

Backbone exponent and annulus crossing probability for 2D percolation via Liouville quantum gravity

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joint work with Pierre Nolin (CityUHK), Wei Qian (CityUHK), Zijie Zhuang (UPenn)
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Antti's Influence on Me

I was academically raised in an environment filled with 6 letters: SLE, LQG. CFT to me was kind of a trick to make SLE sound more attractive.

More than 10 years ago, I listened to a talk by Antti (at Princeton?) describing a CFT related to LQG called Liouville theory.

I didn't understand anything except remembering him saying that it is a very important and special CFT.

In 2015, at a Newton Institute program, Antti and my adviser Scott Sheffield was arguing about whether the Liouville theory he was talking about is really related what we call LQG. Scott asked me to figure it out.

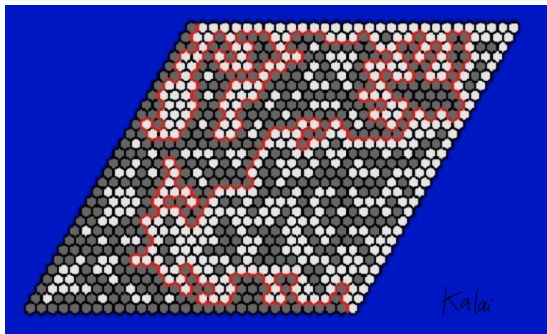
After a decade of work with many people, my conclusion is that:

- there is only one Liouville theory, but two perspectives;
- Combining the two can give new insight on SLE.
- **Most importantly, CFT is a real subject!**

Schramm Loewner Evolution (SLE)

Schramm (1999)

Random interfaces in many 2D statistical physics models should converge to SLE_{κ} with $\kappa > 0$.

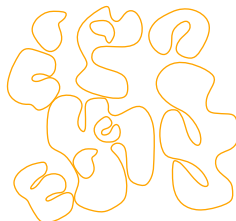
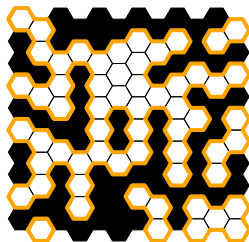


A few scaling limit results, many more conjectures.

Percolation $\rightarrow SLE_6$, Ising model $\rightarrow SLE_3$ (Smirnov et. al.)

Critical Bernoulli percolation

- Bernoulli site percolation on the **triangular lattice**; Color each hexagon either black or white with equal probability independently.

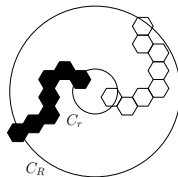


- Full understanding of the scaling limit:
 - Conformal invariance of the quadrangle crossing probability and scaling limit of one interface to **SLE₆** (Smirnov '01).
 - Full scaling limit of interfaces as **CLE₆** (Camia-Newman '06).

Polychromatic arm crossing events and exponents

- C_R : circle of radius R centered at 0.
- Polychromatic j -arm event:

$\mathcal{A}_j(r, R) := \{\exists j \text{ disjoint black/white paths connecting } C_r \text{ to } C_R, \text{ and **not all of the same color** for } j > 1\}.$



Polychromatic 2-arm event $\mathcal{A}_2(r, R)$.

Smirnov-Werner '01, Lawler-Schramm-Werner '02

$\forall j \geq 1,$

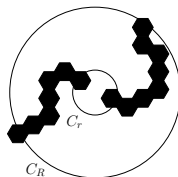
$$\mathbb{P}[\mathcal{A}_j(r, R)] = \left(\frac{r}{R}\right)^{-\alpha_j + o(1)} \quad \text{as } r/R \rightarrow 0$$

with $\alpha_j = \frac{j^2-1}{12}$ for all $j \geq 2$; $\alpha_1 = \frac{5}{48}$.

Monochromatic arm events and exponents

- Monochromatic j -arm event ($j \geq 2$):

$$\mathcal{B}_j(r, R) := \{\exists j \text{ disjoint } \mathbf{black} \text{ paths connecting } C_r \text{ to } C_R\}.$$



Monochromatic 2-arm event $\mathcal{B}_2(r, R)$.

