Quantum analytic Langlands correspondence from probability theory?

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Context and motivation

Probability theory has lead to game-changing progress
on conformal field theories (CFTs) such as Liouville theory.

This sheds light on some of the deepest questions on quantum field theory:

How to construct them **non-perturbatively**, and
how to make **concrete predictions** about them.

This talk is devoted to a **mathematical application:**

A new branch of the geometric Langlands program,
generalising the **analytic Langlands correspondence**
(Etingof, Frenkel, Kazhdan).

Gist of the analytic Langlands-correspondence

Analytic Langlands-correspondence conjecture[1]: A correspondence between two types of geometric objects associated to a Riemann surface C

$$\boxed{(\mathbf{A})\ \left\{\begin{matrix}\text{single-valued } \&\\ \text{square-integrable}\end{matrix}\right\}\ \begin{matrix}\text{eigenfunctions of}\\ \text{Hitchin's Hamiltonians}\end{matrix}} \quad \leftrightarrow \quad \boxed{(\mathbf{B})\ \text{Real opers}}$$

Opers: Differential **oper**ators on C of the form $\partial_z^2 + t(z)$, $t(z) = \frac{1}{2}\{A, z\}$.[2]

Analytic continuation of local solutions along closed curves γ

$$(\partial_z^2 + t(z))\chi_i(z) = 0, \qquad \chi_i(\gamma.z) = \sum_{j=1}^{2} M_{ij}(\gamma)\chi_j,$$

defines **monodromy** of oper.

(B) Real opers: Opers with real monodromy (rare! $M_{ij}(\gamma)$ generically complex).

Correspondence **(A)** $\leftrightarrow$ **(B)**: **geometric** description of solution to spectral problem.

[1] After Etingof-Frenkel-Kazhdan '19; strengthens previous proposal of J.T. '17

[2] $\{A, z\} = A'''/A' - \frac{3}{2}(A''/A')^2$ is the Schwarzian derivative of $A(z)$ we have seen in many other talks.

Hitchin's moduli spaces

Hitchin moduli space $\mathcal{M}_{\rm Hit}(C)$ (C: Riemann surface): Moduli space of pairs $(\mathcal{E},\varphi)$,

- $\mathcal{E}$: holomorphic $G=SL(2)$-bundle on C,
- $\varphi\in H^0(C,\mathrm{End}(\mathcal{E})\otimes K)$.

More concretely:

- Describe $\mathcal{E}$ by cover $U_0\cup\bigcup_{k=1}^N U_k$, U_k: small discs around points $P_k\in C$,

$$\text{transition functions}\;\; g_{0k}=\begin{pmatrix}1 & -x_k\\ 0 & z(P)-z(P_k)\end{pmatrix},\quad k=1,\dots,N,$$

- and describe φ by holomorphic matrix-valued differentials,

$$\varphi_0=\begin{pmatrix}a_0 & b_0\\ c_0 & -a_0\end{pmatrix}dz\;\text{ on }\;U_0,\qquad \varphi_k=g_{0k}\cdot\varphi_0\cdot g_{0k}^{-1},\;\text{ on }\;U_k,$$

  where c_0 can be expanded into "standard functions"[3] $c_0=\sum_{k=1}^N p_k\eta_k(z)$.

The tuples $\boldsymbol{x}=(x_1,\dots,x_N)$ and $\boldsymbol{p}=(x_1,\dots,x_N)$ give coordinates on $\mathcal{M}_{\rm Hit}(C)$.

[3] Technically η_k: sections of line bundle defined by $P_1,\dots,P_N$.

Hitchin's integrable system

Integrability: There exists a canonical Poisson structure on $\mathcal{M}_{\mathrm{Hit}}(C)$, and

$$q(z) = \frac{1}{2}\mathrm{tr}(\varphi^2(z)) = \sum_{r=1}^{3g-3+n} H_r Q_r(z) \in H^0(C, K^2), \qquad \boxed{\{H_r, H_s\} = 0.}$$

Local form:

$$H_r = \sum_{st} C_r^{st}(\boldsymbol{x}) p_s p_t.$$

Quantisation (Beilinson-Drinfeld): There exist differential operators

$$\mathsf{H}_r = -\hbar^2 \sum_{st} C_r^{st}(\boldsymbol{x}) \frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s} + \text{lower degree}, \qquad r = 1, \ldots, 3g-3+n,$$

on $K_{\mathrm{Bun}}^{1/2}$, Bun_G: moduli space of holomorphic $G = SL(2)$-bundles, satisfying

$$[\,\mathsf{H}_r, \mathsf{H}_s\,] = 0, \qquad r, s = 1, \ldots, d, \qquad d = 3g-3+n.$$

