Probabilistic conformal blocks on the torus and the Lamé equation

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Path integrals and friends

Joint work with Promit Ghosal and Andrei Prokhorov
(arXiv:2407.05839)

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Plan

I. Motivation

II. Construction and Methodology

III. Results

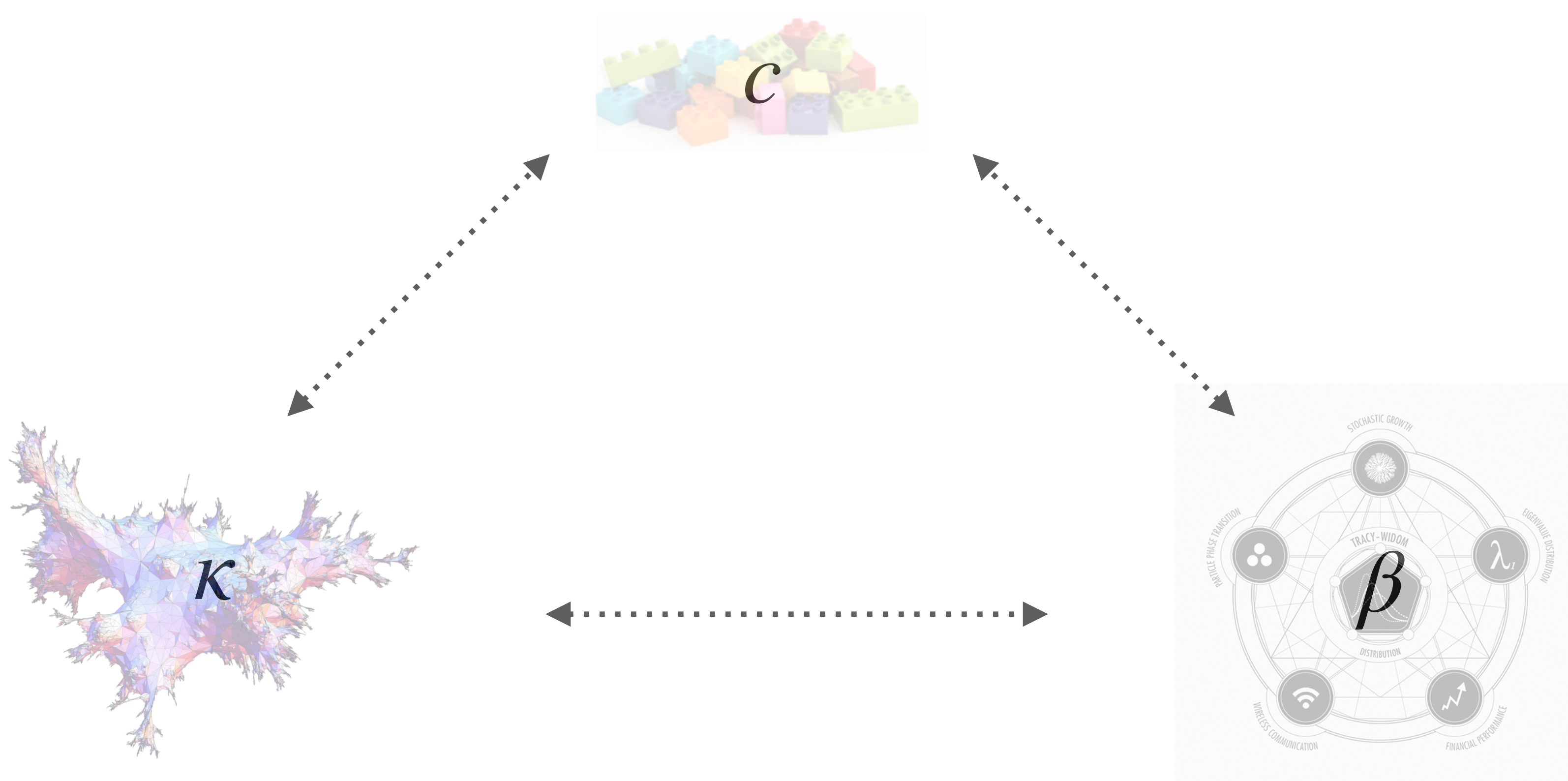
I. Motivation

Conformal Field Theory

Quantum Field Theory + conformal symmetry

CFTs have the following properties:

- infinite set of symmetries (Virasoro algebra), parametrized by $c \in \mathbb{C}$
- rich geometric structure (moduli space of Riemann surfaces)



Conformal Field Theory

Quantum Field Theory + conformal symmetry

CFTs have the following properties:

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- rich geometric structure (moduli space of Riemann surfaces)

Two important results

1. Correspondence between $c = 1$ CFTs and Painlevé equations - initiated by Gamayun, Iorgov, Lisovyi 2012.

   H.D, Joerg Teschner, Julien Roussillon,…
2. Probabilistic definition of Liouville correlation functions for - initiated by Kupiainen, Rhodes, Vargas 2017.

