Entanglement Area-type Bounds for Pure States of Rapid Decorrelation

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The entropy of the restriction of a pure quantum state of a lattice system to a subset is a measure of the entanglement between the system's two components. The talk will focus on conditions that imply an area-type bound on the entanglement in states of quantum lattice models, and the criterion's application to the ground states of the quantum Ising model.

(Joint work with Simone Warzel)

Path Integrals and Friends
Celebrating Antti-Jukka Kupiainen
Helsinki, 3 Sept 2024.

An example: the Quantum Ising Model in transverse field (QIM)

A finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The Hilbert space of state vectors: $\mathcal{H}_{\mathcal{V}} = \otimes \mathbb{C}^2$ ("qbits")

Local spin ops: $\underline{\sigma}_u = (\sigma^x_u, \sigma^y_u, \sigma^z_u)$, with $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The Hamiltonian: $H = \sum_{(u,v)\in\mathcal{E}(\mathcal{G})} J_{u,v}\sigma^z_u\sigma^z_v - \eta \sum_{u\in\mathcal{V}} \sigma^x - \left[h \sum_{u\in\mathcal{V}} \sigma^z\right]$ (we focus on $h = 0$).

If $\mathcal{H}$ has a unique ground state $|\Psi\rangle$ then: $\boxed{\langle\Psi|Q|\Psi\rangle = \lim_{\beta\to\infty} \frac{\mathrm{tr}\ Q\, e^{-\beta\mathcal{H}}}{\mathrm{tr}\ e^{-\beta\mathcal{H}}}}$

More general thermal states: $\boxed{Q \mapsto \langle Q\rangle = \mathrm{tr}\, Q\, e^{-\beta\mathcal{H}}/Z(\beta)}$ $Z_{\mathcal{V}}(\beta) = \mathrm{tr}_{\mathcal{H}_{\mathcal{V}}}\, e^{-\beta\mathcal{H}}$.

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The relevant *path integral* (!) is over the space of spin configurations expressed in a convenient *computational basis*, e.g. of $|(\sigma^z)\rangle = \otimes_u |\sigma^z_u\rangle$.

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Z(\beta) &= \sum_{(\sigma^z)} \langle (\sigma^z) | e^{-\beta H} | (\sigma^z) \rangle = \\
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The quantum state invokes the restriction to $t =$ Const of a $(d+1)$ dimensional Ising model.
Note: i) Ising model's Markov property is lost, ii) quantum phase transition at $T = 0$!)

Quantum states: = expectation value functionals $\boxed{Q \mapsto \langle Q \rangle = \mathrm{tr}\, Q\rho}$ $\quad \rho \geq 0, \quad \mathrm{tr}\, \rho = 1.$

State's entropy: $\quad S(\rho) := -\sum_n \lambda_n \log \lambda_n$ $\qquad$ (λ_n ranging over eigenvalues of ρ)

Pure states: $\quad \rho = |\Psi\rangle\langle\Psi|\,, \quad S(|\Psi\rangle\langle\Psi|) = 0$

Schmidt decomp.: each normalized vector in a product space $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$\boxed{|\Psi\rangle = \sum_{j=1}^{N:=\text{'Schmidt rank'}} \sqrt{p_j}\; |a_j\rangle \otimes |b_j\rangle} \qquad \left(\text{e.g., } \begin{cases} |\Psi\rangle_{(Bell)} = \frac{1}{\sqrt{2}} \left[|+,+\rangle + |-,-\rangle\right] \\ |\Psi\rangle_{(EPR)} = \frac{1}{\sqrt{2}} \left[|+,-\rangle - |-,+\rangle\right] \end{cases} \right)$$

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Thus for pure states $\varrho_A(\psi) := \mathrm{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi| = \sum_j p_j \; |a_j\rangle\langle a_j|$

$$\varrho_B(\psi) := \mathrm{tr}_{\mathcal{H}_A} |\psi\rangle\langle\psi| = \sum_j p_j \; |b_j\rangle\langle b_j|$$

Entanglement entropy: In the above case

$$S(\rho_A) = S(\rho_B) = -\sum_j p_j \log p_j \qquad =: \text{ "}A \leftrightarrow B \text{ entanglement entropy in the state } |\Psi\rangle\langle\Psi|\text{"}$$

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Mutual information: a more general concept

$$\boxed{I_\rho[A,B] = S(\rho_A) + S(\rho_B) - S(\rho_{AB})} \equiv S(A) + S(B) - S(AB)$$