Analytic Langlands correspondence

We may consider the pair of complex conjugate eigenvalue equations

$$\mathsf{H}_r\,\Psi = E_r\Psi, \qquad \bar{\mathsf{H}}_r\,\Psi = \bar{E}_r\Psi, \qquad r = 1,\dots,d. \tag{1}$$

We may then look for smooth[4] solutions in $|K_{\rm Bun}| \equiv K_{\rm Bun}^{1/2} \otimes \bar{K}_{\rm Bun}^{1/2}$, locally of the form

$$\Psi(\boldsymbol{x},\bar{\boldsymbol{x}}) = \sum_{k,l} C_{kl}\psi_k(\boldsymbol{x})\bar{\psi}_l(\bar{\boldsymbol{x}}),$$

($\boldsymbol{x} = (x_1,\dots,x_d)$: local coordinates on $\mathrm{Bun}_G(C)$), which are

single-valued[5], and furthermore **square-integrable**[6].

Conjecture:

$$\boxed{\left\{\begin{array}{l}\text{single-valued } \& \\ L^2\text{-normalisable}\end{array}\right\} \text{ Hitchin eigenfunctions}} \quad \leftrightarrow \quad \boxed{\text{Real opers}}$$

[4] Away from "wobbly" bundles admitting nilpotent Higgs fields.
[5] J.T., 2011, 2017.
[6] Etingof-Frenkel-Kazhdan, 2019.

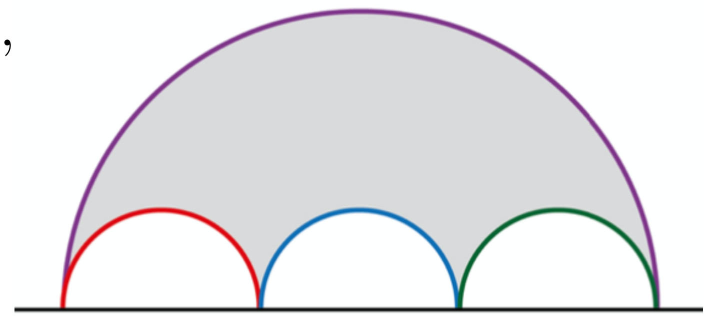
Real opers

A **real** oper is an oper with real **monodromy**. Real opers are in one-to-one correspondence to a certain class of hyperbolic metrics on C, for example:

$$(\partial_u^2 + t_0(u))\phi(u,\bar{u}) = 0, \quad (\bar{\partial}_{\bar{u}}^2 + \bar{t}_0(\bar{u}))\phi(u,\bar{u}) = 0,$$

$$\rightsquigarrow ds^2 = \frac{du d\bar{u}}{(\phi(u,\bar{u}))^2} \text{ has } R = -1,$$

$\rightsquigarrow$ uniformisation, e.g. for $C = C_{1,1}$:



Other real opers obtained by **grafting**.

Classification of real opers[7]:

All real opers can be obtained in this way, classified by counting singularity lines $\simeq$ even[8] half-integer **measured laminations** $\Lambda \in \mathcal{ML}_C^+(\frac{1}{2}\mathbb{Z})$.

Extracting eigenvalues E_r from the real opers: $\boxed{t(z) - t_0(z) = \sum_{r=1}^{3g-3+n} Q_r(z) E_r.}$

[7]W. Goldman; Review of relevant results: Dumas, arXiv:0902.1951.

[8]The natural projection to $H_1(C, \mathbb{Z}/2)$ vanishes.

