How finite model theory came to Finland and what happened next?

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Espoo, August 2022

Part I: Early history

Before FMT came to Finland...

Symposium in mathematical logic in Oulu, summer of 1974

Seppo Miettinen
Per Lindström
Michał Krynicki
Finn Jensen
Dag Westerståhl

A close encounter with FMT

Workshop in Karpacz, Poland, fall of 1974

Hájek, Petr: Generalized quantifiers and finite sets.

PHDs in Helsinki after my own PHD in Manchester 1977 and return to Helsinki in 1978

- 1984: Maaret Karttunen, Model theory for Infinitely deep languages
- 1987: Tapani Hyttinen, Games and infinitary languages
- 1988: Lauri Hella, Definability hierarchies of generalized quantifiers
- 1990: Heikki Tuuri, Infinitary languages and Ehrenfeucht–Fraïssé games
- 1991: Taneli Huuskonen, Comparing notions of similarity for uncountable models
- 1992: Kerkko Luosto, Filters in abstract model theory

The 1990 European Summer Meeting of the Association for Symbolic Logic was held in Finland from July 15 to July 22, 1990. The meeting was called **Logic Colloquium '90** and it took place in the Porthania building of the University of Helsinki as part of the program of the 350th anniversary of the university.

Logic Colloquium '90 speakers

Wilfried Buchholz
Barry Cooper
Patrick Dehornoy
Hans-Dieter Donder
Dov Gabbay
Warren Goldfarb
Jaakko Hintikka
Ian Hodkinson
Ronald Jensen
Haim Judah
Phokion Kolaitis
Richard Laver
Per Martin-Löf
Alan Mekler
Grigori Mints
Yiannis Moschovakis
Tulende Mustafin
Ludomir Newelski
Francoise Point
Jean-Pierre Ressayre
Saharon Shelah
Hugh Woodin

A portion of the program

JAAKKO HINTIKKA (Boston)
Is there completeness in mathematics after Gödel?

IAN HODKINSON (London)
An axiomatisation of the temporal logic with until and since over real numbers

RONALD JENSEN (Oxford)
Remarks on the core model

HAIM JUDAH (Bar-Ilan)
Δ^1_3*-sets of reals*

PHOKION KOLAITIS (Santa Cruz)
1. *Logical definability and complexity classes*
2. *Model theory of finite structures*
3. *0-1 laws*

RICHARD LAVER (Boulder)
Elementary embeddings of a rank into itself

PER MARTIN-LÖF (Stockholm)
Logic and metaphysics

ALAN MEKLER (Vancouver)
Almost free algebras: 20 years of progress

GRIGORI MINTS (Stanford)
Gentzen-type systems and resolution rule for modal predicate logic

Mukkula Logic Summer School, Lahti, Finland, 1991

PhDs in FMT in Finland

- Phokion's mini-courses first in Helsinki in 1990 and then in Mukkula in 1991 marked the beginning of finite model theory in Finland
- Doctoral studies in Finland more or less inspired by FMT: Nurmonen (1996), Kaila (2001), Kontinen Juha (2004), Couceiro (2006), Niemistö (2007),
- ...or team semantics on finite models: Nurmi (2009), Kontinen Jarmo (Amsterdam 2010), Kuusisto (2011), Galliani (Amsterdam 2012), Yang (2014), Virtema (2014), Hannula (2015), Paolini (2016), Rönnholm (2018), Anttila (202?), M. Hirvonen (202?), Iso-Tuisku (202?), Puljujärvi (202?), Quadrellaro (202?), Sandström (202?), Vilander (202?).
- Sorry if I forgot someone!