   Baptiste Cercle, Colin Guillarmou, Xin Sun,…

Painlevé equations

- Six nonlinear second order ODEs on the complex plane.

  Eg: $\partial_\tau^2 u = m^2 \wp'(2u \,|\, \tau)$.

- Describe isomonodromic deformations of a 2×2 matrix valued linear equation

  $$\left(\partial_z - A(z, u, \partial_\tau u, \tau)\right) Y(z, \tau) = 0.$$

- The solutions are famously transcendental, multivalued, and have infinite #poles.

- Have associated Hamiltonians.

  Eg: $H := (\partial_\tau u)^2 - m^2 \wp(2u \,|\, \tau)$.

- Important for random matrices, and Conformal Field Theory (CFT) among others.

  Eg: Universality results in nonlinear physics- Tracy-Widom distribution and Painlevé II

Painlevé/CFT correspondence

CFT correlation functions

Conformal blocks

||

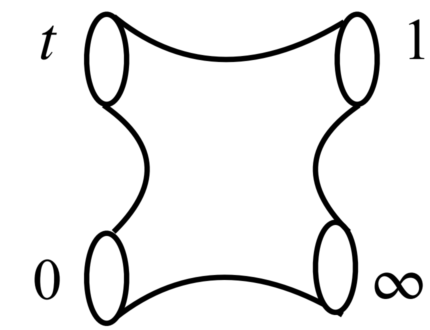
holomorphic functions,

basis of Virasoro representations

For $c = 1$, solve ODEs,

Painlevé equations in specific cases

Painlevé/CFT correspondence



Example:

$$\langle V_1(0)V_2(t)V_3(1)V_4(\infty)\rangle$$

$c = 1$ Conformal blocks ($\mathcal{G}(t;\sigma)$)

Painlevé VI, solution $u_{VI}(t)$

$$u_{VI}(t;\sigma,\eta) \sim \partial_t^2 \log \sum_n e^{in\eta}\mathcal{G}(t;\sigma+n); \quad \mathcal{T}(t;\sigma,\eta) \sim \sum_n e^{in\eta}\mathcal{G}(t;\sigma+n)$$

Consequences:

1. Gives an explicit representation of the Painlevé tau-functions,
2. Exposes geometric (symplectic) structure of the monodromy manifold
3. Solves the ‘connection problem’- ratio of tau-functions at critical points
4. Gives combinatorial expression for joint moments of CUE.

Big picture…

Classical Integrable Systems
(Painlevé equations)

Established!

CFT
$(c = 1)$

Gavrilenko, Lisovyi, Cafasso [CMP 17, LMP 17],
Del Monte, **H.D**, Gavrilenko [CMP 23, JPhys A 23]

Quantum Integrable Systems

$c \in [25,\infty)$

Semi-classical Integrable Systems
(Heun equations)