Beffara-Nolin '11

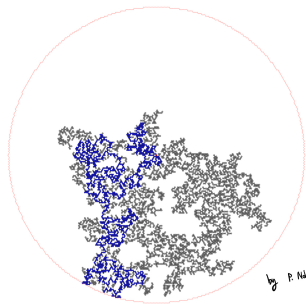
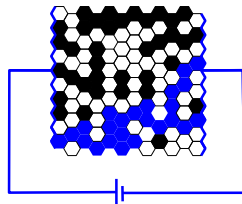
There exists $\beta_j > 0$ such that

$$\mathbb{P}[\mathcal{B}_j(r, R)] = \left(\frac{r}{R}\right)^{-\beta_j + o(1)} \quad \text{as } r/R \rightarrow 0.$$

β_2 : today's topic; β_j ($j \geq 3$): still unsolved.

Backbone exponent β_2

- Monochromatic two-arm exponent β_2 is also called the **backbone exponent**.
- Introduced in physics more than 40 years ago to study electric flow within a critical percolation cluster.
- With high probability, there are $R^{2-\beta_2+o(1)}$ points on the support of electric current.



Support of electric current.

Explicit value of the backbone exponent β_2

Nolin-Qian-S.-Zhuang (2023)

β_2 is the unique solution in the interval $(\frac{1}{4}, \frac{2}{3})$ to the equation

$$\frac{\sqrt{36x+3}}{4} + \sin\left(\frac{2\pi\sqrt{12x+1}}{3}\right) = 0.$$

β_2 is a transcendental number.

$\beta_2 = 0.35666683671288 \dots$

- $\beta_2 = 0.35661 \pm 0.00005$;
Match most recent numerical result, Fang-Ke-Zhong-Deng '22.
- $\beta_2 \neq \alpha_1 + \alpha_2 = \frac{5}{48} + \frac{1}{4} = \frac{17}{48} = 0.354\dots$;
Disprove conjecture by Beffera-Nolin based on earlier numerical work.

Annulus crossing formulae

Cardy has a famous formula for the left-right crossing probability for percolation on a rectangle.

Smirnov proved it for site percolation on the triangular lattice.

What about the exact probabilities for the annulus crossing?

S.-Xu-Zhuang (2024+)

Let $\tau = \frac{1}{2\pi} \log(\frac{R}{r})$ and $\eta(z) = e^{\frac{i\pi z}{12}} \prod_{n=1}^{\infty} (1 - e^{2ni\pi z})$.

$$\mathbb{P}[\mathcal{A}_1(r, R)] = \sqrt{\frac{3}{2}} \cdot \frac{\eta(6i\tau)\eta(\frac{3}{2}i\tau)}{\eta(2i\tau)\eta(3i\tau)};$$

$$\mathbb{P}[\mathcal{A}_2(r, R)] = \sqrt{3} \cdot \frac{\eta(i\tau)\eta(6i\tau)^2}{\eta(3i\tau)\eta(2i\tau)^2}.$$

Predicted by Cardy ('02, '06) using (**non-rigorous**) Coulomb gas approach.

Hard to derive via Ito's calculus, in contrast to the rectangle crossing.

CFT style expressions for annulus crossing probabilities

Closed channel expansion: let $\tau = \frac{1}{2\pi} \log(\frac{R}{r})$ and $q = e^{-2\pi\tau}$.

$$\mathbb{P}[\mathcal{A}_1(r, R)] = \sqrt{\frac{3}{2}} \cdot \frac{\sum_{k \in \mathbb{Z}} (q^{2h_{4k-\frac{1}{2}, 0}} - q^{2h_{4k+\frac{3}{2}, 0}})}{\prod_{n=1}^{\infty} (1 - q^{2n})}.$$

$$\mathbb{P}[\mathcal{A}_2(r, R)] = \sqrt{3} \cdot \frac{\sum_{k \in \mathbb{Z}} (q^{2h_{0, 6k+1}} - q^{2h_{0, 6k+2}})}{\prod_{n=1}^{\infty} (1 - q^{2n})}.$$

$$h_{r,s} = \frac{(3r-2s)^2-1}{24}; \quad \text{Kac table for central charge } c = 0.$$