Workshops on team semantics

1. November 7-9, 2009, First Workshop of the DepLog Group of LINT, Stockholm, Sweden
2. August 16-20, 2010, ESSLLI Workshop on Dependence and Independence in Logic , Copenhagen, Denmark
3. September 22, 2012, Workshop on Dependence Logic and Strategic Reasoning, University of Amsterdam, The Netherlands
4. February 10 – 15 , 2013, Dagstuhl Seminar 13071 "Dependence Logic: Theory and Applications", Dagstuhl, Germany
5. June 17, 2013, Workshop on Inquisitive Logic and Dependence Logic, University of Amsterdam, The Netherlands
6. March 3-5, 2014, KNAW Academy Colloquium "Dependence Logic", Amsterdam, The Netherlands
7. June 21–26 , 2015, Dagstuhl Seminar 15261 "Logics for Dependence and Independence", Dagstuhl, Germany
8. January 13-18, 2019, Dagstuhl Seminar 19031 "Logics for Dependence and Independence", Dagstuhl, Germany
9. August 10-12, 2020, Workshop on Logics of Dependence and Independence, Online
10. August 9-10, 2021, ESSLLI Workshop on Logics of Dependence and Independence (LoDE 2021), Online

Some early developments in FMT in general

- Fagin: Contributions to the model theory of finite structures. University of California, Berkeley. 1973.
- Vardi: Implication problem for data dependencies in the relational model, The Hebrew University in Jerusalem. 1981.
- Gurevich: Toward logic tailored for computational complexity. 1983.
- Kolaitis, Prömel, Rothschild: Asymptotic enumeration and a 0-1 law for m-clique free graphs. 1985, 1987.
- Kolaitis, Vardi, Infinitary logics and 0-1 laws. 1992.
- Dawar, Feasible computation through model theory. Thesis (Ph.D.)–University of Pennsylvania. 1993.
- Makowsky, Pnueli, Oracles and quantifiers. 1994.

Part II. Generalized quantifiers

Finite model theory in Finland: descent from uncountable to finite

- Ehrenfeucht-Fraïssé-games. Pebble games. Bijective games.
- Generalized quantifiers. Their hierarchies.
- Infinitary logic (with finitely many variables).
- Dependence logic.

My third most cited paper:

ELSEVIER

Annals of Pure and Applied Logic 74 (1995) 23–75

ANNALS OF PURE AND APPLIED LOGIC

Generalized quantifiers and pebble games on finite structures

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Received 12 March 1993; revised 25 March 1994; communicated by A. Lachlan

- "What has to be added to first-order logic in order to capture exactly all polynomial-time properties of finite structures?"
- "Expand the framework of abstract model theory in a way that allows for a treatment of finite model theory".

Theorem

Suppose $\mathcal{Q}$ is a finite sequence of simple unary quantifiers on finite models.

1. *The Härtig quantifier I is not expressible in $\mathcal{L}^{\omega}_{\infty\omega}(\mathcal{Q})$.*
2. *The query "is E an equivalence relation with an even number of equivalence classes?" is not expressible in $\mathcal{L}^{\omega}_{\infty\omega}(I, \mathcal{Q})$.*

Proofs used Ramsey theory, such as van der Waerden's Theorem and Folkman's Theorem.

Lauri Hella: **Logical hierarchies in PTIME.** Information and Computation 129 (1996).

- For each n, there is a polynomial time computable query which is not definable in any extension of fixpoint logic by n-ary quantifiers.
- This rules out the possibility of characterizing PTIME in terms of definability in fixpoint logic extended by a finite set of generalized quantifiers.

Hella: "I also give my special thanks to Phokion Kolaitis, from whom I have learned everything I know about fixpoint logics, DATALOG, and finite variable logics."

- Hella, Kolaitis, Luosto, Almost everywhere equivalence of logics in finite model theory. 1996
- Hella, Kolaitis, Luosto, How to define a linear order on finite models. 1997
- Dawar, Gottlob, Hella, Capturing relativized complexity classes without order. 1998
- Hella, Imhof, Enhancing fixed point logic with cardinality quantifiers. 1998
- Dawar, Hella, Seth, Ordering finite variable types with generalized quantifiers. 1998.
- Hella, Libkin, Nurmonen, Notions of locality and their logical characterizations over finite models. 1999.

A hierarchy result for generalized quantifiers on finite models

$Qx\phi(x)$	type	(1)
$Qxy\phi(x,y)$	type	(2)
$Qxy,z\phi(x,y)\psi(z)$	type	$(2,1)$
etc		

Theorem (Hella, Luosto and V. 1996)

For each similarity type s there is a generalized quantifier Q of type s so that Q is not definable in the extension of first order logic by all generalized quantifiers of type lower than s.

