How does the supercritical GMC converge?

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Gaussian Multiplicative Chaos

Basic construction: Take a log-correlated Gaussian field X:

$$\mathbf{E}X(x)X(y) = g(|x-y|)$$

with $g(r) = |\log r| + \mathcal{O}(1)$ as $r \to 0$. (Example: massive GFF.)

Mollifier ϱ_ε, set $X^{(\varepsilon)} = \varrho_\varepsilon \star X$ and consider

$$\mu_\varepsilon^{(\gamma)}(dx) = \varepsilon^{\gamma^2/2} e^{\gamma X^{(\varepsilon)}(x)}\, dx$$

Kahane '85: for $\gamma \in (0, \sqrt{2d})$, the limit $\lim_{\varepsilon\to 0} \mu_\varepsilon^{(\gamma)}$ exists in probability and is non-trivial. For $\gamma \geq \sqrt{2d}$, it vanishes. Very robust (Shamov '16, etc).

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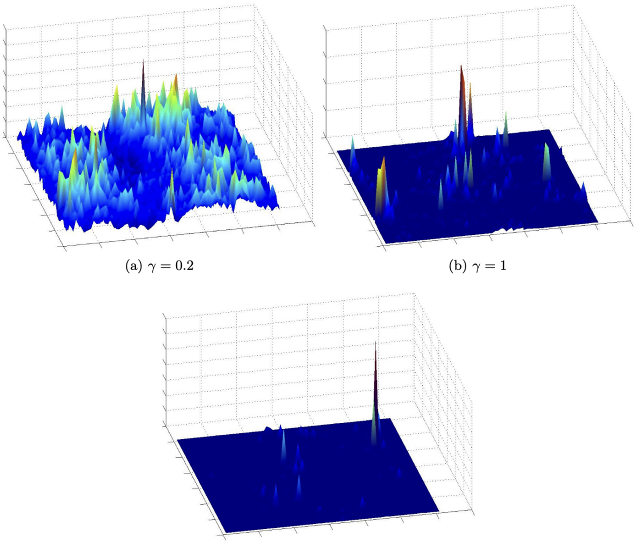
Some pictures (Rhodes & Vargas)



(a) $\gamma = 0.2$ (b) $\gamma = 1$

(c) $\gamma = 1.8$

What about larger γ?

For $\gamma = \sqrt{2d}$, convergence still holds with slightly different renormalisation:

$$\mu'_\varepsilon(dx) = \sqrt{|\log\varepsilon|}\varepsilon^d e^{\sqrt{2d}X^{(\varepsilon)}(x)}\,dx$$

Convergence more delicate, but still in probability. One has $\mu' \propto \frac{d}{d\gamma}\mu^{(\gamma)}|_{\gamma=\sqrt{2d}}$.

What about $\gamma > \sqrt{2d}$ and what is special about $\sqrt{2d}$? Recall max of N iid $\mathcal{N}(0,1)$ vars about $\sqrt{2\log N}$.

Rough cartoon for $X^{(\varepsilon)}$ is about ε^{-d} iid $\mathcal{N}(0,|\log\varepsilon|)$ vars, so maximum around $\sqrt{|\log\varepsilon|}\sqrt{2\log\varepsilon^{-d}} = \sqrt{2d}|\log\varepsilon|$. Contribution of each large peak to $\int e^{\gamma X^{(\varepsilon)}(x)}\,dx$ about $\varepsilon^d e^{\gamma\sqrt{2d}|\log\varepsilon|} = \varepsilon^{d-\gamma\sqrt{2d}}$. Balances $\varepsilon^{-\gamma^2/2}$ when $\gamma = \sqrt{2d}$.

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The supercritical case

For $\gamma > \sqrt{2d}$, consider the Poisson point process Γ on $\mathbf{R}^d \times \mathbf{R}_+$ with intensity measure $\mu'(dx)s^{-1-\alpha}ds$ with $\alpha = \frac{\sqrt{2d}}{\gamma}$. Then (Madaule, Rhodes, Vargas '16),

$$\mu_\varepsilon^{(\gamma)}(dx) = |\log \varepsilon|^{\frac{3}{2\alpha}} \varepsilon^{\gamma\sqrt{2d}-d} e^{\gamma X^{(\varepsilon)}(x)} dx$$

converges stably for some $c > 0$ to the atomic measure $\mathrm{PP}_\alpha(\mu') = c \sum_{(x,s)\in\Gamma} s\delta_x$.