Open channel expansion: let $\tilde{q} = e^{-\pi/\tau}$.

$$\mathbb{P}[\mathcal{A}_1(r, R)] = \frac{\sum_{k \in \mathbb{Z}} (\tilde{q}^{h_{1, 4k+1}} - \tilde{q}^{h_{1, 4k+3}})}{\prod_{n=1}^{\infty} (1 - \tilde{q}^n)}.$$

$$\mathbb{P}[\mathcal{A}_2(r, R)] = \frac{\sum_{k \in \mathbb{Z}} (\tilde{q}^{h_{1, 6k+2}} + \tilde{q}^{h_{1, 6k+4}} - 2\tilde{q}^{h_{1, 6k+3}})}{\prod_{n=1}^{\infty} (1 - \tilde{q}^n)}.$$

A CFT derivation is missing.

Monochromatic two-arm crossing formulae

S.-Zhuang (2024+)

Let $\tau = \frac{1}{2\pi} \log(\frac{R}{r})$ and $q = e^{-2\pi\tau}$.

$$\mathbb{P}[\mathcal{B}_2(r, R)] = \frac{q^{-\frac{1}{12}}}{\prod_{n=1}^{\infty} (1 - q^{2n})} \sum_{s \in \mathcal{S}} \frac{-\sqrt{3} \sin(\frac{2\pi}{3} \sqrt{3s}) \sin(\pi \sqrt{3s})}{\cos(\frac{4\pi}{3} \sqrt{3s}) + \frac{3\sqrt{3}}{8\pi}} q^s,$$

$\mathcal{S} = \{0.440, 2.194 \pm 0.601i, 5.522 \pm 1.269i, 10.361 \pm 2.020i, \dots\}$.

all (complex) solutions to $\sin(4\pi\sqrt{\frac{x}{3}}) + \frac{3}{2}\sqrt{x} = 0$, except 0 and $\frac{1}{3}$.

Relation to a CFT with complex and transcendental spectrum?

Open channel expansion: let $\tilde{q} = e^{-\pi/\tau}$.

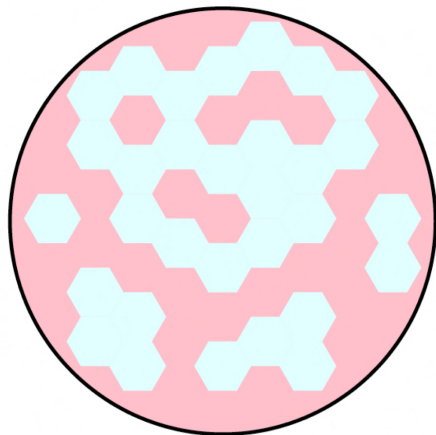
$\mathbb{P}[\mathcal{B}_2(r, R)] = 1 - (1 + \frac{3\sqrt{3}}{4}\tau^{-1})\tilde{q}^{\frac{1}{3}} + \text{remaining terms of similar form.}$

Logarithmic structure. Relation to a log CFT?

Quantum gravity derivation of one-arm exponent

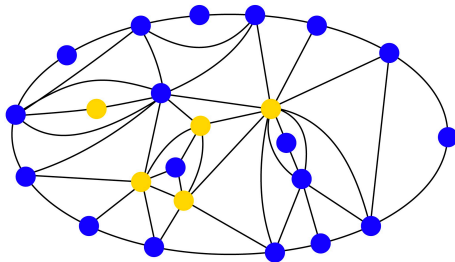
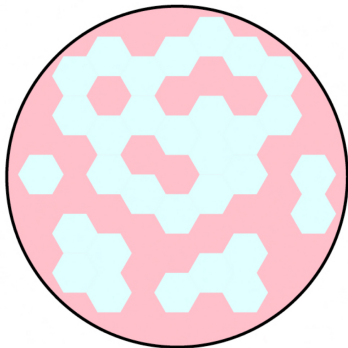
$\mathbb{P}[\text{one arm event}] \sim n^{-\frac{5}{48}}$ is equivalent to:

If we have n^2 vertices on the lattice region, the size of the **boundary connecting cluster** is $\sim n^{91/48}$.
($n^2 \times n^{-\frac{5}{48}} = n^{\frac{91}{48}}$.)