An idea ...

"Pseudo-finite model theory", *Matematica Contemporanea*, (V. 2003)

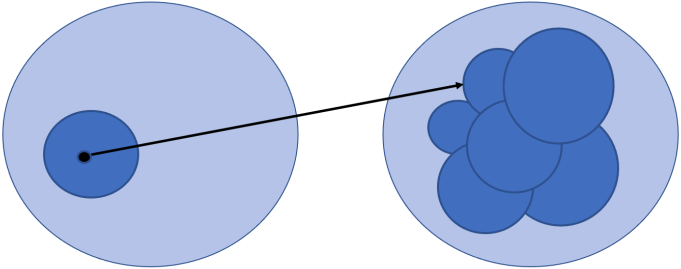
"We consider the restriction of first–order logic to models, called pseudo-finite, with the property that every first–order sentence true in the model is true in a finite model. We argue that this is a good framework for studying first-order logic on finite structures. We prove a Lindström Theorem for extensions of first order logic on pseudo-finite structures."

Part III: Team semantics.

How I learned to stop worrying ... and understand team semantics

- 11th LMPS 1999, Cracow: Michał Krynicki gave a talk about the Hodges semantics (“team semantics”) for so-called IF-logic. Petr Hájek stood up. Flight back home.
- On the semantics of informational independence, Log. J. IGPL, 2002.
- Dependence logic, CUP 2007.
- From infinite to finite models.

Team semantics



The set of assignments satisfying a formula

The set of teams satisfying a formula

Tarski's semantics

Team semantics

Dependence logic $\mathcal{D}$—A one slide sketch

- A team T, is a set (any set) of assignments of values to a fixed set of variables.
- A team T satisfies a *dependence atom* $=(x, y)$ if the values of the variables x completely determine (in T) the values of the variables y i.e. $\forall s, s' \in T(s(x) = s'(s) \to s(y) = s'(y))$.
- We can build a logic where dependence atoms are the atomic formulas, as well as the usual ones $x = y, x \neq y, R(x), \neg R(x)$.
- We have the 'usual' first order logical operations $\wedge, \vee, \forall, \exists$. (They agree with their *usual* meaning, if no dependence atoms are present.)
- The resulting logic is called **dependence logic** $\mathcal{D}$.

Paradigm shift

- The original (1999) paradigm of a *team* was a set of plays in a semantic game.
- Connection to database dependency theory only unfolded six years later (2005), when Peter van Emde Boas pointed out to me that my $=\!(x, y)$ is well known in computer science as functional dependence $x \Rightarrow y$.
- Current paradigms are database and experimental data.

NP

Theorem (Kontinen & V. 2009)

The properties of teams definable in $\mathcal{D}$ are exactly the downward closed NP properties of teams.

PTIME

- If we start from *inclusion* dependency instead of functional dependency, we get **inclusion logic**.
- On finite models this is in a precise sense equal in expressive power to fixpoint logic, i.e. on finite ordered models to **PTIME**. (Galliani-Hella CSL 2013)

Fragments, fragments,...

- Lauri Hella and Phokion Kolaitis: **Dependence logic vs. constraint satisfaction**. CSL 2016.
- Identified a natural fragment of universal dependence logic and showed that, in a precise sense, the fragment captures constraint satisfaction.
- "During the past decade, dependence logic has emerged as a formalism suitable for expressing and analyzing notions of dependence and independence that arise in different scientific areas." (Hella & Kolaitis, CSL 2016)

