V = V[\langle y_\alpha : \alpha < \kappa^+, \alpha \neq \gamma \rangle]$ be the new ground model).

Claim

Then $V[y] \vDash (\Vdash_{\mathbb{M}(\mathcal{C})_\lambda}$ "$\check{y}$ splits $\dot{x}$").

Proof

- Suppose not. Then $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \setminus \varepsilon \subseteq \dot{y}$ or $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \cap \dot{y} \subseteq \varepsilon$ for some $\varepsilon \in \kappa$ and $(p, \bar{a}, \dot{q}) \in \mathbb{C} * \dot{\mathbb{M}}(\mathcal{C})_\lambda$.
- Suppose $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \setminus \varepsilon \subseteq \dot{y}$.
- Let y be the $\mathbb{C}$-generic over V with p in the generic filter, i.e. $p \subseteq y$.
- Define $y' \in 2^\kappa$ by letting $y'(i) = p(i) = y(i)$ for $i \in \mathrm{dom}(p)$ and $y'(i) = 1 - y(i)$ otherwise.
- Then $V[y] = V[y'] =: W$, but possibly $q := \dot{q}[y] \neq \dot{q}[y'] = q'$.
- In W, $(\bar{a}, q)$ and $(\bar{a}, q')$ are compatible, because their stems are the same. Let $(\bar{a}, r) \in \mathbb{M}(\mathcal{C})_\lambda$ be their common extensions.
- Now let $(\bar{b}, s) \leq (\bar{a}, r)$ and $\delta \in \kappa \setminus \bigcup\{\varepsilon, \mathrm{dom}(p)\}$ be such that $(\bar{b}, s) \Vdash \delta \in \dot{x}$.

- As $y' \cap y \subseteq \varepsilon$ we have $\delta \notin y$ or $\delta \notin y'$.
- Suppose $\delta \notin y$, then whenever G is $\mathbb{M}(\mathcal{C})_\lambda$-generic over W containing $(\bar{b}, s)$, $W[G] \vDash \dot{x}[G] \setminus \varepsilon \nsubseteq y$.
- This is a contradiction because $(p, \bar{a}, q)$ is in the corresponding $\mathbb{C} * \mathbb{M}(\mathcal{C})_\lambda$-generic over V. Similarly for $\delta \notin y'$.
- So suppose $(p, \bar{a}, \dot{q}) \Vdash \dot{x} \cap \dot{y} \subseteq \varepsilon$.
- Then again as $y' \cap y \subseteq \varepsilon$ we have $\delta \in y$ or $\delta \in y'$.
- Suppose $\delta \in y$, then whenever G is $\mathbb{M}(\mathcal{C})_\lambda$-generic over W containing $(\bar{b}, s)$, $W[G] \vDash \dot{x}[G] \cap y \nsubseteq \varepsilon$.
- This is a contradiction because $(p, \bar{a}, q)$ is in the corresponding $\mathbb{C} * \mathbb{M}(\mathcal{C})_\lambda$-generic over V.
- Similarly for the case $\delta \in y'$. □

Theorem (F., Mejia, 2025*)

Let κ be a supercompact. It is relatively consistent that

$$\kappa^+ < \mathfrak{s}(\kappa) < \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) < \mathfrak{r}(\kappa) = 2^\kappa.$$

Thank you for your attention!