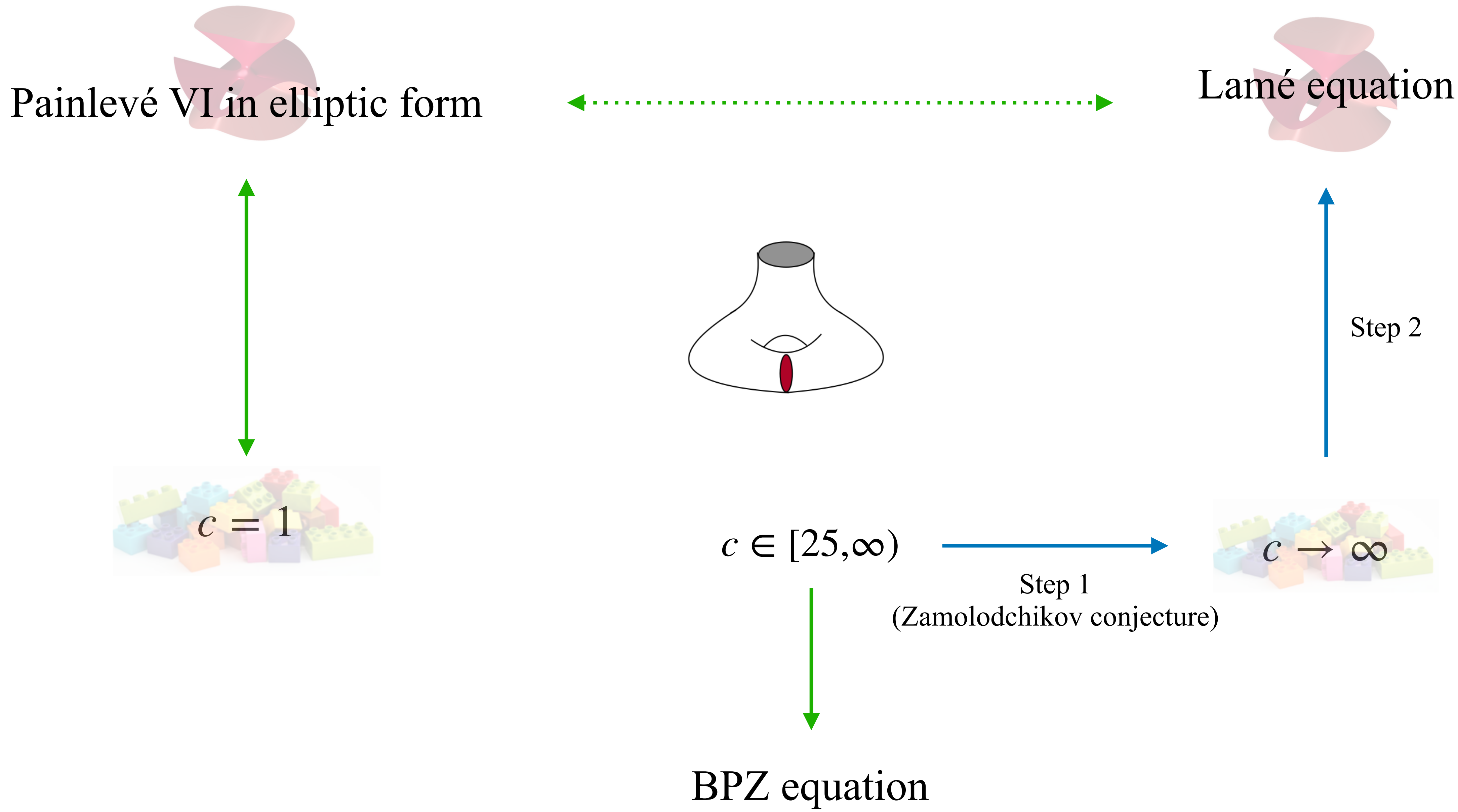
Conjectured…

CFT
$(c \to \infty)$

Naiduik, Lisovyi [LMP 21, JPhys A 22],
Bonelli, Iossa, Lichtig, Tanzini [CMP 23],
Bershtein, Gavrilenko, Grassi [CMP 22]