A richer picture

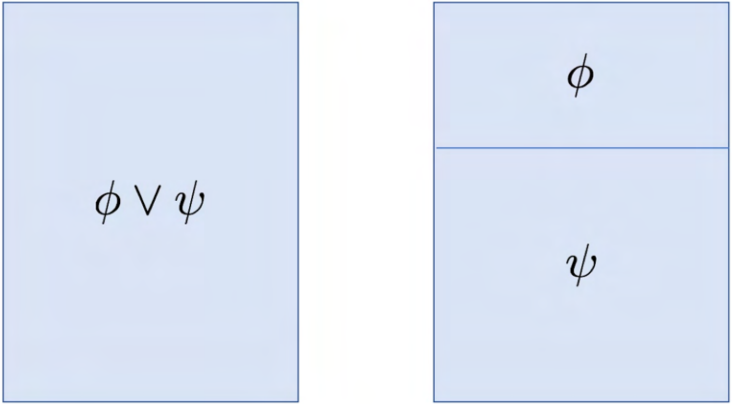
- **First order literals** θ: $M \models_T \theta$ if and only if $M \models_s \theta$ for all $s \in T$.
- **Dependence atom**: $M \models_T =\!(\vec{x}, y)$ if and only if $s(\vec{x}) = s'(\vec{x})$ implies $s(y) = s'(y)$ for all $s, s' \in T$.
- **Constancy atom**: $M \models_T =\!(y)$ if and only if $s(y) = s'(y)$ for all $s, s' \in T$.
- **Exclusion atom**: $M \models_T \vec{x} \mid \vec{y}$ if and only if for every $s, s' \in T$ we have $s(\vec{x}) \neq s'(\vec{y})$.
- **Inclusion atom**: $M \models_T \vec{x} \subseteq \vec{y}$ if and only if for every $s \in T$ there is $s' \in T$ such that $s(\vec{x}) = s'(\vec{y})$.
- **Anonymity atom**: $M \models_T \vec{x} \Upsilon y$ if and only if for every $s \in T$ there is $s' \in T$ such that $s(\vec{x}) = s'(\vec{x})$ and $s(y) \neq s'(y)$.
- **Independence atom**: $M \models_T \vec{x} \perp \vec{y}$ if and only if for every $s, s' \in T$ there is $s'' \in T$ such that $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$.

The basic logical operations, others will follow...

In a model $\mathcal{M}$, a team X satisfies:

$\phi \wedge \psi$	iff	X satisfies ϕ and ψ
$\phi \vee \psi$	iff	$X = Y \cup Z$ s.t. Y satisfies ϕ and Z satisfies ψ
$\exists x\phi$	iff	$X(F/x)$ satisfies ϕ for some $F : X \to \mathcal{P}^*(M)$
$\forall x\phi$	iff	$X(M/x)$ satisfies ϕ
$Qx\phi$	iff	$(\exists F : X \to \mathcal{P}(M^2))(M \models_{X(F/xy)} \phi$ and $\forall s \in X((M, F(s)) \in Q)$.

Notation: $\mathcal{P}^*(M) = \mathcal{P}(M) \setminus \{\emptyset\}$. $s(a/x)$ is like s except at x the value is a.
$X(F/x) = \{s(a/x) : s \in X, a \in F(s)\}$. $X(M/x) = \{s(a/x) : s \in X, a \in M\}$.


$\phi \vee \psi$
ϕ
ψ

Some team logics

New atom	New logic	Sent.	Formulas
$=\!(x)$	Constancy logic	FO	$\neq$FO
$=\!(x, y)$ $x \mid y$	Dependence logic = Exclusion logic	NP	$\downarrow$-closed NP
$x \Upsilon y$ $x \subseteq y$	Anonymity logic = Inclusion logic	P on o. f.	$\subset$ Additive P on o. f.
$x \perp y$	Independence logic	NP	NP

Definability of an atom from other atoms

Lemma (Galliani)

(a) *The k-ary dependence atom* $=\!(x,y)$ *is definable from the $k+1$-ary exclusion atom* $xz|yu$ *and also in terms of the $k+1$-ary independence atom* $xz \perp yu$. *The k-ary exclusion atom is definable from the k-ary dependence atom.*

(b) *The k-ary exclusion atom is definable in terms of the k-ary inclusion and the k-ary independence atoms.*

(c) *The k-ary inclusion atom is definable from the (k,2)-ary independence atom.*

(d) *The k-ary anonymity atom is definable in terms of the $k+1$-ary inclusion atom.*

Problem: How to show that such definability results cannot be essentially improved?

Part VI: Dimension theory

Joint ongoing work with Lauri Hella and Kerkko Luosto.