Separation of variables – making ALC computable

Claim:[9] There exist explicit integral transformations of the form

$$\Psi(\boldsymbol{x}) = \int d^2\boldsymbol{u}\, K(\boldsymbol{x},\boldsymbol{u})\,\Phi(\boldsymbol{u}), \tag{2}$$

$\mathbf{u} = (u_1,\dots,u_d;\bar{u}_1,\dots,\bar{u}_d)$, such that

$$\begin{aligned}
\mathsf{H}_r\,\Psi(\boldsymbol{x}) &= E_r\Psi(\boldsymbol{x}), \\
\bar{\mathsf{H}}_r\,\Psi(\boldsymbol{x}) &= \bar{E}_r\Psi(\boldsymbol{x}),
\end{aligned}
\quad\Leftrightarrow\quad
\begin{aligned}
(\partial_{u_r}^2 + t(u_r))\Phi(\boldsymbol{u}) &= 0, \\
(\bar{\partial}_{\bar{u}_r}^2 + \bar{t}(\bar{u}_r))\Phi(\boldsymbol{u}) &= 0,
\end{aligned}
\qquad r = 1,\dots,d.$$

The proof generalises a paradigm from the theory of quantum integrable models called Separation of Variables (SOV).

$\Phi(\boldsymbol{u})$ and therefore $\Psi(\boldsymbol{x})$ are **single-valued iff** monodromy of $\partial_u^2 + t(u)$ is **real** (conjugate to representation $\rho : \pi_1(C) \to \mathrm{SL}(2,\mathbb{R})$). It follows that

$$\Phi(\boldsymbol{u}) = \prod_{r=1}^{D} \phi(u_r,\bar{u}_r), \qquad
\begin{aligned}
(\partial_u^2 + t(u))\phi(u,\bar{u}) &= 0, \\
(\bar{\partial}_{\bar{u}}^2 + \bar{t}(\bar{u}))\phi(u,\bar{u}) &= 0.
\end{aligned} \tag{3}$$

[9] Ambrosino-J.T. for $g = 0$, Dinh-J.T. for $g > 0$ (in preparation). Building upon work of Sklyanin; Frenkel '95; Enriquez-Feigin-Roubtsov; Enriquez-Roubtsov; Felder-Schorr; Ribault-J.T., Frenkel-Gukov-J.T.

Deformation of Hitchin eigenvalue equations I

Hamiltonians H_r depend on moduli τ_r of $C \Rightarrow$ can define a second dynamics:

$$\text{Deform} \quad \varphi \to \nabla_{\epsilon_2} = \epsilon_2 \partial_z + \varphi, \quad \text{and identify} \quad \boxed{t_r = \frac{1}{\epsilon_2}\tau_r.}$$

The resulting **non-autonomous** dynamical system,

$$\frac{\partial}{\partial \tau_r}\varphi(z) = \{\varphi(z), H_r\},$$

describes **monodromy-preserving** deformations of $(\mathcal{E}, \nabla_{\epsilon_2})$.

Compare the two types of **time-dependent** Schrödinger/heat equations:

a) $\quad i\hbar \dfrac{\partial}{\partial t_r}\Psi_k(\boldsymbol{x}, \boldsymbol{t}; \boldsymbol{\tau}) = \mathsf{H}_r \Psi_k(\boldsymbol{x}, \boldsymbol{t}; \boldsymbol{\tau}),$ $\quad$ solved by spectral decomposition (ALC),

n) $\quad i\hbar\, \epsilon_2 \dfrac{\partial}{\partial \tau_r}\Psi_k(\boldsymbol{x}, \boldsymbol{\tau}) = \mathsf{H}_r \Psi_k(\boldsymbol{x}, \boldsymbol{\tau}),$ $\quad$ or with $\quad \mathcal{D}_r := -\dfrac{1}{\hbar^2}\mathsf{H}_r, \quad k := \dfrac{\epsilon_2}{i\hbar} - 2,$

$$\Rightarrow \quad \boxed{(k+2)\frac{\partial}{\partial \tau_r}\Psi_k(\boldsymbol{x}, \boldsymbol{\tau}) = \mathcal{D}_r \Psi_k(\boldsymbol{x}, \boldsymbol{\tau}) \qquad \textbf{(KZB)}}$$

