Random Planar Surfaces and Trees

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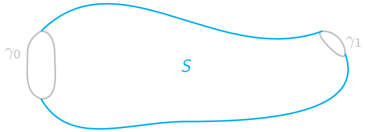
1. Contents

a) Are there any viable discrete approximations to string theory path integrals?

Path of string $\gamma_t \quad = \quad$ surface S:



- Hypercubic lattice models
- Dynamically triangulated models
- Scaling limits - appearance of tree-like structures.
- Relation to $2D$-gravity.

b) Ensembles of infinite planar trees

- Local limits of generic trees
- Uniform tree and $2D$-gravity
- Non-generic trees with size- and height-dependent weights.

2. Surface models
Hypercubic surfaces in $\mathbb{Z}^d$

$\mathcal{S}(\gamma_1, \ldots, \gamma_k) = \{$ surfaces in $\mathbb{Z}^d$ obtained by gluing elementary plaquettes along edges, having topology of S^2 with k holes $\gamma_1, \ldots, \gamma_k\}$.

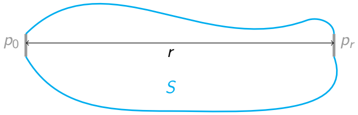
$|S|$ = area of S = number of plaquettes in S.

- $\sharp\{S \in \mathcal{S}(\gamma_1, \ldots, \gamma_k) \mid |S| = A\} \sim \text{cst.} A^{\gamma+k-3} e^{\beta_0 A} \quad (A \to \infty)$

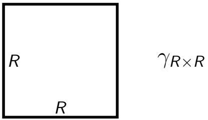
Define k-loop function:

$G_\beta(\gamma_1, \ldots, \gamma_k) = \sum_{S \in \mathcal{S}(\gamma_1, \ldots, \gamma_k)} e^{-\beta |S|}, \quad \beta > \beta_0.$

Mass: $m(\beta) = -\lim_{r \to \infty} r^{-1} \log G_\beta(p_0, p_r)$



String tension: $\tau(\beta) = -\lim_{R\to\infty} R^{-2} \log G_\beta(\gamma_{R\times R})$



Scaling limit requires $m(\beta) \to 0$ as $\beta \to \beta_0$:

$$m(\beta) \sim (\beta - \beta_0)^\nu \qquad (\beta \to \beta_0)$$

Susceptibility: $\chi(\beta) = \sum_p G_\beta(p_0, p)$

- If $\chi(\beta)$ diverges at β_0 then $m(\beta)$ vanishes at β_0.

Conjecture: $\chi(\beta)$ diverges at β_0 for all $d \geq 2$. Numerical results confirm this for β_0 for $d = 2$, and $d = 3$. Mean field approximation supports it for large d.

Continuum k-point function: Choose $\beta(a)$, $a =$ lattice spacing, such that $a^{-1}m(\beta(a)) = m_{\rm ph} > 0$ and define, $x_1, \ldots, x_k \in \mathbb{R}^d$,

$$G(x_1, \ldots, x_k) = \lim_{a\to 0} a^{-\alpha_{k,d}} G_{\beta(a)}(p_{a^{-1}x_1}, \ldots, p_{a^{-1}x_k})\,.$$

Theorem (D-Fröhlich-Jonsson, 1984) If $\chi(\beta)$ diverges at β_0, then $\nu = \frac{1}{4}$ and $G(x_1, x_2)$ exists with $\alpha_{2,d} = d - 2$, and it equals the standard Euclidean propagator:

$$\hat{G}(p) = \frac{\text{cst.}}{p^2 + m_{ph}^2}, \quad p \in \mathbb{R}^d.$$

- If $\chi(\beta)$ diverges at β_0 then $\tau(\beta) \to \tau(\beta_0) > 0$ as $\beta \to \beta_0$ and

$$\tau(\beta) - \tau(\beta_0) \sim (\beta - \beta_0)^{\frac{1}{2}}.$$

Consequently, the continuum string tension

$$\tau_{ph} = \lim_{a \to 0} a^{-2} \tau(\beta(a)) = \infty,$$

indicating that tree-like surfaces dominate the scaling limit.

- Hausdorff dimension $d_H = \nu^{-1} = 4$.

Conjecture:

$$\sharp\{S \in \mathcal{S}(p) \mid |S| = A\} \sim \text{cst.} A^{-3/2} e^{\beta_0 A} \qquad (A \to \infty)$$

for all $d \geq 2$.

Dynamically triangulated surfaces

• Replace $\mathcal{S}(\gamma_1, \ldots, \gamma_k)$ by triangulated surfaces in $\mathbb{R}^d$ with quadratic dependence of action on embedding coordinates.

• Explicit proof that string tension does not vanish at β_0.

• Can be (formally) extended to $d \leq 1$. For $d = 0$ one obtains a model of (abstract) two-dimensional (simplicial) complexes, believed to be equivalent to 2D (Liouville) quantum gravity.