Quantum lattice systems: $\boxed{\mathcal{H} = \otimes_{u\in W}\mathcal{H}_u}$, with $\boxed{W \subset \mathbb{Z}^d}$, $\mathcal{H}_u \approx \mathbb{C}^\nu$ ('qdits')

Restriction of **pure state state** $|\psi\rangle$ of **quantum lattice system** to $A \subset W \subset \mathbb{Z}^d$:

$$\varrho_A(\psi) := \mathrm{tr}_{\mathcal{H}_{W\setminus A}} |\psi\rangle\langle\psi|$$

Entanglement entropy: $S(\varrho_A) := -\,\mathrm{tr}_{\mathcal{H}_A}\, \varrho_A \ln \varrho_A = \sum_{j=1}^{\dim \mathcal{H}_A} \lambda_j^{\downarrow}(\varrho_A) \log \lambda_j^{\downarrow}(\varrho_A)^{-1}$

$$I_\varrho(A \,|\, A^c) \begin{cases} \propto \quad \mathcal{O}(|A|) & \text{under } \rho = |\Psi\rangle\langle\Psi| \text{ at generic } \Psi \in \mathcal{H}_W \qquad \textit{(Page's law)} \\ \leq \quad \beta\, \mathcal{O}(|\partial A|) & \text{under } \rho = e^{-\beta H}/Z_{W,\beta}\text{ , } H \text{ of finite range} \\ & \qquad \textit{(Wolf, Verstraete, Hastings, Cirac)} \\ \propto \quad \mathcal{O}(|\partial A|) & \text{expected for non-critical ground states } \Psi \qquad \textit{(area law)} \end{cases}$$

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A state is said to be **stoquastic** if, in a basis of $\mathcal{H}_w$ based on configurations $\boxed{\boldsymbol{\sigma} = (\sigma_u)_{u\in W}}$

$$|\Psi\rangle = \sum_{\boldsymbol{\sigma}=(\sigma_u)_{u\in W}} \sqrt{p(\boldsymbol{\sigma})}\;|\boldsymbol{\sigma}\rangle \quad \text{with: } |\boldsymbol{\sigma}\rangle = \otimes_u |\sigma_u\rangle, \quad p(\boldsymbol{\sigma}) \equiv p((\sigma_u)_{u\in W}) \geq 0\,.$$

We may refer to such states as $|P\rangle$ – in terms of the probability function $p(\boldsymbol{\sigma})$.

Stoquastic state (= sign-problem free): $|P\rangle = \sum_{\boldsymbol{\sigma}} \sqrt{p(\boldsymbol{\sigma})} \otimes_u |\boldsymbol{\sigma_u}\rangle$ with $p((\{\sigma_u\}) \geq 0$

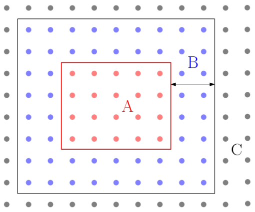
Examples:

- $|\psi_\beta\rangle \propto \exp(-\beta H)\,|\widetilde{\boldsymbol{\sigma}}\rangle$ for positivity preserving H
- ground state of the **quantum Ising model** $H = -\sum_{u,v} J_{u,v}\, S_u^{(3)} S_v^{(3)} - \eta \sum_u S_u^{(1)}$
- p from **classical Ising model's Gibbs equilibrium measure** sampled e.g. along hyperplanes
- ground state of antiferromagnetic spin S chains with the **projection based interaction**

$$H = -(2S+1) \sum_{u\sim v} K_{u,v}^{(0)}$$

which for $S = 1/2$ coincides with the d-dimensional **quantum Heisenberg model**, …

Intuition: surrounding $A \subset W$ by a buffer zone B



- $|\psi_W\rangle$ non-critical ground state in $W = A \cup B \cup C$
$$\Rightarrow \quad \varrho_A(\psi_W) \approx \varrho_A(\psi_{AB})$$
- $\mathrm{rank}\,\varrho_A(\psi_{AB}) \sim 2^{|B|}$ Compared with $\varrho_A(\psi_{AB})$ this could be low rank !!!! Is it of high fidelity ?
- *Potentially Yes* – if conditioned on $\boldsymbol{\sigma}_B$ the state's correlations of A and C are "typically" *weak*

Hence, seeking **high-fidelity, low complexity approximation** we are led to:

$$\text{approximate} \quad |P\rangle \quad \longleftrightarrow \quad p(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_C) = p(\boldsymbol{\sigma}_B) \cdot p(\boldsymbol{\sigma}_C|\boldsymbol{\sigma}_B) \cdot p(\boldsymbol{\sigma}_A|\boldsymbol{\sigma}_B\, \boldsymbol{\sigma}_C)$$

$$\text{by} \quad |P_{(B)}\rangle \quad \longleftrightarrow \quad p_{(B)}(\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_C) = p(\boldsymbol{\sigma}_B) \cdot p(\boldsymbol{\sigma}_A|\boldsymbol{\sigma}_B) \cdot p(\boldsymbol{\sigma}_C|\boldsymbol{\sigma}_B)$$