Deformation of Hitchin eigenvalue equations II

KZB is a natural deformation of Hitchin eigenvalue equations,

$$(k+2)\frac{\partial}{\partial \tau_r}\Psi_k(\boldsymbol{x},\boldsymbol{\tau})=\mathsf{H}_r\Psi_k(\boldsymbol{x},\boldsymbol{\tau})$$

also in the sense that

$$\left[\begin{array}{c} \Psi(\boldsymbol{x},\boldsymbol{\tau})=e^{\frac{1}{k+2}S(\boldsymbol{\tau})}\psi(\boldsymbol{x},\boldsymbol{\tau})+\dots \\ \mathsf{H}_r\Psi(\boldsymbol{x},\boldsymbol{\tau})=(k+2)\frac{\partial}{\partial z_r}\Psi(\boldsymbol{x},\boldsymbol{\tau}) \end{array}\right] \underset{k\to -2}{\Rightarrow} \left[\begin{array}{c} \frac{\partial}{\partial \tau_r}S(\boldsymbol{\tau})=E_r(\boldsymbol{\tau}), \\ \mathsf{H}_r\psi(\boldsymbol{x},\boldsymbol{\tau})=E_r(\boldsymbol{\tau})\psi(\boldsymbol{x},\boldsymbol{\tau}). \end{array}\right]$$

Quantum analytic Langlands conjecture: (Gaiotto, J.T.)

$$\boxed{\left\{\begin{array}{l}\text{single-valued } \& \\ L^2\text{-normalisable}\end{array}\right\} \text{ KZB solutions}} \quad \leftrightarrow \quad \boxed{\begin{array}{l}\text{half-integer measured} \\ \text{laminations } \Lambda\end{array}}$$

Main problem: How to **construct** the relevant solutions to (KZB)?

Probability theory to the rescue

H_3^+-WZNW model[10] (see talk by C. Guillarmou):

$$\Psi_k(\boldsymbol{x},\boldsymbol{\tau}) = \Big\langle \prod_{k=1}^{n} \Phi^{j_k}(x_k|z_k) \Big\rangle_{\mathrm{W}} := \int_{h:\mathbb{C}\to\mathbb{H}_3^+} \mathcal{D}[h]\; e^{-S_{\mathrm{WZ}}[h]} \prod_{k=1}^{n} \phi^{j_k}(h(z_k);x_k),$$

$$\phi^j(h;x) = \frac{2j+1}{\pi}\left((1,-x)\cdot h\cdot \begin{pmatrix}1\\ -\bar{x}\end{pmatrix}\right)^{2j}.$$

For fixed $C = C_{g,n}$: $\Psi_k(\boldsymbol{x},\boldsymbol{\tau}) \sim$ Distinguished density on $\mathrm{Bun}_G(C)$.

Conjectures:

a) $\Psi_k(\boldsymbol{x},\boldsymbol{\tau})$ is single-valued solution to **(KZB).**

b) If $k+2\in i\mathbb{R}$, $\Psi_k(\boldsymbol{x},\boldsymbol{\tau})$: square-integrable section of $|K|^{-\frac{k}{2}}$.

Puzzles:

a) This is **one** solution. How do we generate an infinite family of solutions labelled by measured laminations $\Lambda\in\mathcal{ML}_C^+(\frac{1}{2}\mathbb{Z})$ ($\leftrightarrow$ **real opers**)?

b) Can we define correlation functions at $k+2\in i\mathbb{R}$ by analytic continuation?

[10] Gawedzki-Kupiainen; J.T. 1997, 1999

Address puzzles by mapping to Liouville theory

There exists a one-parameter family of integral transformations[11]

$$\Psi_k(\boldsymbol{x},\boldsymbol{\tau}) = \int d^2\boldsymbol{u}\; \mathcal{K}(\boldsymbol{x},\boldsymbol{u};b)\Phi_b(\boldsymbol{u},\boldsymbol{\tau}), \qquad k+2 = -\frac{1}{b^2}, \tag{4}$$

with coordinates $\boldsymbol{\tau} = (z_1,\dots,z_d)$ for $\mathcal{T}(C)$, intertwining solutions to

KZB-equations

$$(k+2)\frac{\partial}{\partial z_r}\Psi_k(\boldsymbol{x},\boldsymbol{\tau}) \overset{(*)}{=} \mathsf{H}_r\Psi_k(\boldsymbol{x},\boldsymbol{\tau}),$$

representation theory of $\widehat{\mathfrak{sl}}_{2,k}$

BPZ-equations – quantum opers

$$\left(b^2\frac{\partial^2}{\partial u_r^2} + \mathcal{T}_r(u_r)\right)\Phi_b(\boldsymbol{u},\boldsymbol{\tau}) = 0.$$

representation theory of Virasoro algebra

The probabilistic construction of the H_3^+-WZNW model (see talk by C. Guillarmou) can probably establish (4) in a completely different way.

