Dynamics of NLS in the Infinite Volume Limit

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Path Integrals and Friends

Celebrating Antti Kupiainen's 70'th Birthday

Introduction

This is joint work with **Ben Dodson and Avy Soffer**:
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This is a new field for me. I got interested because of the work of Damanik - Goldstein on 1D KdV with **small** amplitude quasi-periodic data.

We wondered what would happend we consider NLS smooth random data or KdV with larger amplitude q-p data? As well see, these are challenging open questions and there are no answers as yet. Instead, I will review some related questions on the lattice and the continuum.

The Nonlinear Schrödinger Equation on $\mathbb{R}^d$ or $\mathbb{Z}^d$

$$i\partial_t u(t,x) = -\Delta u(t,x) + u(t,x) + 2\lambda |u|^2 u(t,x),$$

$$\text{where } u(t,x) \in \mathbb{C}, \quad \lambda \geq 0.$$

Initial data $u(0,x) = u_0(x),\ x \in \mathbb{R}$ *or* $\mathbb{Z}$.

On $\mathbb{R}^d$, assume that $u_0(x)$ is uniformly smooth and bounded.
For example:

$$u_0(x) = acos(x) + bcos(\sqrt{2}x).$$

On $\mathbb{Z}^d$, Δ denotes the finite difference Laplacian.

Initial data, $u_0(x)$ is bounded or random

Related Dispersive Equations

The KdV equation:

$$\partial_t u(t,x) = -\partial_x^3 u(t,x) + 6u\partial_x u$$

The Anharmonic Crystal Hamiltonian:

$$\frac{1}{2}\sum_j \{p_j^2 + (q_{j+1} - q_j)^2 + q_j^2 + \lambda q_j^4\}$$

Examples of Initial Data

Examples on $\mathbb{R}^d$. Mostly we consider d=1

1) $u_0(x)$ Periodic function

2) Quasi-periodic $u_0(x) = a\cos(x) + b\cos(\alpha x)$, α *irrational*

3) Smooth Random $u_0(x) = \sum_{j\in\mathbb{Z}^d} a_j e^{-(x-j)^2}$, $x \in \mathbb{R}$, $|a_j| \le 1$

4) Slowly decaying: $|u_0(x)| \le (|x|+1)^{-\epsilon}$, $\epsilon > 0$

5) Random: distributed by equilibrium measure e^{-H}

Some Results in the Continuum

Bourgain (2000) Established **Global existence** for 1D defocussing cubic NLS with initial data given by the **invariant measure** as $L \Rightarrow \infty$.

Bringmann, Deng, Nahmod, and Yue, prove that the ϕ_3^4 Euclidean field model is invariant under a renormalized wave equation with cubic non linearity. The NLS case is still open.

Recent preprint of **Deng-Hani** establishes the Kinetic Wave Equation for NLS in 3D with random initial data up to blowup time of the Kinetic theory. Non-equilibrium data.

Examples on $\mathbb{Z}$

1) Bounded: $|u_0(x)| \le C,\ x \in \mathbb{Z}$.

2) Quasi-periodic: $u_0(x) = a\cos(\alpha\pi x),\ x \in \mathbb{Z},\ \alpha$ *irrational*

3) Random: $u_0(x)$ independent identically distributed (iid), $x \in \mathbb{Z}$.

4) $u_0(x)$ in equilibrium $\sim e^{-\beta\sum_x\{|\nabla u(x)|^2+|u(x)|^2+\lambda|u|^4(x)\}}Du$

5) $u_0(x)$ with distribution $\sim e^{-\sum_x|u(x)|^2}Du$

$$Du \equiv c\prod_j du(j)\,d\bar{u}(j)\,.$$

Conservation laws for NLS

Mass: M(t)= $\int_R |u(t,x)|^2 dx$

Energy: E(t)= $\frac{1}{2}\int_R \{|\nabla u(t,x)|^2 + |u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4\} dx$

On the lattice $\mathbb{Z}$ the integrals are replaced by sums.

Conservation law : M(t) and E(t) are **independent of t**.
They are essential for proving global existence of solutions when they are finite.

However, Mass and Energy are **infinite** for the cases above.

Motivation:

In some physical situations such as light transmission in very long fibers, or dynamics of ocean waves it is natural to replace the finite energy assumption by **finite energy per unit volume.**

Of course, one could look at very large volume of side L but then the question is how the solution depends L.

Key Questions for NLS:

Does there exist a unique solution? Yes, for discete case. In the continuum??

What is its growth the maximum as a function of time? Finite volume L dependence of solution as L gets large?

Describe properties of the solution. Eg. Space-time correlations as in Lukkarinen-Spohn.

For DNLS how to describe ensemble of **breather like, localized** solutions which occur when energy/vol $\gg$ mass/vol ? Micro-canonical ensemble.