In other words: we approximate the reduced state $\rho_A(P)$ by

$$\varrho_A(P_{(B)}) = \sum_{\boldsymbol{\sigma}_B} p(\boldsymbol{\sigma}_B)\, |\varphi_A(\boldsymbol{\sigma}_B)\rangle\langle\varphi_A(\boldsymbol{\sigma}_B)| \qquad |\varphi_A(\boldsymbol{\sigma}_B)\rangle := \sum_{\boldsymbol{\sigma}_A} \sqrt{p(\boldsymbol{\sigma}_A|\boldsymbol{\sigma}_B)}\, |\boldsymbol{\sigma}_A\rangle\,.$$

Fidelity of quantum states: $F(\varrho_1, \varrho_2) \stackrel{def}{=} \|\sqrt{\varrho_1}\sqrt{\varrho_2}\|_1^2$

It controls the trace distance: $[1 - \sqrt{F(\varrho_1, \varrho_2)}] \leq \frac{1}{2}\|\varrho_1 - \varrho_2\|_1 \leq 2[1 - \sqrt{F(\varrho_1, \varrho_2)}]$.

Uhlmann's variational principle: $F\Big(\rho_A(|\Psi\rangle\langle\Psi|), \varrho_A(|\Phi\rangle\langle\Phi|)\Big) \geq |\langle\Psi|\Phi\rangle|^2$.

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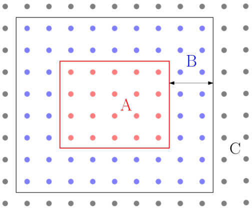
As a tool for estimating the fidelity of the approximation of P by $P_{(B)}$ we employ the following measure of **conditional correlation** between A and C, conditioned on $\boldsymbol{\sigma}_B$:

$$\delta_B(A \,|\, C) := \sum_{\boldsymbol{\sigma}_B} p(\boldsymbol{\sigma}_B)\, \widehat{\delta}_{\boldsymbol{\sigma}_B}(A \,|\, C)$$

with $\widehat{\delta}$ the **total variation measure for conditional dependence:**

$$\widehat{\delta}_{\boldsymbol{\sigma}_B}(A \,|\, C) := \sum_{\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_C} [p(\boldsymbol{\sigma}_A \boldsymbol{\sigma}_C | \boldsymbol{\sigma}_B) - p(\boldsymbol{\sigma}_A | \boldsymbol{\sigma}_B)\, p(\boldsymbol{\sigma}_C | \boldsymbol{\sigma}_B)]_+$$

Note: $\widehat{\delta}_{\boldsymbol{\sigma}_B}(A \,|\, C) = 0$ for p classical Gibbs measures, if range $<$ width B.



Theorem 1 ***(our fidelity estimate)*****:** For any stoquastic vector $|P\rangle$ and its approximation $|P_{(B)}\rangle$

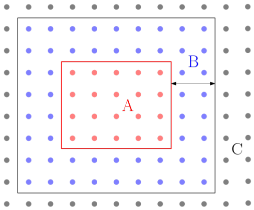
$$\boxed{\frac{1}{4}\|\rho_A(P) - \varrho_A(P_{(B)})\|_1 \;\leq 1 - |\langle P_{(B)}|P\rangle|^2 \leq \; 2\, \delta_B(A \,|\, C)}$$

Recap: As a measure of conditional correlation between A and C ,conditioned on $\boldsymbol{\sigma}_B$, we employ:

$$\delta_B(A\,|\,C) := \sum_{\boldsymbol{\sigma}_B} p(\boldsymbol{\sigma}_B)\; \widehat{\delta}_{\boldsymbol{\sigma}_B}(A\,|\,C)$$

with

$$\widehat{\delta}_{\boldsymbol{\sigma}_B}(A\,|\,C) := \sum_{\boldsymbol{\sigma}_A,\boldsymbol{\sigma}_C} [p(\boldsymbol{\sigma}_A\boldsymbol{\sigma}_C|\boldsymbol{\sigma}_B) - p(\boldsymbol{\sigma}_A|\boldsymbol{\sigma}_B)\; p(\boldsymbol{\sigma}_C|\boldsymbol{\sigma}_B)]_+$$