[11] Frenkel-Gukov-J.T. for $g=0$, Dinh-J.T. (in preparation) for $g>0$.
Building on previous work of Feigin-Frenkel-Stoyanovsky; Ribault-J.T.; Hikida-Schomerus.

Single valued solutions to BPZ-equations

... are the Liouville correlation functions[12]:

$$\Phi_b(\boldsymbol{u},\boldsymbol{\tau}) = \left\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r) \prod_{l=1}^{d} V_{-1/2b}(u_l) \right\rangle_{\rm Liou},$$

$$\left\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r) \right\rangle_{\rm Liou} := \int_{\phi:C\to\mathbb{R}} \mathcal{D}[\phi]\; e^{-S_{\rm L}[\phi]} \prod_{r=1}^{n} e^{2\alpha_r\phi(z_r)},$$

$$S_{\rm L} = \frac{1}{4\pi}\int_C d^2x \left(|\nabla\phi(x)|^2 + 4\pi\mu e^{2b\phi(x)} \right).$$

Liouville theory $\sim$ Quantum uniformisation[13]

$$\Phi_b(\boldsymbol{u},\boldsymbol{\tau}) \sim e^{-b^2 S(\boldsymbol{\tau})} \prod_{r=1}^{d} \phi_{\rm uni}(u_r,\bar{u}_r), \qquad \begin{aligned} (\partial_u^2 + t(u))\phi_{\rm uni}(u,\bar{u}) &= 0, \\ (\bar{\partial}_{\bar{u}}^2 + \bar{t}(\bar{u}))\phi_{\rm uni}(u,\bar{u}) &= 0. \end{aligned}$$

such that $\quad ds^2 = \dfrac{du d\bar{u}}{(\phi_{\rm uni}(u,\bar{u}))^2}\quad$ is the **uniformising metric**.

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[12] Rigorously constructed by David, Kupiainen, Rhodes, Vargas in 2014.

[13] Polyakov; Takhtajan-Zograf; Zamolodchikov-Zamolodchikov; J.T. 2003; Vartanov-J.T. 2015.

Quantum grafting $\simeq$ Verlinde line operators

Idea of definition: Insert observables defined from **quantised real opers**.

If γ: simple closed curve on C,

$$\left\langle\, L_\gamma \prod_{r=1}^{n} V_{\alpha_r}(z_r) \,\right\rangle_{\mathrm{Liou}} := \int_{\phi:C\to\mathbb{R}} \mathcal{D}[\phi]\; e^{-S_{\mathrm{L}}[\phi]} \mathcal{L}_\gamma[\phi] \prod_{r=1}^{n} e^{2\alpha_r \phi(z_r)},$$

$$\mathcal{L}_\gamma[\phi] = \mathrm{Tr}(\mathcal{M}_\gamma[\phi]), \qquad \mathcal{M}_\gamma = \mathcal{P}\exp\left(\int_\gamma A\right), \qquad A = \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix},$$

where $t(z) = -(\partial_z\phi)^2 + Q\partial_z^2\phi$. More general **Verlinde line operators**[14]

can be labelled by **measured laminations** $\Lambda \in \mathcal{ML}_C(\frac{1}{2}\mathbb{Z})$!

Key result:

$$\boxed{\text{Limit } b^2 \to \infty \text{ of } L_\Lambda \;\simeq\; \text{grafting along } \Lambda.}$$

[14] E. Verlinde; Alday-Gaiotto-Gukov-Tachikawa-Verlinde; Drukker-Gomis-Okuda-J.T., ...

Analogy with classical Langlands correspondence

The wave functions $\Psi_\Lambda(\boldsymbol{x})$ are good analogs of **automorphic forms**, densities on the double quotient

$$\mathrm{Bun}_G(C) \simeq G_{\mathrm{out}} \backslash G(\!(t)\!) \,/\, G[\![t]\!],$$

labelled by laminations $\Lambda \simeq$ representations of fundamental group associated to real opers $\sim$ analogs of Galois representations.

Analytic Langlands	**Classical Langlands**
$\mathrm{Bun}_G(C) \simeq G_{\mathrm{out}}\backslash G(\!(t)\!)/G[\![t]\!]$	symmetric space $\Gamma \backslash G \,/\, K$
Hitchin Hamiltonians	Laplacians
Hecke operators	Hecke operators
Hitchin/Hecke eigenfunctions	Automorphic forms
Holonomies of real opers	Galois representations

Happy birthday, Antti!