A vertex v is in the boundary connecting cluster if the one arm event occurs at v .

Knizhnik-Polyakov-Zamolodchikov (KPZ) Relation



A KPZ derivation of the one-arm exponent

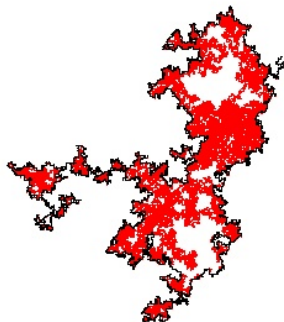
- 1 On random triangulation, the answer is $n^{\text{quantum exponent}}$.
- 2 $91/48 = \text{KPZ}(\text{quantum exponent})$

Enumeration on RHS is simpler;

$\text{KPZ}(\cdot)$ is explicit quadratic.

History of KPZ

- Derived from the CFT description of LQG. (KPZ '88).
- “Verified” by enumeration of planar maps. (around '90)
David, Douglas, Gross, Kazakov, Kostov, Migdal, Shenker ...
- Provide a powerful framework to study fractals.



Conjecture

Mandelbrot '82

Frontier of planar Brownian motion
has fractal dimension $4/3$.

- physics “proof” by KPZ. Duplantier '98.
- rigorous proof via SLE_6 .
Lawler-Schramm-Werner '00.

Duplantier-Sheffield (2011): First rigorous KPZ relation.
Starting point of **Liouville quantum gravity** in probability

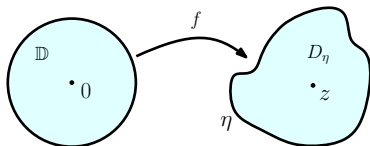
Quantum gravity approach beyond KPZ

The quantum exponent corresponds to β_2 is still a transcendental number.
A standard application of KPZ is hard for evaluating β_2 .

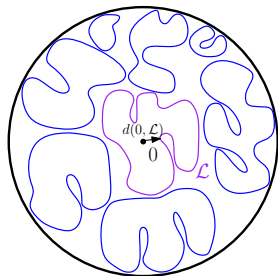
Two crucial tools in our approach:

- Conformal radius encoding of arm-exponent;
- integrability of Liouville CFT that governs LQG surfaces.

Conformal radius: for a loop η and a point z , $CR(z, \eta) := |f'(0)|$.



Example: One-arm exponent



- \mathcal{L} : outermost CLE_6 loop.
- $\mathbb{P}[d(0, \mathcal{L}) \leq \epsilon] = \mathbb{P}[\mathcal{A}_1(\epsilon, 1)] \approx \epsilon^{\alpha_1}$.
- $\alpha_1 = \inf\{x : \mathbb{E}_{\text{CLE}_6}[d(0, \mathcal{L})^{-x}] = \infty\}$.
- Koebe 1/4 theorem:
 $\frac{1}{4}\text{CR}(0, \mathcal{L}) \leq d(0, \mathcal{L}) \leq \text{CR}(0, \mathcal{L})$.

$$\alpha_1 = \inf\{x : \mathbb{E}_{\text{CLE}_6}[\text{CR}(0, \mathcal{L})^{-x}] = \infty\}.$$

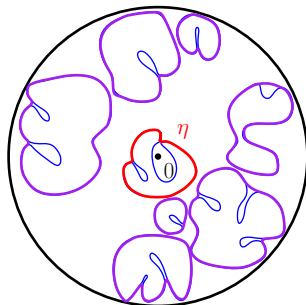
Schramm-Sheffield-Wilson '09

$$\mathbb{E}_{\text{CLE}_6}[\text{CR}(0, \mathcal{L})^{-x}] = \frac{1}{2 \cos(\frac{\pi}{3} \sqrt{12x+1})}, \quad \text{hence } \alpha_1 = \frac{5}{48}.$$

Conformal Radius Encoding for the Backbone exponent

Consider the **outer boundaries of CLE_6 loops**: a collection of simple loops.
 η : outermost one surrounding 0; locally looks like an $\text{SLE}_{8/3}$ curve.