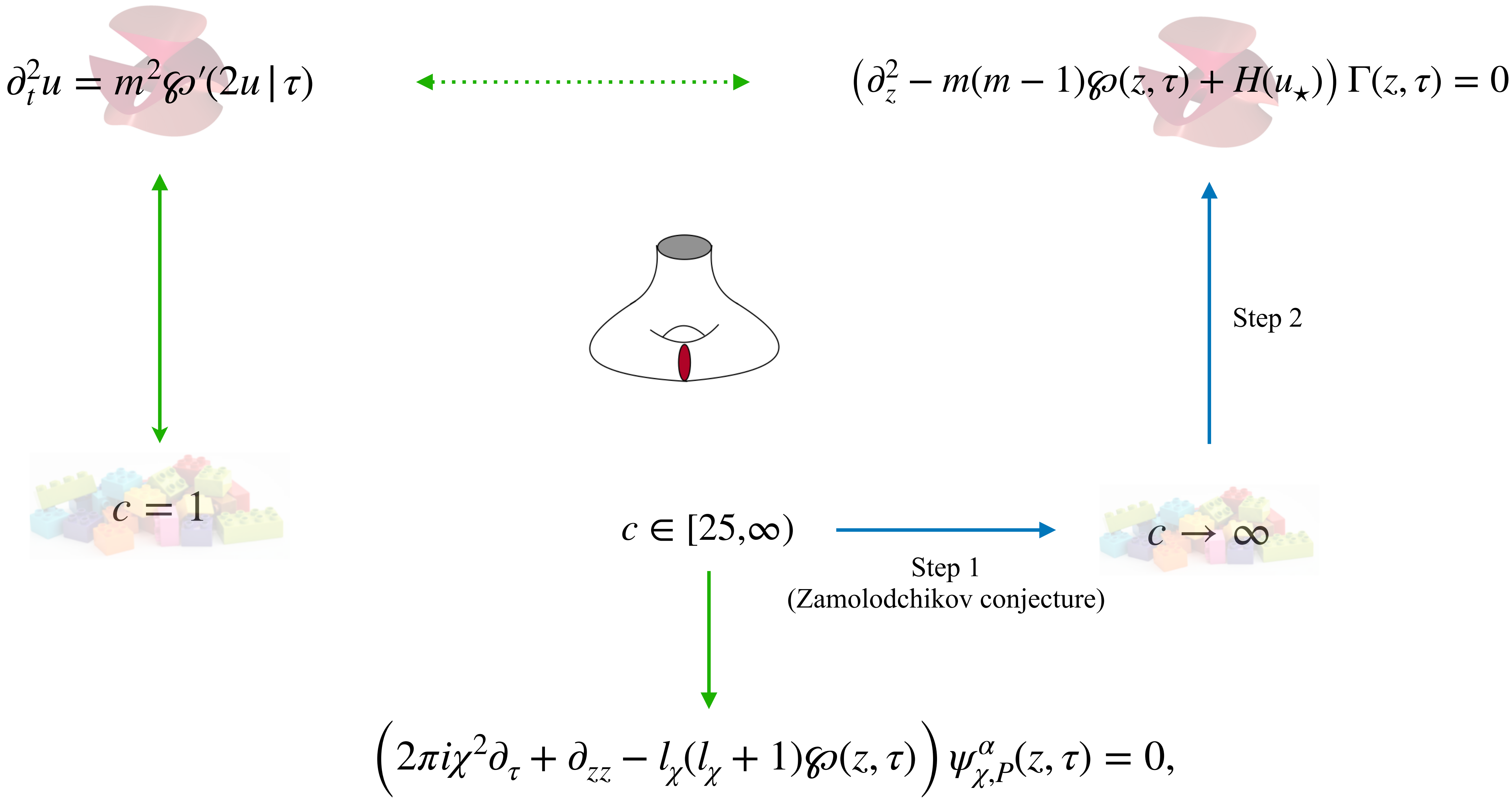
 -Character variety

Strategy



Probabilistic conformal blocks available on the one-point torus!
(Ghosal, Remy, Sun, Sun; Duke Math. J. 24)

Strategy



$\partial_t^2 u = m^2 \wp'(2u \,|\, \tau)$
$\left(\partial_z^2 - m(m-1)\wp(z,\tau) + H(u_\star)\right)\Gamma(z,\tau) = 0.$
Step 2
$c = 1$
$c \in [25, \infty)$
$c \to \infty$
Step 1
(Zamolodchikov conjecture)
$\left(2\pi i \chi^2 \partial_\tau + \partial_{zz} - l_\chi(l_\chi + 1)\wp(z,\tau)\right)\psi^{\alpha}_{\chi,P}(z,\tau) = 0,$

Enter… Lamé equation

Painlevé VI (elliptic form): $\partial_t^2 u = m^2 \wp'(2u \,|\, \tau)$

$\left(\partial_z - A\right) Y(z,\tau) = 0.$

Lax equation

$H := (\partial_\tau u)^2 - m^2 \wp(2u \,|\, \tau)\,.$

Hamiltonian

$u(\tau_\star) = 0$

Lamé equation: $\left(\partial_z^2 - m(m-1)\wp(z,\tau) + H(\tau_\star)\right)\Gamma(z,\tau) = 0.$

Enter… Lamé equation

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Accessory parameter

Transcendental in general

Semi-classical conformal field theory and the Lamé equation

[DGP ’24] Proof of Zamolodchikov conjecture for semi-classical conformal blocks on torus. **H.D**, Promit Ghosal, and Andrei Prokhorov; arXiv:2407.05839

Motivating question:

Can we establish semi-classical Integrability/CFT correspondence for the one-point torus?