• Combinatorial correspondences between two-dimensional complexes and trees turn out important for carrying out calculations (... G. Schaeffer, ...) and as a theoretical tool (... J.-F. Le Gall, ...). Example (CDT):

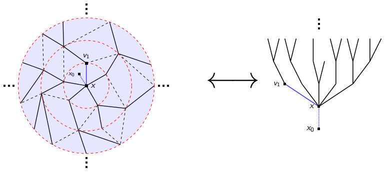
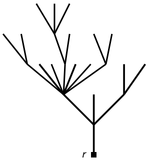


Figure: Causal triangulation of the disk and the corresponding tree.

3. Tree models

Generic planar trees

$\mathcal{T}_N = \{$planar rooted trees with N edges$\}$; root r of degree 1
$T \in \mathcal{T}_N$: $\sigma_i =$ degree of vertex $i \in T$
$B_R(T) =$ ball of radius R in T around the root



$$\mathcal{T} = \bigcup_{N\geq 1} \mathcal{T}_N \cup \mathcal{T}_\infty$$
$$d(T, T') = \inf\{\tfrac{1}{R} \mid B_R(T) = B_R(T')\}$$
$$(\mathcal{T}, d)\,,\text{complete separable metric space}$$

Weak convergence of measures: $\mu_N \to \mu$ if $\int f d\mu_N \to \int f d\mu$ as $\mathbb{N} \to \infty$, f bounded continuous fct. on $\mathcal{T}$
Equivalently: $\mu_N(\mathcal{B}_a(T)) \to \mu(\mathcal{B}_a(T))\,, T \in \mathcal{T}, a > 0$, where $\mathcal{B}_a(T)$ is the ball of radius a around T.

Branching (or offspring) probabilities:
$p_n \geq 0,\ n = 0, 1, 2, \ldots \quad \sum_n p_n = 1.$

• Corresponding Galton-Watson branching process is concentrated on finite trees if avarage offspring $m = \sum_n np_n$ equals 1 (critical case) or is < 1 (subcritical case).

Finite size partition function: $Z_N = \sum_{T \in \mathcal{T}_N} \prod_{i \in T \setminus r} p_{\sigma_i - 1}$

Generating function: $Z(g) = \sum_{N \geq 1} Z_N g^N$

Finite size measure: $\mu_N(T) = \frac{\prod_{i \in T \setminus r} p_{\sigma_i - 1}}{Z_N},\ T \in \mathcal{T}_N$

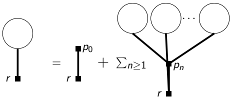
Example (uniform tree): $p_n = 2^{-(n+1)},\ n \geq 0,$

$$\prod_{i \neq r} p_{\sigma_i - 1} = 2^{\sum_{i \neq r} \sigma_i} = 2^{-2|T|+1}, \qquad Z_N = 2^{-2N+1} \sharp \mathcal{T}_N,$$

$$\mu_N(T) = \frac{1}{\sharp \mathcal{T}_N},\ T \in \mathcal{T}_N$$

Basic equation for partition functions:

$$\boxed{Z(g) = g\ f(Z(g))}$$

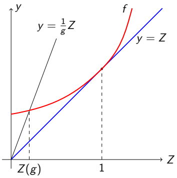


where $\quad f(z) = \sum_{n=0}^{\infty} p_n z^n$

Generic assumption:

a) (p_n) is critical, i.e. $\sum_{n=0}^{\infty} n p_n = 1$
(valid for uniform tree)

b) Convergence radius for f is $\rho > 1$
($\rho = 2$ for uniform tree)

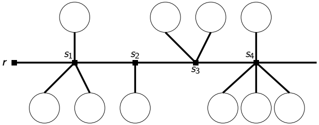


Then

$$Z(g) = 1 - \sqrt{\frac{2}{f''(1)}}(1-g)^{\frac{1}{2}} + O(1-g)$$

$$Z_N = (2\pi f''(1))^{-\frac{1}{2}} N^{-\frac{3}{2}}(1 + O(N^{-1})\ (N \to \infty)$$

Theorem (D-Jonsson-Wheater, 2007) $\mu_N \to \mu$ as $N \to \infty$ where μ is concentrated on one-ended trees whose branches at spine vertices s_i are i.i.d. Galton-Watson trees with offspring probabilities p_n and σ_{s_i} are i.i.d. with probability $(\sigma_{s_i} - 1)p_{\sigma_{s_i}-1}\,,\ \sigma_{s_i} \geq 2.$



- Hausdorff dimension d_h: $|B_R(T)| \sim R^{d_h}$ for $R \to \infty$
- Spectral dimension d_s : $q_{2n}(T) \sim n^{-\frac{d_s}{2}}$ as $n \to \infty$,

where q_{2n} is the probability that simple random walk on T starting at r returns to r after $2n$ steps.

Theorem (D-Jonsson-Wheater, 2007) For any generic tree $d_h = 2$ and $d_s = \frac{4}{3}$ hold μ - a.s.

- The local limit of causal dynamical triangulations exists and $d_h = 2$ almost surely. Also known that $d_s = 2$ almost surely.