Exist two monochromatic arms in the annulus bounded by η and $\partial\mathbb{D}$.
 $\mathbb{P}[\text{CR}(0, \eta) \leq \epsilon] \approx p_{BB}(\epsilon, 1) \approx \epsilon^{\beta_2}$.



$$\beta_2 = \inf\{x : \mathbb{E}[\text{CR}(0, \eta)^{-x}] = \infty\}.$$

$$\mathbb{E}[\text{CR}(0, \eta)^{-x}] = \frac{3\sqrt{3}}{4} \sin\left(\frac{\pi}{2}\sqrt{12x+1}\right) \left(\frac{\sqrt{36x+3}}{4} + \sin\left(\frac{2\pi\sqrt{12x+1}}{3}\right) \right)^{-1}.$$

Derivation of $\mathbb{E}[\text{CR}(0, \eta)^{-x}]$ via Liouville quantum gravity on the disk, applying method from Ang- Holden-S. (2021).

Nolin-Qian-S.-Zhuang '23

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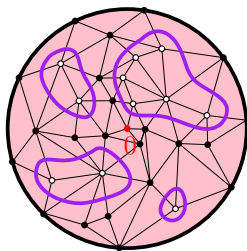
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Derivation of $\mathbb{P}[\mathcal{B}_2(r, R)]$ via Liouville quantum gravity on the annulus, applying method from Ang-Remy-S. (2022).

2D Quantum Gravity coupled with percolation



- Sample random triangulation of the disk decorated with a percolation.
- Conformally embed the random surface on the unit disk \mathbb{D} .

Continuum limit: LQG on \mathbb{D} with $\gamma = \sqrt{8/3}$ decorated with CLE_6 .

A random geometry on \mathbb{D} with area measure $e^{\gamma\phi} d^2x$ on \mathbb{D} ;
length measure $e^{\frac{\gamma}{2}\phi} dx$ on $\partial\mathbb{D}$. (Gaussian multiplicative chaos)

ϕ : a random generalized function, locally looks like a Gaussian free field.

(Convergence proved by Holden-S. under a certain conformal embedding.)

Liouville field description of the law of the canonical ϕ

The law of the field ϕ is “ $e^{-S_L[\phi]}D\phi$ ”; $S_L[\phi]$: Liouville action on \mathbb{D} .

$$S_L[\phi] = \int_{\mathbb{D}} \left(\frac{1}{4\pi} |\nabla \phi|^2 + \mu e^{\gamma \phi} \right) d^2x + \int_{\partial \mathbb{D}} \left(\frac{Q\phi}{2\pi} + \nu e^{\frac{\gamma}{2}\phi} \right) dl.$$

$Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, μ and ν : cosmological constants.

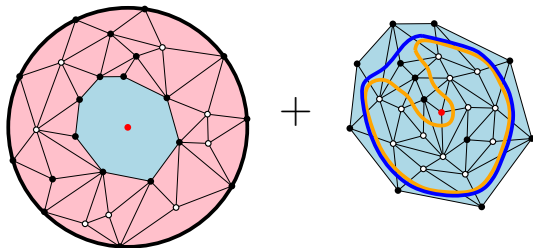
“ $e^{-\text{Liouville action}} D\phi$ ” defines a quantum field theory called Liouville theory, made rigorous by David-Kupiainen-Rhodes-Vargas and follow-up.

Liouville theory is a conformal field theory, with rich integrability:

- DOZZ formula: Kupiainen-Rhodes-Vargas.
- Conformal bootstrap on closed surface: Guillarmou-KRV.
- We need the **disk analog of DOZZ (with $\mu = 0$)** solved by Remy-Zhu.
- We need the annulus conformal bootstrap by B. Wu.

