Exercises on conformally invariant probability

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The following is a list of exercises intended for students who are learning about the Gaussian free field, Liouville quantum gravity, SLE, etc. Exercises are sorted by topic, but otherwise are in no particular order.

1 General probability

Problem 1.1. Give an example of a sequence of pairs of non-constant random variables $(X_n, Y_n)_{n\in\mathbb{N}}$ and a pair of non-constant random variables (X, Y) with the following properties:

- $(X_n, Y_n) \to (X, Y)$ in law.
- Y_n is a measurable function of X_n for each $n \in \mathbb{N}$.
- X and Y are independent.

Problem 1.2. Let X, Y, and $\{Y_n\}_{n\in\mathbb{N}}$ be random variables defined on the same probability space. Assume that $(X, Y_n) \to (X, Y)$ in law. Show that if Y is a measurable function of X, then $Y_n \to Y$ in probability.

Problem 1.3. Let $(X_n, Y_n)_{n\in\mathbb{N}}$ and (X, Y) be pairs of random variables, each taking values in a product of separable metric spaces $\Omega_1 \times \Omega_2$. Assume that $(X_n, Y_n) \to (X, Y)$ in law and that there is a family of probability measures $\{P_x\}_{x\in\Omega_1}$ on Ω_2, indexed by Ω_1, such that for each bounded continuous function $f : \Omega_2 \to \mathbb{R}$, we have

$$(\mathbb{E}[f(Y_n)|X_n], X) \to \left(\int_{\Omega_1} f(y)dP_X(y), X\right)$$

in law. show that a.s. P_X is the regular conditional law of Y given X.

2 Brownian motion

Problem 2.1. Let $\mathcal{B}$ be a standard planar Brownian motion started from 0, let L be a straight line, and let $B_\varepsilon(L)$ be the ε-neighborhood of L. Show that there are universal constants $c_0, c_1 > 0$ such that for each $\varepsilon > 0$,

$$\mathbb{P}[\mathcal{B}([0,1]) \subset B_\varepsilon(L)] \leq c_0 e^{-c_1/\varepsilon^2}.$$

Problem 2.2. Let $\mathcal{B}$ be a standard planar Brownian motion started from 0. Let $\eta : [0,1] \to \mathbb{R}^2$ be a deterministic continuous path started from 0. Show that for each $\varepsilon > 0$, it holds with positive probability (allowed to depend on η and ε) that the uniform distance between $\mathcal{B}|_{[0,1]}$ and η is at most ε.

Problem 2.3. Let $\mathcal{B}^1$ and $\mathcal{B}^2$ be independent planar Brownian motion started from $(-1, 0)$ and $(1, 0)$, respectively. For $R > 1$ and $i \in \{1, 2\}$, let τ_R^i be the exit time of $\mathcal{B}^i$ from the ball $B_R(0)$. Show that there is an exponent $\alpha > 0$ such that for each $R \geq 2$,

$$\mathbb{P}\left[\mathcal{B}^1([0, \tau_R^1]) \cap \mathcal{B}^2([0, \tau_R^2]) = \emptyset\right] \leq R^{-\alpha}.$$

Remark: *it follows from [LSW01] that the optimal value of α is $5/4$. The proof is based on SLE.*

Problem 2.4. Let $\mathcal{B}$ be a standard planar Brownian motion started from 0. Let $K \subset \mathbb{C}$ be a closed set with empty interior such that $0 \in K$. Let $\tau = \inf\{t : \mathcal{B}_t \notin K\}$ be the exit time of $\mathcal{B}$ from K. Does one have $\tau = 0$ a.s.? Give a proof or a counterexample.

3 SLE

Problem 3.1. Let $\kappa > 0$ and let η be a chordal SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$. Show that a.s. η intersects every vertical ray $\{x + iy : y \geq 0\}$ for $x \in \mathbb{R}$.