When random initial condition is **not** given by an invariant measure, it is **more difficult** to control long time behavior.

Simpler Related ODE Question:

$$\ddot{x}(t) + 2\lambda x^3 + f(t)x = 0, \quad f(t) \geq 1 \, and \, |\dot{f}(t)| \leq C.$$

Let

$$E(t) = \dot{x}^2 + \lambda x^4 + f(t)x^2, \quad \lambda > 0$$

Then

$$\frac{dE(t)}{dt} = \dot{E} = \dot{f}x^2 \leq CE^{1/2} \Rightarrow E(t) \leq Ct^2$$

Conjecture: If f(t) is smooth and periodic, $E(t)$ is bounded - KAM.

Remark: If $\lambda = 0$ and f(t) = E+V(t) is usual Quantum potential and u may grow exponentially fast!

Conjecture: If f(t) is smooth, quasi-periodic, then $|E(t)|$ grows very slowly when $\lambda > 0$.

Dynamics of NLS on Torus

The **Periodic case** can be formulated on a circle or torus - $\mathbb{T}^d$ **finite energy**.

Bourgain, then Colliander, Keel, Staffilani, Takaoka and Tao proved polynomial upper bounds in time on the growth of $H^s(\mathbb{T}^d)$ for smooth initial conditions.

Note that $H^1(\mathbb{T}^d)$ is bounded by conservation of energy but when $s > 1$ the H^s norm may grow - "wave turbulence" unless there are other conservation laws. Eg integrable case.

Results for Quasi-Periodic Data on $\mathbb{R}$

$$u_0(x) = a\cos(x) + b\cos(\sqrt{2}\,x)$$

Tadahiro Oh: Local existence proved for quasi-periodic data and global existence for a class of *limit periodic* data.

Damanik, Goldstein, Binder, Lukic:
Global existence of **KdV** for **small amplitude** data with good Diophantine frequency. Solution bounded and almost periodic in time. Relies of **integrability** of 1D KdV. Deift conjecture.

Remark: Global existence for large qp data **not known** for KdV or NLS. If bound states appear in the Lax operator - analysis breaks down - unstable"breathers" or solitons may appear .

Wei-Min Wang: There exist small amplitude solutions to 1D NLS-type equations which are quasi-periodic in space and time. KAM method. Applies to non-integrable equations.

Nonlinear Wave equation (NLW) on $\mathbb{R}$

$$u_{tt}(t,x) - u_{xx}(t,x) + u^3(t,x) = 0, \;\; u(0,x),\, u_t(0,x) \in C^2(R)$$

Finite Propagation Speed:

$u(t, x_0)$ only depends on $u_0(x)$ for $|x - x_0| \leq t$.

Thus the energy is effectively finite at any time.

$$E(u, u_t) = \frac{1}{2}\int (\partial_x u(t,x))^2 dx + \frac{1}{2}\int (u_t(t,x))^2 dx + \frac{1}{4}\int u(t,x)^4 dx.$$

Lemma: If the data uniformly in $C^2(R)$ then NLW has a unique solution and $|u(t,x)| \leq Ct^{1/3}$.

Problem: NLS and KdV **do not** have finite propagation speed.
Rough data can propagate very rapidly.
Solution may become rough via nonlinearity.

Dynamics of Linear Schrödinger $\mathbb{R}$

$$i\,\dot{u}(t,x) = -\Delta u(t,x), \quad u(0,x) = u_0(x), \textit{ bounded}$$

Consider:

$$u_0(x) = \sum_j a_j e^{-(x-j)^2}, \;\; a_j \in \mathbb{C}, |a_j| \le 1$$

then $|u(t,x)| \le Ct^{1/2}$. The a_j can be chosen to cancel the phases so that you cannot improve $t^{1/2}$. Similar results hold on the lattice.

If the a_j are independent random variables, of mean 0 then $\mathbb{E}[|u(t,x)|^{2p}] \le C, \;\; 0 \le p < \infty$.

Dynamics of NLS on the Lattice $\mathbb{Z}$

Let $-\Delta = \partial^* \partial$ be the finite difference Laplacian on $\mathbb{Z}$. Here

$$\partial f(x) = f(x+1) - f(x), \text{ and } \partial^* f(x) = f(x-1) - f(x)$$

The lattice NLS is given by

$$i\frac{\partial}{\partial t}\psi(t,x) = i\dot{\psi}(t,x) = -\Delta\psi(t,x) + |\psi|^2\psi(t,x), \; x \in \mathbb{Z}$$

Proposition: If $|u(0,x)| \leq A$, then $|u(t,x)| \leq C\,A\,t^{1/4}$. Moreover, the following space average is bounded:

$$\frac{1}{t}\sum_{|x-x_0|\leq t} |u(t,x)|^4 \leq Const. \quad all\ x_0 \in \mathbb{Z}$$

Earlier Results for Anharmonic Crystal due to

Butta, Caglioti, di Ruzza, Marchioro, Pulvirenti, and others
Establish bounds on solutions with initial data which grow slowly as $|x| \to \infty$

Note these bounds are needed when data is given by the invariant measure.