Cut and glue random surfaces: discrete

η : the outermost outer boundary surrounding a marked interior point.
 η cuts the triangulation decorated by percolation into:



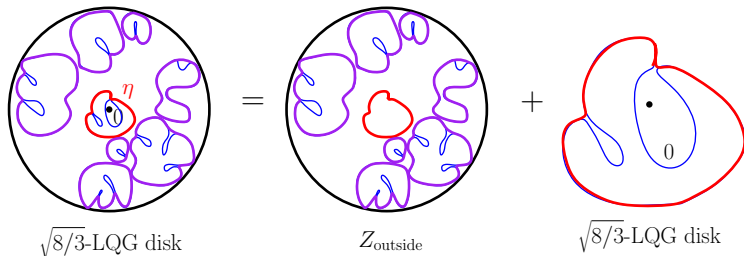
Sample random triangulation according to Boltzmann weight $a^n b^L$;
 n : number of triangles L : number of boundary edges.

$$\sum_{\ell} Z_{\text{outside}}(L, \ell) \times \ell \times Z_{\text{inside}}(\ell) = Z_{\text{whole}}(L).$$

ℓ : the length of η . $Z_{\text{outside}}, Z_{\text{inside}}, Z_{\text{whole}}$: surface partition functions.

Cut and glue random surfaces: continuum

$Z_{\text{whole}}(L) \propto L^{-3/2}$ in the continuum limit for critical weights a and b .
Partition function for the most canonical random disk in $\sqrt{8/3}$ -LQG with one interior marked point and boundary length L .



Using SLE/LQG coupling (Sheffield's quantum zipper):

The inside of η is another copy of $\sqrt{8/3}$ -LQG disk; $Z_{\text{inside}}(\ell) \propto \ell^{-3/2}$.

$$\int_0^\infty Z_{\text{outside}}(L, \ell) \times \ell \times \ell^{-3/2} d\ell = L^{-3/2}. \quad (\text{proved purely from continuum})$$

Conformal Radius from Deforming the Bulk Insertion

$$\int_0^\infty Z_{\text{outside}}(L, \ell) \times \ell \times \ell^{-\frac{3}{2}+a} d\ell = L^{-\frac{3}{2}+a} \cdot \mathbb{E}[\text{CR}(0, \eta)^{-\frac{1}{3}a(a-1)}], \quad \text{for all } a.$$

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From $L^{-\frac{3}{2}}$ to $L^{-\frac{3}{2}+a}$ using the field description of the LQG disk.

$$\int_{\phi: \mathbb{D} \rightarrow \mathbb{R}} F\left(\int_{\partial \mathbb{D}} e^{\frac{\gamma}{2}\phi} dx\right) e^{\gamma\phi(0)} e^{-S_L[\phi]} D\phi \propto \int_0^\infty F(L) e^{-\nu L} L^{-\frac{3}{2}} dL.$$

ν : boundary cosmological constant in the Liouville action $S_L[\phi]$
 $e^{\gamma\phi(0)}$ is inserted since 0 is the marked point on the $\sqrt{8/3}$ -LQG disk.

$$\int_{\phi: \mathbb{D} \rightarrow \mathbb{R}} F\left(\int_{\partial \mathbb{D}} e^{\frac{\gamma}{2}\phi} dx\right) e^{\alpha\phi(0)} e^{-S_L[\phi]} D\phi \propto \int_0^\infty F(L) e^{-\nu L} L^{-\frac{3}{2}+a(\alpha)} dL.$$

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Same reason for $\ell^{-\frac{3}{2}}$ to $\ell^{-\frac{3}{2}+a}$, except **the smaller disk is bounded by η instead of $\partial \mathbb{D}$** . This difference is compensated by a power of conformal derivative at 0, which is $\text{CR}(0, \eta)^{-\frac{1}{3}a(a-1)}$.