Problem 3.2. Let $\kappa \geq 8$ and let η be a chordal SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$. Show that for each $a, b > 0$,

$$\mathbb{P}[\eta \text{ hits } -a \text{ before } b] > 0.$$

Remark: *the probability is computed exactly in [Bef12, Theorem 10].*

Problem 3.3. Let $\kappa \in (0, 4]$ and let η_0 (resp. η_1) be a chordal SLE$_\kappa$ in $\mathbb{H}$ from 0 to ∞ (resp. from 1 to ∞). Show that

$$\mathbb{P}[\eta_0 \cap \eta_1 \neq \emptyset] = 1.$$

Problem 3.4. Let $K \subset \overline{\mathbb{H}}$ be a compact connected set such that $\mathbb{H} \setminus K$ is simply connected. Show that K has positive half-plane capacity, i.e., show that $\lim_{y\to\infty} y\mathbb{E}^{iy}[\operatorname{Im} B_\tau] > 0$, where B is a planar Brownian motion and τ is its exit time from $\mathbb{H} \setminus K$.

Problem 3.5. Let $K \subset \overline{\mathbb{H}}$ be a compact connected set such that $0 \notin K$ and $\mathbb{H} \setminus K$ is simply connected. For $\kappa > 0$, let η_κ be an SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$. Is $\kappa \mapsto \mathbb{P}[\eta_\kappa \cap K \neq \emptyset]$ a non-decreasing function of κ? Give a proof or a counterexample.

Problem 3.6. Let $\kappa > 0$ and let η be an SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$. Show that there exists a deterministic constant $d > 0$, depending only on κ, such that for each $t > 0$, the Hausdorff dimension of $\eta([0, t])$ is a.s. equal to d. ***Remark:*** *One can show that $d = \min\{1 + \kappa/8, 2\}$ [Bef08].*

Problem 3.7. Let $\kappa \in (4, 8)$ and let η be an SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$. Show that for each $t > 0$, a.s. $\mathbb{H} \setminus \eta([0, t])$ has infinitely many connected components.

Problem 3.8. Let $\kappa > 0$ and let η be an SLE$_\kappa$ from 0 to ∞ in $\mathbb{H}$. Let $U \subset \mathbb{H}$ be open. Show that $\mathbb{P}[\eta \cap U \neq \emptyset] > 0$.

4 Brownian loop soup

Problem 4.1. Let $\mathcal{L}$ be a Brownian loop soup on $\mathbb{D}$ of intensity $\lambda > 0$ and let $K \subset \mathbb{D}$ be compact.

a. Show that for any value of λ, it holds with positive probability that there is a loop in $\mathcal{L}$ which disconnects K from $\partial\mathbb{D}$.

b. Show that the probability that such a loop exists tends to 1 as $\lambda \to \infty$.

Problem 4.2. Let $\lambda > 0$ and let $\mathcal{L}$ be a Brownian loop soup on the unit disk $\mathbb{D}$ of intensity λ. Consider the following two sets.

1. Let G_1 be the closure of the connected component of $\overline{\mathbb{D}} \setminus \bigcup_{\ell \in \mathcal{L}} \ell$ which contains $\partial \mathbb{D}$.
2. A *cluster* of $\mathcal{L}$ is a connected component of $\bigcup_{\ell \in \mathcal{L}} \ell$. The *outer boundary* of a cluster C is the boundary of the connected component of $\overline{\mathbb{D}} \setminus C$ which contains $\partial \mathbb{D}$. Let G_2 be the closure of the union of the outer boundaries of the clusters of $\mathcal{L}$.

Show that $G_1 = G_2$ a.s. This set is called the *gasket* of $\mathcal{L}$.

Problem 4.3. Let $\mathbf{c}_1, \mathbf{c}_2 > 0$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be independent Brownian loop soups on $\mathbb{D}$ with parameters $\mathbf{c}_1$ and $\mathbf{c}_2$. Let G_1 and G_2 be the corresponding gaskets. Show that if $\mathbf{c}_1 + \mathbf{c}_2 > 1$, then $G_1 \cap G_2$ is totally disconnected.

Problem 4.4. Let $\lambda > 0$ and let $\mathcal{L}$ be a Brownian loop soup on the unit disk $\mathbb{D}$ of intensity λ. Show that there exist universal constants $C, \beta > 0$ such that for each $z \in B_{1/2}(0)$, the probability that there is a loop $\ell \in \mathcal{L}$ which disconnects z from $\partial \mathbb{D}$ is at least $1 - C\varepsilon^{\beta\lambda}$.