"Almost" finite propagation speed is proved - assuming an **invariant measure:** Let f_j, g_k be functions localized near $j, k \in \mathbb{Z}^d$ and let $\{\cdot\}$ be Poisson bracket, then

$$\{f_j, g_k^t\} \approx 0 \;\; when \;\; |j-k| \geq Ct \log^p(t)$$

Recent work of A Vuoksenmaa on NLS allows data to grow at polynomial rate.

Proof of Bounds on $|u(t,x)|^2$

Define a **Local Mass**:

$$M(t) = \sum_x |u(t,x)|^2 e^{-F(t,x)}, \quad F(t,x) = \frac{\sqrt{(x-x_0)^2+1}}{(2t_0-t)}, \; t_0 \geq t$$

Then using the fact that Δ **is a bounded operator** on the lattice

$$\frac{dM(t)}{dt} \leq \frac{3M(t)}{(2t_0-t)} \rightarrow M(t_0) \leq (3\ln 2)M(0) \rightarrow$$

$$\sum_{|x-x_0|\leq t_0} |u(t,x)|^2 \leq C \sum_x e^{-F} A^2 \leq CA^2 t.$$

Thus $|u(t,x)|^2 \leq t$

Global existence for Regularized NLS on $\mathbb{R}$

Consider the regularized Hamiltonian, formally given by

$$\int_{\mathbb{R}} \{\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u_\phi(t,x)|^4\} dx$$

Where $u_\phi = u \star \phi(x)$ with ϕ smooth symmetric function eg. e^{-x^2}.

Theorem DSS If $u_0(x)$ lies in C^4 then there is a global solution with $|u(t,x)| \leq Ct^{8/3}$.

The proof uses a local energy norm:

$$E(t,x_0) = \int_{\mathbb{R}} \{\chi(\frac{x-x_0}{R})[\frac{1}{2}|u_x^2| + \frac{1}{4}|u_\phi|^4]\} \, dx$$

where $\chi(x) \geq 0$ is smooth and compactly supported.

Remark: The proof is more complicated because the derivative is unbounded.

Local existence for smooth Data

Theorem DSS If $u_0(x)$ is bounded and analytic in a strip of width 3 then NLS has a unique real analytic solution for **short time**.

For example: analytic random or quasi-periodic data.

Idea: Apply Newton iteration to solve NLS by solving time dependent linear equations.

R. Schippa and F. Klaus later established **local** well-posedness of NLS in modulation spaces $M^0_{\infty,q}$.

Very recent work of **Bringmann and Staffilani** establishes existence and uniqueness **locally in time** when the data is uniformly in $C^2(R)$.

Global existence of NLS with smooth data in $L^p, p < \infty$

Theorem DSS If $u_0(x)$ and its $p/2$ deriviatives lie in $L^p, 2 < p < \infty$ then there is a **global solution** to NLS.

R. Schippa and F. Klaus have established global well-posedness in $M^s_{p,q}$ for s=1, $p < \infty$.

To prove *global* solutions we need to assume NLS is defocussing: $\lambda > 0$.

Some Questions and Comments

How to improve $|u(t,x)| \leq t^{1/4}$ when u_0 is random on $\mathbb{Z}$?

For random iid data on a *periodic* box of side L,
the **average local energy** is bounded in time uniformly in L:

$$\langle |u(t,x)|^4 \rangle \leq Const.$$

Proof by translation invariance: Expectation is independent of x.

If the data are in **equilibrium** then Expectation of **all moments** of the local energy are can be estimated using the invariant measure and are bounded in t.

Problem: How to analyse growth and interaction of "breather modes" which should occur when Energy/vol $\gg$ Mass/vol.

Breathers are a form of "energy localization" arising from large ℓ^4 norm with small ℓ^2 norm.

In grand-canonical ensemble the Gibbs weight is $e^{-\beta H + \mu A}$ and $A = \sum_j |u_j|^2$

If we set $\langle A/N \rangle_{\beta,\mu} \equiv a$ and $\langle H/N \rangle_{\beta,\mu} = h$ then one can show that

$$h \leq Ca^2 \, .$$

This means that the Grand-Canonical ensemble does not hold when $h \gg a^2$. This is the regime in which breathers are expected to dominate the dynamics.

Aubry, Iubini, Flach, Kevrekidis, Livi, Majumdar, Politi, Rasmussen ...

Happy Birthday Antti!