Solving the conformal radius and the backbone exponent

Equations like (\star) relating boundary length partition function and conformal radius were systematically derived in Ang- Holden-S. (2021).

$$\int_0^\infty Z_{\text{outside}}(L, \ell) \times \ell \times \ell^{-\frac{3}{2}+a} d\ell = L^{-\frac{3}{2}+a} \mathbb{E}[\text{CR}(0, \eta)^{-\frac{1}{3}a(a-1)}]. \quad (\star)$$

No formula for $Z_{\text{outside}}(L, \ell)$ is available for Nolin-Qian-S.-Zhuang (2023) to solve $\mathbb{E}[\text{CR}(0, \eta)^{-\frac{1}{3}a(a-1)}]$ using (\star) .

NQSZ found an effective variant of $\mathbb{E}[\text{CR}(0, \eta)^{-\frac{1}{3}a(a-1)}]$ such that:

- still encodes the backbone exponent;
- has an equation like (\star) with a solvable $Z_{\text{outside}}(L, \ell)$.

The NQSZ variant is defined in terms the **SLE₆ bubble measure**.

The counterpart for $Z_{\text{outside}}(L, \ell)$ is solved using the **boundary analog of DOZZ** due to Remy-Zhu.

The ratio between $\mathbb{E}[\text{CR}(0, \eta)^{-\frac{1}{3}a(a-1)}]$ and its NQSZ variant is another conformal radius moment solved using the AHS method, [D. Wu '23].

$$\mathbb{E}[\text{CR}(0, \eta)^{-x}] = \frac{3\sqrt{3}}{4} \sin\left(\frac{\pi}{2}\sqrt{12x+1}\right) \left(\frac{\sqrt{36x+3}}{4} + \sin\left(\frac{2\pi\sqrt{12x+1}}{3}\right) \right)^{-1}.$$

Plugging into (\star) we get

$$\int_0^\infty Z_{\text{outside}}(L, \ell) \ell^{ix} d\ell = \frac{3\sqrt{3}}{4} \frac{\sinh(\pi x)}{\sinh(\frac{4\pi x}{3}) + \frac{\sqrt{3}}{2}x} L^{ix-1} \quad \text{for all } x.$$

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$Z_{\text{outside}}(L, \ell) = Z^{\text{nt}}(L, \ell) + Z^{\text{t}}(L, \ell).$ Z_{outside} : surface between η and $\partial\mathbb{D}$

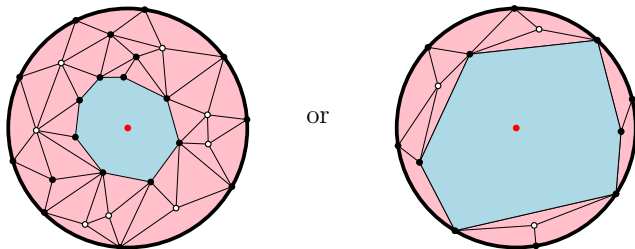
$Z^{\text{nt}}(L, \ell)$: η does not touch $\partial\mathbb{D}$.

$Z^{\text{t}}(L, \ell)$: η touches $\partial\mathbb{D}$.

$$\int_0^\infty Z^{\text{t}}(L, \ell) \ell^{ix} d\ell = \frac{3\sqrt{3}}{4} \frac{\sinh(\frac{\pi x}{3})}{\sinh(\frac{2\pi x}{3})} L^{ix-1}; \quad \text{still by Ang-Holden-S. method,}$$

We now use $Z^{\text{nt}}(L, \ell)$ to solve $\mathbb{P}[\mathcal{B}_2(r, R)]$.

Quantum gravity on the annulus



$Z^{\text{nt}}(L, \ell)$: partition function of the random triangulation of the annulus
coupled with a percolation with monochromatic two-arm crossing.

Random triangulation of the annulus without any decoration converges to the most canonical random annulus in $\sqrt{8/3}$ -LQG (Brownian annulus).

with partition function: $Z^{\text{B}}(L, \ell) = \frac{1}{\sqrt{L\ell}(L+\ell)}$.