Guiding principle:

A rigorous definition of conformal block on the torus for $c \geq 25$ was given by Ghosal, Remy, Sun, Sun in 2020. Control over its semi-classical limit would be the key!

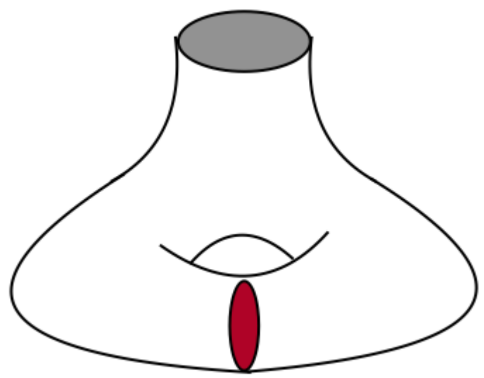
Summary of results:

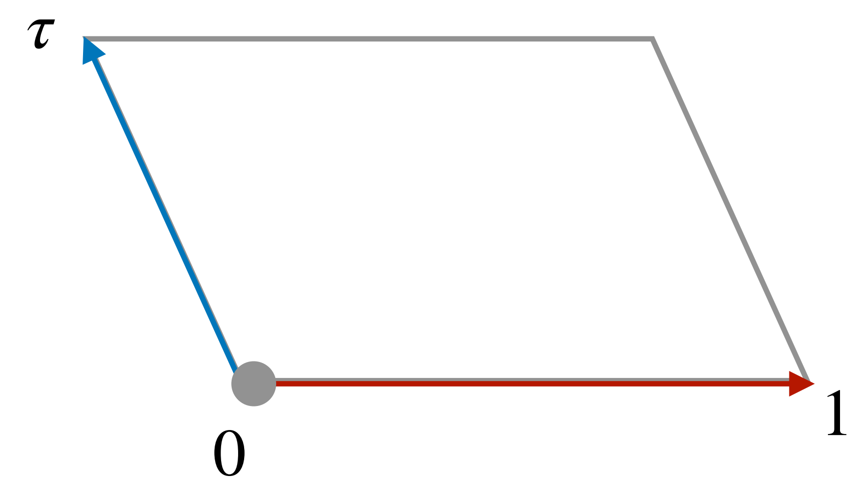
1. Proves a long-standing conjecture known as the Zamolodchikov conjecture (1986) that posits the structure of semi-classical conformal blocks.

2. Provides closed form expressions for the general (two parameter) solution and accessory parameter of the Lamé equation.

II. Construction and Methodology

Random fields on the torus





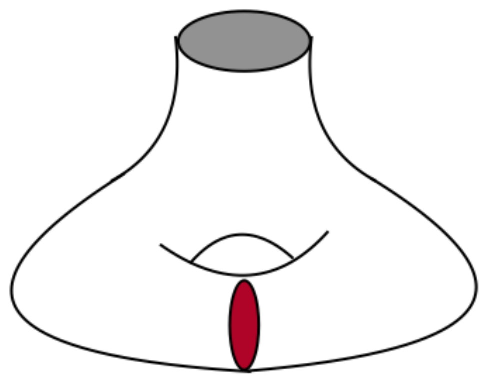
On the circle [0,1], define

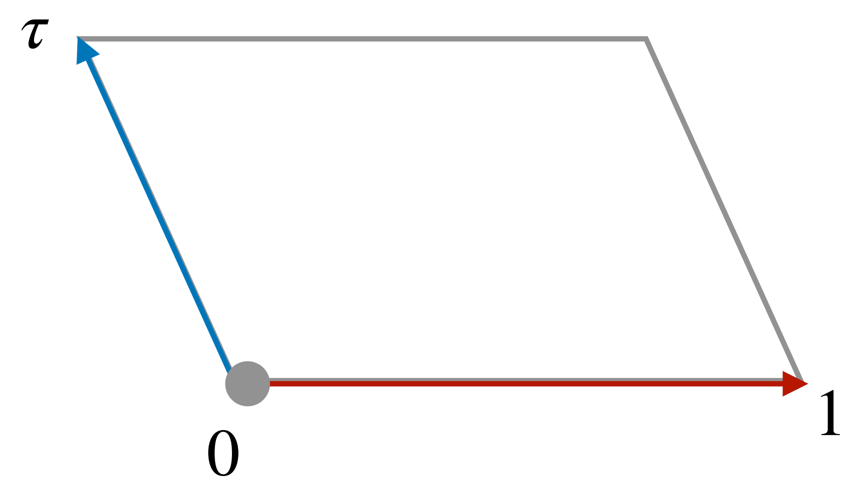
1. the τ independent field: $Y(x) = \sum_{n\geq 1}\sqrt{\frac{2}{n}}\left(\alpha_n \cos(2\pi n x) + \widetilde{\alpha}_n \sin(2\pi n x)\right),$