Theorem 2: (Area law bound, general dimension) If for all A and buffers B_l, of widths $l > 0$,

$$\delta_{B_l}(A\,|\,C) \leq \exp\left(-[l - l_0(A)]_+/\xi\right), \quad \xi \in (0,\infty)$$

then

$$\boxed{S(\varrho_A(P)) \leq C\,|\partial A|\; l_0(A)} \quad \Big(\text{in cases of interest } l_0(A) \approx \log|\partial A|\Big)$$

$\Big[$A weaker alternative condition: $\delta_{B_l}(A\,|\,C) \leq (1 + [l - l_0(A)]_+/\xi)^{-\alpha}$ (power law decay)

with some $\alpha > 2$ and $\xi > 0$ $\Big]$

Theorem 3: *(*FKG boost) For stoquastic states whose probability distributions p has the FKG property

$$\delta_B(A \,|\, C) \le \frac{1}{4} \sum_{u \in A, v \in C} \max_{D \supset B} \max_{\boldsymbol{\sigma}_D} \langle \sigma_u ; \sigma_v \rangle_{\boldsymbol{\sigma}_D}$$

(inspired by Lebowitz '72)

A specific example:

For the quantum Ising model in transverse field one has: $\boxed{\langle \sigma_u ; \sigma_v \rangle_{\boldsymbol{\sigma}_D} \le \langle \sigma_u \sigma_v \rangle}$ (an extension of the classical model's recent **correlation inequality** of Ding-Song-Sun '22).

Based on that, and the model's known "sharpness of the phase transition", the model's sub-critical ground states satisfy:

$$\delta_{B_l}(A \,|\, C) \le c_\xi \, |\partial A| \, \exp\left(-l/\xi\right), \quad \xi \in (0, \infty)$$

We conclude that the QIM's ground states exhibit area-size entanglement – **up to the model's quantum phase transition**.

- **Some references of relevance:**

  1D non-critical ground states: Hastings '07 , Brandao-Horodecki '15, …

  Subset of sub-critical 1D Quantum Ising model (by other means)
  Grimmet-Osborn-Scudo '08 &'20

  Case $d \in \{1, 2\}$ and ground-states of gapped, frustration free Hamiltonians:
  detectibility argument by Anshu-Arad-Gosset '22

  Random current method for Quantum Ising: Björnberg-Grimmett, Crawford-Ioffe '09

- **Expectation:**

  Strict area law of non-critical ground-states of local Hamiltonians vs.
  log-corrected area law at criticality (Rényi-2 entropy). Calabrese-Cardy '04

- **Further references and results:** Aizenman-Warzel (arXiv preprint '24)

Thank you for your attention!

Congratulations Antti
on your multiple accomplishments,
and best wishes for challenges and joys ahead !!

Some key estimates in the proof of

Theorem 2: (Area law bound) Exponential decoupling over buffers B_l of widths $l > 0$

$$\delta_{B_l}(A \mid C) \leq \exp\left(-[l - l_0(A)]_+/\xi\right), \quad \xi \in (0, \infty)$$

implies $\boxed{S(\varrho_A(P)) \leq C\,|\partial A|\, l_0(A)}$

To bound the entropy, break $S(\varrho_A(P)) = -\sum_{j=1}^{\dim \mathcal{H}_A} \lambda_j \ln \lambda_j$ into:

$$-\sum_{j=1}^{\dim \mathcal{H}_{B_l}} \lambda_j \ln \lambda_j \leq \ln \dim \mathcal{H}_{B_l} = |B_l| \ln \nu$$

$$-\sum_{j>\dim \mathcal{H}_{B_l}} \lambda_j \ln \lambda_j \leq \mu \ln \frac{\dim \mathcal{H}_A}{\mu} \qquad \text{with} \quad \mu_l := \sum_{j>\dim \mathcal{H}_{B_l}} \lambda_j .$$

From **fidelity bound** with buffer of width l:

$$\begin{aligned}
\mu_l &= 1 - \max\left\{\operatorname{tr} \varrho_A P_N \mid P_l \text{ orthogonal projection of rank } P \leq \dim \mathcal{H}_{B_l}\right\} \\
&= 1 - \max\left\{F(\rho_A, \widehat{\varrho}_A) \mid \widehat{\varrho}_A \text{ is a state on } A \text{ with rank } \widehat{\varrho}_A \leq \dim \mathcal{H}_{B_l}\right\} \\
&\leq 2\, \delta_{B_l}(A \mid C) \leq 2 \exp\left(-[l - l_0(A)]_+/\xi\right) .
\end{aligned}$$

Optimize, and apply the bound on a range of scales ...