No convergence in probability!

Question: Does the measure-valued process $\mu_{\varepsilon,t}^{(\gamma)} = \mu_{\varepsilon e^{-t}}^{(\gamma)}$ converge to a non-trivial limit? What does it look like?

Theorem: There exists a process Z with stationary increments (negative drift!) so that the mass s of each atom moves according to iid copies of $\exp(Z)$.

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$\star$-scale invariant fields

A special type of log-correlated fields:

$$\mathbf{E}X_t(x)X_s(y) = \int_0^{s\wedge t} K(e^r(x-y))\,dr\ , \quad K(0) = 1\ .$$

Has independent increments $X_{s,t}$ and $X_{s,t}(\bullet) \stackrel{\text{law}}{=} X_{0,t-s}(e^s\bullet)$. Think of $X_t = X^{(\varepsilon)}$ with $\varepsilon = e^{-t}$.

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Behaviour of $X_{s,t}$

$X_{s,t}$ behaves roughly like e^{ds} copies of X_{t-s}, rescaled by e^s. Maximum M of X_{t-s} (on order 1 region) about $M_* \approx \sqrt{2d}(t-s)$ (plus log-correction) with tails $\mathbf{P}(M \geq M_* + K) \approx \exp(-\sqrt{2d}K)$.

Yields highest local maxima of order $\sqrt{2d}(t-s) + \sqrt{d/2}s$ and heights distributed according to $\mathrm{Exp}(\sqrt{2d})$.

Since $X_{s,t}$ has correlation length of order e^{-t}, each local maximum gives a contribution to $\int e^{\gamma X_{s,t}}$ of order e^{-dt}.

Suggests that, modulo logs, $e^{(d-\sqrt{2d}\gamma)t+\sqrt{d/2}\gamma s} e^{\gamma X_{s,t}}$ is close to $\mathrm{PP}_\alpha(dx)$.

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Modulations of stable PPP's

Lemma: For any positive function f, one has $f\,\mathrm{PP}_\alpha(dx) = \mathrm{PP}_\alpha(f^\alpha\,dx)$.

Recall $\alpha = \frac{\sqrt{2d}}{\gamma}$, so that

$$e^{(d-\sqrt{2d}\gamma)t} e^{\gamma X_t} \approx e^{(d-\sqrt{2d}\gamma)t+\sqrt{d/2}\gamma s} e^{\gamma X_{s,t}} \big(e^{-ds} e^{\sqrt{2d}X_s}\big)^{1/\alpha}$$
$$\approx \mathrm{PP}_\alpha\big(e^{-ds} e^{\sqrt{2d}X_s}\big) \approx \mathrm{PP}_\alpha(\mu')\;.$$

Explains what the limit looks like.

The process Z

Based on a suitable "cluster process" describing the behaviour of X_t for very large t near its largest local maxima.

The view from the top

Construction of $\Upsilon_t = X_t(e^{-t}\bullet)$, viewed from a local maximum, as $t \to \infty$.

1. Condition on $\Upsilon_t(0) \approx \sqrt{2d}t$, yields Gaussian field Υ_∞ with covariance $\mathfrak{a}(x) + \mathfrak{a}(y) - \mathfrak{a}(x-y)$ and mean $-\sqrt{2d}\mathfrak{a}$.
2. Condition Υ_∞ on $\sup_x \Upsilon_\infty(x) \leq \lambda$, yields $\tilde{\Upsilon}_\lambda$.
3. There is a unique (in law) field Ψ with maximum 0 at 0 such that

$$\mathbf{E}\big[\mathcal{F}(\tilde{\Upsilon}_\lambda)\big] \propto \mathbf{E}\Big[\int_{\mathbf{R}^d} \mathcal{F}(\tau_x \Psi) e^{\sqrt{2d}\Psi(x)} \mathbf{1}_{\{\Psi(x) \geq -\lambda\}} dx\Big] \,.$$

(Independent of λ!)

Thanks for your attention

Happy birthday Antti!