2. the τ dependent fields with $q = e^{i\pi\tau}$:
$$\begin{cases} F_\tau(x) = 2\sum_{n,m\geq 1}\frac{q^{nm}}{\sqrt{n}}\left(\beta_n \cos(2\pi n x) + \widetilde{\beta}_n \sin(2\pi n x)\right), \\ Y_\tau(x) = Y(x) + F_\tau(x), \qquad \lim_{q\to 0} Y_\tau(x) = Y(x)\,. \end{cases}$$

Note that $\alpha_n, \widetilde{\alpha}_n; \beta_n, \widetilde{\beta}_n$ are i.i.d random variables, and $Y(x)$, $Y_\tau(x)$ are log-correlated Gaussian fields.

Random fields on the torus





1. τ independent field:

$$\mathbb{E}[Y(x)Y(y)] = -2\log|e^{2i\pi x} - e^{2i\pi y}|,$$

2. τ dependent fields with $q = e^{i\pi\tau}$:

$$\mathbb{E}[Y_\tau(x)Y_\tau(y)] = -2\log|\theta_1(x-y)| + 2\log|q^{1/6}\eta(q)|.$$

General structure of conformal blocks

Definition: For $\gamma \in (0,2)$, $q \in (0,1)$, the one point conformal block on the torus

$$\mathscr{G}^{\alpha}_{\gamma,P}(q) := \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 dx\, e^{\frac{\gamma}{2}Y_\tau(x)}\; \theta_1(x)^{-\frac{\alpha\gamma}{2}}\; e^{\gamma\pi P x}\right)^{-\alpha/\gamma}\right].$$

Gaussian multiplicative chaos (GMC) measure

vertex operator

See: Probabilistic conformal blocks for Liouville CFT on the torus; Ghosal, Remy, Sun, Sun; *Duke Math.J.'24,* arXiv:2003.03802

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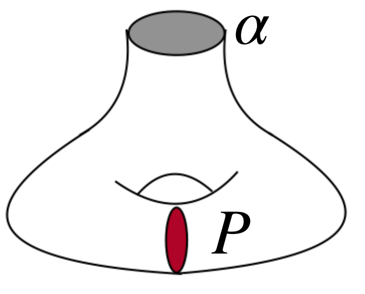

- α is the weight of the marked point = monodromy around the singularity.
- P is the integration parameter of bootstrap integral = A-cycle monodromy.
- Z is the partition function.
- The central charge

$$c = 1 + 6Q^2, \qquad Q = \frac{\gamma}{2} + \frac{2}{\gamma}. \qquad\qquad (c \to \infty \Rightarrow \gamma \to 0)$$

See: Probabilistic conformal blocks for Liouville CFT on the torus; Ghosal, Remy, Sun, Sun; *Duke Math.J.'24,* arXiv:2003.03802

The tale of two blocks

One point conformal block: $$\mathscr{G}^{\alpha}_{\gamma,P}(\tau) = \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 dx e^{\frac{\gamma}{2}Y_\tau(x)}\theta_1(x)^{-\frac{\alpha\gamma}{2}}e^{\gamma\pi P x}\right)^{-\alpha/\gamma}\right].$$

Properties: Zamolodchikov recursion, Nekrasov-Okounkov partition function, DOZZ formula
(Series rep.) (Combinatorial rep.)

Two point conformal block* (with one degenerate insertion) is defined as

$$\psi^{\alpha}_{\chi,P}(z,\tau) := \mathscr{W}(\tau)e^{\chi P z\pi}\mathbb{E}\left[\mathscr{V}^{\alpha}_{\gamma,P}(z,\tau)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right],$$

where,

$$\mathscr{V}^{\alpha}_{\gamma,P}(z,q) := \int_0^1 dx\, e^{\frac{\gamma}{2}Y_\tau(x)}\theta_1(z+x)^{\frac{\gamma\chi}{2}}\theta_1(x)^{-\frac{\alpha\gamma}{2}}\theta_1(z)^{-\frac{\gamma\chi}{2}}e^{\pi\gamma P x}.$$