Trees with height dependent weights (with M. Ünel, 2023)

$\mathcal{T}_{m,N} = \{$planar rooted trees of size N and height $h(T) \leq m\}$

$$A_{m,N} = \sharp \mathcal{T}_{m,N}$$

Define $\mu_N^{(\zeta)}(T) = \frac{e^{\zeta h(T)}}{Z_N^{(\zeta)}}, \; T \in \mathcal{T}_N,$

where $Z_N^{(\zeta)} = \sum_{T \in \mathcal{T}_N} e^{-\zeta h(T)} = \sum_{m=1}^{\infty} e^{-\zeta m}(A_{m,N} - A_{m-1,N})$

Note For $\zeta = 0$ the measure $\mu^{(0)}$ coincides with the uniform measure on $\mathcal{T}_N$.

We have:

$$A_{m,N} = 4^N \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{m+1} \tan^2 \frac{\pi k}{m+1} \Big(1 + \tan \frac{\pi k}{m+1}\Big)^{-N}, \; N \geq 2$$

For $\zeta > 0$, the asymptotic form of $Z_N^{(\zeta)}$ is obtained by a saddlepoint approximation:

$$Z_N^{(\zeta)} = (e^{\zeta} - 1)\sqrt{\frac{\pi}{B}}\frac{\zeta}{2} e^{-AN^{\frac{1}{3}}} N^{-\frac{5}{6}} 4^N (1 + N^{-\delta}) \; (\delta > 0)$$

$$A = 3\Big(\frac{\pi\zeta}{2}\Big)^{\frac{2}{3}}, \qquad B = 3\Big(\frac{\zeta^2}{4\pi}\Big)^{\frac{2}{3}}$$

Theorem Assume $\zeta > 0$. Then $\mu_N^{(\zeta)} \to \mu^{(\zeta)}$ as $N \to \infty$, where $\mu^{(\zeta)}$ is concentrated on $\mathcal{T}_\infty$ and determined by

$$\mu^{(\zeta)}(\mathcal{B}_{\frac{1}{r}}(T)) = e^{-\zeta(r-1)} 4^{-|T|} 2^{K+1} \sum_{R=1}^{K} \binom{K}{R} \frac{\mu^{R-1}}{(R-1)!}$$

for any finite tree T of height r and with K vertices at height r.

Description of $\mu^{(\zeta)}$:

The *spine* of $T \in \mathcal{T}_\infty$ is defined as the subtree of T spanned by vertices of *infinite type*, i.e. vertices having infinitely many descendants in T.

Let $\mathcal{T}^s = \{ T \in \mathcal{T}_\infty \mid \text{all vertices in } T \text{ are of infinite type} \}$, which is a closed subset of $\mathcal{T}$. The spine map $\chi : T \to T^s$ is Borel-measurable. Set $\tilde{\mu}^{(\zeta)}(E) = \mu^{(\zeta)}(\chi^{-1}(E))\,,\ E \subseteq \mathcal{T}^s$.

Theorem $\tilde{\mu}^{(\zeta)}$ is a Poisson tree defined by

$$\tilde{\mu}^{(\zeta)}(\mathcal{B}^s_{\frac{1}{r}}(T)) = e^{-\zeta(r-1)} \frac{\zeta^{R-1}}{(R-1)!}\,,\ r \geq 0$$

for any finite tree T of height r with R leaves, all at height r.

Corollary The random variables

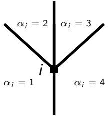
$$\tau_r(T^s) = |B_{r+1}(T^s)| + |B_{r-1}(T^s)| - 2|B_r(T^s)|$$

are i.i.d. with distribution

$$\tilde{\mu}^{(\zeta)}(\tau_r = n) = e^{-\zeta}\frac{\zeta^n}{n!}\,, \quad n \geq 0.$$

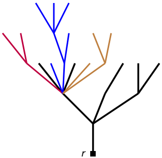
- This implies $d_h(T^s) = 2$ for $\tilde{\mu}^{(\zeta)}$ - a.e. T^s.

Vertex i in T^s has σ_i angular sectors:



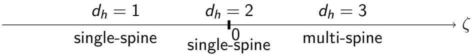
Vertex i of degree 4
with 4 angular sectors.

Theorem $\mu^{(\zeta)}$ is obtained from $\tilde{\mu}^{(\zeta)}$ by decorating T^s with branches T_{i,α_i}, $\alpha_i = 1, \ldots, \sigma_i$, in each angular sector, independently and identically distributed as Galton-Watson trees with offspring probability $p_n = 2^{-(n+1)}$.



Theorem $d_h(T) = 3$ holds $\mu^{(\zeta)}$ - a.s.

The case $\zeta < 0$: $\lim_{N\to\infty} \mu_N^{(\zeta)} = \mu^{(\zeta)}$ exists and is concentrated on single-spine trees with i.i.d. subcritical branches. In particular, $d_h(T) = 1$ for $\mu^{(\zeta)}$ - a.e. T.



4. Outlook

- Spectral dimensions of $\mu^{(\zeta)}$ and $\tilde{\mu}^{(\zeta)}$ for $\zeta > 0$ not known.
- Generalisations: introduce height-dependence for general generic GW-trees.
- Implications for randomly triangulated models, in particular CDT coupled to e.g. spin systems or to loop models.