Brownian annulus and random modulus

When the Brownian annulus is conformally embedded to the flat annulus \mathcal{C}_τ of modulus τ , we still get Liouville field on \mathcal{C}_τ ,
but the modulus τ itself is random.

[Ang-Remy-S. '22] Exact Law of τ : Liouville + ghost

$$\int \int_0^\infty e^{-\nu_1 L - \nu_2 \ell} Z^B(L, \ell) d\ell dL = \int_0^\infty Z_{\text{Liouville}}^{\nu_1, \nu_2}(\tau) \cdot Z_{\text{ghost}}(\tau) d\tau$$

$Z_{\text{Liouville}}^{\nu_1, \nu_2}(\tau) = \int_{\phi: \mathcal{C}_\tau \rightarrow \mathbb{R}} e^{-S_L(\phi)} D\phi$; solved by B. Wu (2022).

$$S_L(\phi) = \int_{\mathbb{D}} \left(\frac{1}{4\pi} |\nabla \phi|^2 + \mu e^{\gamma \phi} \right) d^2x + \int_{\partial_{\text{in}} \mathcal{C}_\tau} \nu_1 e^{\frac{\gamma}{2} \phi} dl + \int_{\partial_{\text{out}} \mathcal{C}_\tau} \nu_2 e^{\frac{\gamma}{2} \phi} dl$$

The formula for $Z^B(L, \ell)$ gives $Z_{\text{ghost}}(\tau) = \eta(2i\tau)^2$,
as predicted in the bosonic string theory, ghost CFT has $c = -26$;
math conjecture formulated by Guillarmou-Rhodes-Vargas; Remy.

From $Z^B(L, \ell)$ to $Z^{\text{nt}}(L, \ell)$

$$\int \int_0^\infty e^{-\nu_1 L - \nu_2 \ell} Z^{\text{nt}}(L, \ell) d\ell dL = \int_0^\infty Z_L(\tau) \cdot \mathbb{P}[\mathcal{B}_2(e^{-2\pi\tau}, 1)] \cdot Z_{\text{ghost}}(\tau) d\tau.$$

$$\int_0^\infty \mathbb{P}[\mathcal{B}_2(e^{-2\pi\tau}, 1)] \eta(2i\tau) e^{-\frac{2\pi x^2 \tau}{3}} d\tau = \frac{\sqrt{3}}{x} \left(\frac{\sinh(\frac{2}{3}\pi x) \sinh(\pi x)}{\sinh(\frac{4}{3}\pi x) + \frac{\sqrt{3}}{2}x} - \sinh(\frac{1}{3}\pi x) \right).$$

S.-Zhuang (2024+)

$$\mathbb{P}[\mathcal{B}_2(e^{-2\pi\tau}, 1)] = \frac{q^{-\frac{1}{12}}}{\prod_{n=1}^\infty (1 - q^{2n})} \sum_{s \in \mathcal{S}} \frac{-\sqrt{3} \sin(\frac{2\pi}{3} \sqrt{3}s) \sin(\pi \sqrt{3}s)}{\cos(\frac{4\pi}{3} \sqrt{3}s) + \frac{3\sqrt{3}}{8\pi}} q^s,$$

$\mathbb{P}[\mathcal{A}_1(r, R)]$ and $\mathbb{P}[\mathcal{A}_2(r, R)]$ are derived using the same strategy.

- Determine β_j for $j \geq 3$? (NQSZ variant not found yet.)
- CFT interpretations of the β_2 and $\mathbb{P}[\mathcal{B}_2(r, R)]$ formulas?

One-arm and dichromatic two-arm exponents naturally appear in a percolation related CFT proposed recently in physics.

Physics: Jacobsen, Rilbault, Saluer, et.al.: **bootstrap formulas** for the probability that **four points** are on the same cluster or the same loop

Math: Ang, Cai, S., Wu.: rigorously derive the **three-point** probability formulas involving the **Imaginary DOZZ** formula.

Happy Birthday, Antti!

Thank you for all your inspirations!