Note: The two conformal blocks coincide in the limit

$$\lim_{z\to 0}\left(\theta_1(z)^{l_\chi}\psi^{\alpha}_{\chi,P}\right) = \mathscr{W}(\tau)Z^{\alpha-\chi}(\tau)\mathscr{G}^{\alpha-\chi}_{\gamma,P}(\tau).$$

*Shifting α by χ and omiting some technicalities about the range of α

The tale of two blocks

One point conformal block: $$\mathscr{G}^{\alpha}_{\gamma,P}(\tau) = \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 dx e^{\frac{\gamma}{2}Y_\tau(x)}\theta_1(x)^{-\frac{\alpha\gamma}{2}}e^{\gamma\pi P x}\right)^{-\alpha/\gamma}\right].$$

No PDE!

Two point conformal block*
(with one degenerate insertion) : $$\psi^{\alpha}_{\chi,P}(z,\tau) := \mathscr{W}(\tau)e^{\chi P z\pi}\mathbb{E}\left[\mathscr{V}^{\alpha}_{\gamma,P}(z,\tau)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right].$$

Theorem [GRSS '24]: The conformal block $\psi^{\alpha}_{\chi}(z,\tau)$ solves the BPZ equation

$$\left(2\pi i\chi^2\partial_\tau + \partial_{zz} - l_\chi(l_\chi+1)\wp(z,\tau)\right)\psi^{\alpha}_{\chi,P}(z,\tau) = 0, \quad \chi \in \left\{\frac{\gamma}{2}, \frac{2}{\gamma}\right\}, \qquad l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}.$$

*Shifting α by χ and omiting some technicalities about the range of α

Semi-classical limit

Goal: Prove the existence of $\lim_{\gamma \to 0} \gamma^2 \log \mathcal{G}^{\alpha}_{\gamma,P}(\tau)$.

Step 1. Scale $\alpha = \frac{\alpha_0}{\gamma}$, $P = \frac{P_0}{\gamma}$: $\quad \mathcal{G}^{\alpha_0}_{\gamma,P_0}(\tau) = \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 dx e^{\frac{\gamma}{2}Y_\tau(x)}\theta_1(x)^{-\frac{\alpha_0}{2}} e^{\pi P_0 x}\right)^{-\alpha_0/\gamma^2}\right]$.

Step 2. Rewrite $\quad \mathcal{G}^{\alpha_0}_{\gamma,P_0}(\tau) = \frac{1}{Z}\mathbb{E}\left[e^{-\frac{\alpha_0}{\gamma}X}\right] \sim e^{\frac{\alpha_0^2}{\gamma^2} const.}, \quad X = \frac{1}{\gamma}\log\left(\int_0^1 dx e^{\frac{\gamma}{2}Y_\tau(x)}\theta_1(x)^{-\frac{\alpha_0}{2}} e^{\pi P_0 x}\right)$.

Step 3. Note that $\gamma \to 0$ limit of X dictates the semi-classical limit of the block.

By mean value theorem: $\lim_{\gamma \to 0} X \sim \int_0^1 \partial_\gamma GMC \; f(\,.\,)$.

→ Derivative GMC measure

Modified goal: Analyse the DGMC measure.

DGMC measure

Definition (DGMC measure): For a bounded test function $f(x,\tau) \in L_\infty([0,1])$,

$$(\mathsf{DM}_{\gamma,N} f)(\tau) := \int_0^1 : Y_{\tau,N}(x) : e^{\gamma :Y_{\tau,N}(x):} f(x,\tau), \qquad (\mathsf{DM}_\gamma f)(\tau) = \lim_{N\to\infty} (\mathsf{DM}_{\gamma,N} f)(\tau)\,.$$

Theorem [H.D, Ghosal, Prokhorov '24]: For a bounded test function $f(x,\tau) \in L_\infty([0,1])$,

$$\mathbb{P}\left[(\mathsf{DM}_\gamma f)(\tau) \leq -v\right] \leq c_1(\tau) e^{-c_2(\tau) v^2}\,.$$

Strategy of the proof:

1. Show that $(\mathsf{DM}_{\gamma,N} f)(\tau)$ is a martingale.
2. Quadratic variation of the martingale gives the bound $c_1(\tau)e^{-c_2(\tau)v^2}$.
3. Analyse the $N \to \infty$ limit of the above steps.