The background

- 2009-2020: Ciardelli, Hella, Luosto, Lück, Sano, Stumpf, Vilander and Virtema.
- Matroid rank, Vapnik–Chervonenkis- or VC-dimension.

Dimension of a family $\mathcal{A}$ of (arbitrary) sets

- *Convex* if for all $S, T \in \mathcal{A}$,

$$A \subseteq C \subseteq B \Rightarrow C \in \mathcal{A}.$$

- *Dominated (by $\bigcup \mathcal{A}$)* if $\bigcup \mathcal{A} \in \mathcal{A}$.
- $\mathcal{G} \subseteq \mathcal{A}$ *dominates* $\mathcal{A}$ if there exist dominated convex families $\mathcal{D}_G$, $G \in \mathcal{G}$, such that $\bigcup_{G \in \mathcal{G}} \mathcal{D}_G = \mathcal{A}$ and $\bigcup \mathcal{D}_G = G$, for each $G \in \mathcal{G}$.
- The *dimension* of the family $\mathcal{A}$ is

$$\mathrm{D}(\mathcal{A}) = \min\{|\mathcal{G}| \mid \mathcal{G} \text{ dominates the family } \mathcal{A}\},$$

Some relevant operators on families of sets

- The intersection operator $\mathcal{A} \cap \mathcal{B}$.
- The tensor disjunction operator: $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.
- Let $f \colon X \to Y$, where $X = X_0 \times \cdots \times X_{m-1}$ and $Y = X_0 \times \cdots X_{i-1} \times X_{i+1} \times \cdots \times X_{m-1}$ be defined by $f(a_0, \ldots, a_{m-1}) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m-1})$. The projection operator is $\Delta^X_{\exists i}(\mathcal{A}) = \{f[A] : A \in \mathcal{A}\}$.
- Given a set $B \in \mathcal{P}(Y)$, let

$$B[X_i/i] = \begin{array}{l} \{(a_0, \ldots, a_{m-1}) \in X \mid \\ (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m-1}) \in B, a_i \in X_i\}. \end{array}$$

  The universal quantifier operator is: $\Delta^X_{\forall i}(\mathcal{A}) = \{B \in \mathcal{P}(Y) \mid B[X_i/i] \in \mathcal{A}\}$.

Semantics via operators

We denote by $\|\phi\|^{M,\vec{x}}$ the set of teams T such that $M \models_T \phi$ when ϕ is a formula with free variables among $\vec{x}$, $\mathrm{len}(\vec{x}) = m$.

$$
\begin{aligned}
\|\phi \wedge \psi\|^{M,\vec{x}} &= \|\phi\|^{M,\vec{x}} \cap \|\psi\|^{M,\vec{x}} \\
\|\phi \vee \psi\|^{M,\vec{x}} &= \|\phi\|^{M,\vec{x}} \vee \|\psi\|^{M,\vec{x}} \\
\|\exists x_i \phi\|^{M,\vec{x}^-} &= \Delta^{M^m}_{\exists i}(\|\phi\|^{M,\vec{x}}) \\
\|\forall x_i \phi\|^{M,\vec{x}^-} &= \Delta^{M^m}_{\forall i}(\|\phi\|^{M,\vec{x}}),
\end{aligned}
$$

where $\vec{x}^-$ is the tuple obtained from $\vec{x}$ by deleting the component x_i.

Dimension function $\mathrm{Dim}_{\phi,\vec{x}}$

$$\mathrm{Dim}_{\phi,\vec{x}}(n) = \sup\left\{ \mathrm{D}(\|\phi\|^{M,\vec{x}}) \mid M \text{ is a model}, |M| = n \right\}.$$

Recall:

$$\mathrm{D}(\mathcal{A}) = \min\{|\mathcal{G}| \mid \mathcal{G} \text{ dominates the family } \mathcal{A}\},$$

First order — dimension is 1.

For every (classical) first order formula ϕ we have

$$\|\phi\|^{M,\vec{x}} = \mathcal{P}(T_\phi),$$

where $T_\phi = (\|\phi\|^M =)\{s \in M^m \mid M \models_s \phi\}$. Thus for first order ϕ the family $\|\phi\|^{M,\vec{x}}$ is dominated (by T_ϕ), downward closed, and convex. So $\mathrm{Dim}_{\phi,\vec{x}}(n) = 1$.

Explicit dimension function computations

1. $\mathrm{Dim}_{\phi,\vec{x}}(n) = 1$ for every first order ϕ.
2. $\mathrm{Dim}_{=(y),y}(n) = n$.
3. $\mathrm{Dim}_{=(\vec{x},y),\vec{x}y}(n) = n^{n^k}$, where $\mathrm{len}(\vec{x}) = k$.
4. $\mathrm{Dim}_{\vec{x}|\vec{y},\vec{x}\vec{y}}(n) = 2^{n^m} - 2$, where $\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = m$.
5. $\mathrm{Dim}_{\vec{x}\subseteq\vec{y},\vec{x}\vec{y}}(n) = 2^{n^k} - n^k$, where $\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = k$.
6. $\mathrm{Dim}_{\vec{x}\perp\vec{y},\vec{x}\vec{y}}(n) = (2^{n^m} - n^m - 1)(2^{n^k} - n^k - 1) + n^m + n^k$, where $\mathrm{len}(\vec{x}) = k$, and $\mathrm{len}(\vec{y}) = m$.

Dimension under relevant operators

Definition ([Lüc20])

Let X and Y be nonempty sets. A function $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ is a Kripke-operator, if there is a relation $\mathcal{R} \subseteq \mathcal{P}(Y) \times \mathcal{P}(X)^n$ such that

$$B \in \Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) \iff$$
$$\exists A_0 \in \mathcal{A}_0 \ldots \exists A_{n-1} \in \mathcal{A}_{n-1} : (B, A_0, \ldots, A_{n-1}) \in \mathcal{R}.$$

- Intersection of families is a Kripke-operator.
- Tensor disjunction on X is a Kripke-operator.
- $\Delta^{M^m}_{\exists i}$ and $\Delta^{M^m}_{\forall i}$ are Kripke-operators.

Operators preserving dimension

Definition
Let $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ be an operator. We say that Δ *weakly preserves dominated convexity* if $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$ is dominated and convex or $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) = \emptyset$ whenever $\mathcal{A}_i$ is dominated and convex for each $i < n$.

Theorem
Let $\Delta_{\mathcal{R}}\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ be a Kripke-operator, and let $\mathcal{A} = \Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$. If Δ weakly preserves dominated convexity then $\mathrm{D}(\mathcal{A}) \leq \mathrm{D}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}(\mathcal{A}_{n-1})$.

Theorem
The operators $\Delta_{\cap}^{M^m}$, $\Delta_{\vee}^{M^m}$, $\Delta_{\exists i}^{M^m}$ and $\Delta_{\forall i}^{M^m}$ weakly preserve dominated convexity. Hence they preserve dimension.

The first main result — A strong Hierarchy Theorem

Definition

- The atom $=(\vec{x}, y)$ is k-ary, if $\mathrm{len}(\vec{x}) = k$,
- The atom $\vec{x} \subseteq \vec{y}$ is k-ary if $\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = k$,
- The atom $\vec{t}_2 \perp \vec{t}_3$ is $\max(k, l)$-ary, if $\mathrm{len}(\vec{t}_2) = k,$ and $\mathrm{len}(\vec{t}_3) = l$.

Theorem

Dependence logic, inclusion logic, and independence logic each has a proper definability hierarchy (even in the empty vocabulary) for formulas based on the arity of the non-first order atoms.

The same for exclusion and conditional independence atoms.

Answers a question of Durand & Kontinen 2012.

The second main result — A Hierarchy Theorem across atoms

Theorem

- *The k-ary dependence atom is* **not** *definable in the extension of first order logic by $<k$-ary dependence (or any other $<k$-ary) atoms, $\leq k$-ary independence, inclusion, constancy atoms, and any Lindström quantifiers.*
- *The k-ary inclusion atom is* **not** *definable in the extension of first order logic by $<k$-ary inclusion, dependence, or constancy (or any other $<k$-ary) atoms, and any Lindström quantifiers.*
- *The k-ary independence atom: respectively.*

Intuitionistic implication

$$M \models_T \phi \to \psi \iff \forall Y \subseteq T (M \models_Y \phi \Rightarrow M \models_Y \psi).$$

$$\models =\!(x_1, \ldots, x_n, y) \equiv (=\!(x_1) \wedge \ldots \wedge =\!(x_n)) \to =\!(y)$$

Hence, $\phi \to \psi$ increases (in some cases) dimension exponentially.