See also: Path integral for quantum Mabuchi K-energy; Lacoin, Rhodes, Vargas; *Duke Math.J.'22,* arXiv:1807.01758

III. Results

Undeformed blocks

Theorem: For $\alpha_0 \in (-4, 2)$, $P_0 \in \mathbb{R}$, the undeformed conformal block

$$\lim_{\gamma \to 0} \gamma^2 \log \mathcal{G}^{\alpha}_{\gamma,P}(q) = -\alpha_0 \mathcal{L}(\alpha_0, P_0, q) + \lim_{\gamma \to 0} \mathcal{M}(P_0, \alpha_0, q),$$

Deterministic Probabilistic

Strategy of the proof:

1. Convergence follows from the properties of DGMC measure.
2. Uniqueness is proved in two stages:
   - deformed blocks are unique due to their relation to Lamé equation,
   - use the relation between undeformed and deformed blocks.

Theorem: There exists $r_0 > 0$ which is independent of α_0, P_0 such that the semi-classical limit of the torus conformal block $\mathcal{G}^{\alpha}_{\gamma,P}(q)$ is well defined and analytic as a function of q for all $|q| < r_0$.

Deformed blocks

Theorem ($\chi = \frac{\gamma}{2}$):

1. (Existence) The limit exists and is z-independent $\lim_{\gamma \to 0} \gamma^2 \log \psi_{\gamma/2}^{\alpha_0/\gamma}(z, \tau) =: \phi(\tau)$.

   The above statement is also known as the Zamolodchikov conjecture.

2. (Lamé) The limit $\gamma \to 0$ of the BPZ equation

$$\left(\partial_z^2 - \frac{\alpha_0}{4} \left(\frac{\alpha_0}{4} - 1 \right) \wp(z, \tau) + \frac{\pi i}{2} \partial_\tau \phi(\tau) \right) \Gamma(z; \alpha_0, P_0, \tau) = 0.$$

   with $\Gamma(\,.\,)$ being the subleading term of the limit above.

Similar theorem can be proved for $\chi = \frac{2}{\gamma}$. We get a Hamilton-Jacobi equation instead!

New insights into the Lamé equation

$$\lim_{\gamma\to 0} BPZ$$

$\chi = \frac{\gamma}{2}$ → Lamé equation:

$$\left(\partial_z^2 - \frac{\alpha_0}{4}\left(\frac{\alpha_0}{4} - 1\right)\wp(z) + \frac{\pi i}{2}\ \mathscr{E}(\alpha_0, P_0, q)\ \right)\Gamma(z; \alpha_0, P_0, q) = 0$$

$\chi = \frac{2}{\gamma}$ → Hamilton-Jacobi equation related to the action of Painlevé VI in elliptic form

See also: Quantum spectral problems and isomonodromic deformations. Bershtein, Gavrilenko, Grassi; CMP'22, arXiv:2105.00985

New insights into the Lamé equation

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Accessory parameter

$\chi = \frac{2}{\gamma}$ → Hamilton-Jacobi equation related to the action of Painlevé VI in elliptic form

Classical action

Two important points:

1. For the above relation to work, the interplay between the undeformed and deformed conformal blocks is crucial.
2. The classical action we obtain here is already regularized!

See also: Quantum spectral problems and isomonodromic deformations. Bershtein, Gavrilenko, Grassi; CMP'22, arXiv:2105.00985

New insights into the Lamé equation

$\lim_{\gamma\to 0} BPZ$

$\chi = \frac{\gamma}{2}$ → Lamé equation:

$$\left(\partial_z^2 - \frac{\alpha_0}{4}\left(\frac{\alpha_0}{4}-1\right)\wp(z) + \frac{\pi i}{2}\,\mathscr{E}(\alpha_0, P_0, q)\right)\Gamma(z;\alpha_0,P_0,q) = 0$$

$\chi = \frac{2}{\gamma}$ → Hamilton-Jacobi equation related to the action of Painlevé VI in elliptic form

The solution $\Gamma(z;\alpha_0,P_0,q) := e^{\frac{P_0 z\pi}{2}} \exp\left(-\frac{\alpha_0\int_0^1 \log\left(\frac{\theta_1(z+x)}{\theta_1(z)}\right)\theta_1(x)^{-\frac{\alpha_0}{2}} e^{\pi P_0 x} dx}{4\int_0^1 \theta_1(x)^{-\frac{\alpha_0}{2}} e^{\pi P_0 x} dx}\right).$

Happy 70th birthday Antti!