Note: $\to$ has second order strength (F. Yang 2013).

Exists-1 and forall-1

- The $\exists^1$-quantifier is defined as follows: $M \models_T \exists^1 x\phi$ if for some $a \in M$ we have $M \models_{T[\{a\}/x]} \phi$.
- The $\forall^1$-quantifier is defined as follows: $M \models_T \forall^1 x\phi$ if for all $a \in M$ we have $M \models_{T[\{a\}/x]} \phi$.
- The non-empty atom NE is defined by $M \models_T \mathrm{NE}$ if and only if $T \neq \emptyset$.

Theorem

The logical operations $\mathbin{\text{\textcircled{$\vee$}}}$*,* $\forall^1$*,* $\exists^1$*, and* $\rightarrow$ *all increase dimension.* NE *has upper dimension 1 but it is not first order.*

Proof.

1. $D(x = y \mathbin{\text{\textcircled{$\vee$}}} \neg x = y) = 2.$
2. $=\!(x_1, ..., x_k, y) \equiv \forall^1 z_1 ... \forall^1 z_k (z_1 \neq x_1 \vee \ldots \vee z_k \neq x_k \vee =\!(y))$
3. $\exists^1 x \phi \equiv \exists x (=\!(x) \wedge \phi)$. $=\!(x) \equiv \exists^1 x (x = y)$.
4. $=\!(x_1, \ldots, x_k, y) \equiv (=\!(x_1) \wedge \ldots \wedge =\!(x_k)) \rightarrow =\!(y)$

$\square$

Corollary

1. $\forall^1$ *does not have a uniform definition in dependence logic (Galliani 2012).*
2. $\forall^1$ *and* $\exists^1$ *are not lifts of Lindström quantifiers from Tarski semantics to team semantics.*

Summary of dimension theory

- With our dimension concept one can prove hierarchy results for formulas, not just sentences[1].
- Dimension reveals subtle qualitative differences between logical operations (cf. $\forall^1, \rightarrow, \underline{\vee}$).
- Our method is very general, applies to arbitrary families of sets in a finite domain.

[1]... and in team semantics there is a big difference!

Summary of the talk

- From generalized quantifiers to generalized atoms.
- General theory of team semantics on finite domains.
- Now also multi- and probabilistic teams.
- Still open: Is there a logic for PTIME?

Thank you Phokion!

The Helsinki logicians look forward to proving many new theorems with you! And Happy Birthday!

Thank you!

L. A. Aslanyan.
Length of the shortest disjunctive normal form of weakly defined Boolean functions.
In *Applied mathematics, No. 2*, pages 32–40, 141–142. Erevan. Univ., Erevan, 1983.

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From IF to BI: a tale of dependence and separation.
Synthese, 167(2, Knowledge, Rationality & Action):207–230, 2009.

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Arnaud Durand and Juha Kontinen.
Hierarchies in dependence logic.
ACM Trans. Comput. Log., 13(4):Art. 31, 21, 2012.

Pietro Galliani.

Inclusion and exclusion dependencies in team semantics—on some logics of imperfect information.
Ann. Pure Appl. Logic, 163(1):68–84, 2012.

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Hierarchies in independence logic.
In Simona Ronchi Della Rocca, editor, *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2-5, 2013, Torino, Italy*, volume 23 of *LIPIcs*, pages 263–280. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013.

~~V. V. Glagolev.~~
An estimate of the complexity of the contracted normal form for almost all functions of the logic of algebra.
Dokl. Akad. Nauk SSSR, 158:770–773, 1964.

~~Miika Hannula.~~
Hierarchies in inclusion logic with lax semantics.
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The expressive power of modal dependence logic.
In *Advances in modal logic. Vol. 10*, pages 294–312. Coll. Publ., London, 2014.

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