Problem 4.5. Show that there exists $\lambda_* \in (0, \infty)$ such that the following is true for each $\lambda > \lambda_*$. If $\mathcal{L}$ is a Brownian loop soup on $\mathbb{D}$ of intensity λ, then a.s. for each $z \in \mathbb{D}$ there exists $\ell \in \mathcal{L}$ such that ℓ disconnects z from $\partial \mathbb{D}$. ***Remark:*** *it can be shown (unpublished) that the optimal value of λ_* is $\lambda_* = 10$, which corresponds to central charge 20. Note that the set of points which are not disconnected from $\partial \mathbb{D}$ by a single loop is larger than the "gasket" of $\mathcal{L}$, which consists of points which are not disconnected from $\partial \mathbb{D}$ by any finite union of loops.*

5 GFF

Throughout, the additive constant for the whole-plane GFF is always chosen so that the circle average $h_1(0) = 0$, unless otherwise stated.

Problem 5.1. Let $G = (V, E)$ be a connected graph with a set of boundary vertices $\partial V \subset V$. Let h be the zero-boundary GFF on G. Let $x, y \in V \setminus \partial V$ be two distinct vertices which are not joined by an edge. Find the conditional joint distribution of $\{h(x), h(y)\}$ given $\{h(z) : z \in V \setminus \{x, y\}\}$.

Problem 5.2. Let $G = (V, E)$ be a connected graph with a set of boundary vertices $\partial V \subset V$. Let h be the zero-boundary GFF on G. Let $f : V \to \mathbb{R}$ be a deterministic function which vanishes on ∂G. Find the Radon-Nikodym derivative of the law of $h + f$ with respect to the law of h.

Problem 5.3. Let h be a zero-boundary GFF on a bounded domain $U \subset \mathbb{C}$ and let $\{h_r\}_{r>0}$ be its circle average process. Show that for each $\zeta > 0$, it holds with probability tending to 1 as $\varepsilon \to 0$ that

$$\max\big\{|h_\varepsilon(z)| : z \in U \cap (\varepsilon\mathbb{Z}^2),\ \mathrm{dist}(z, \partial U) \geq \varepsilon\big\} \leq (2 + \zeta) \log \varepsilon^{-1}.$$

Problem 5.4. Let h be a zero-boundary GFF on a domain $U \subset \mathbb{C}$ and let $\{h_r\}_{r>0}$ be the circle average process. For $\alpha \in \mathbb{R}$, a point $z \in U$ is called an *α-thick point* if

$$\limsup_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{\log \varepsilon^{-1}} = \alpha.$$

Show that for each $\alpha \in \mathbb{R}$, either the set of α-thick points is a.s. empty or the set of α-thick points is a.s. dense in U. ***Remark:*** *it can be shown that the set of α-thick points is non-empty iff $\alpha \in [-2, 2]$, see [HMP10].*

Problem 5.5. Let $U \subset \mathbb{C}$ be an open domain and let $z \in U$. Let $\mathcal{H}(U)$ be the Hilbert space completion of the space of smooth, compactly supported functions on U with respect to the Dirichlet inner product $(f,g)_\nabla = \int_U \nabla f(z) \cdot \nabla g(z)\, dz$. Show that the space of smooth functions on U which are supported on a compact subset of $U \setminus \{z\}$ is dense in $\mathcal{H}(U)$.

Problem 5.6. Let h be a zero-boundary GFF on a domain $U \subset \mathbb{C}$. Fix $z \in U$. Show that the tail σ-algebra $\bigcap_{\varepsilon>0} \sigma(h|_{B_\varepsilon(z)})$ is trivial.

Problem 5.7. Let h be the whole-plane GFF. Show that the inversion $h(1/\cdot)$ agrees in law with h.

Problem 5.8. Let $U \subset \mathbb{C}$ be an open set and let h (resp. h^0) be a free-boundary (resp. zero-boundary) GFF on U. Show that for each bounded open set V with $\overline{V} \subset U$, the laws of $h^0|_V$ and $h|_V$ are mutually absolutely continuous.

Problem 5.9. Let $U \subset \mathbb{C}$ be a simply connected open set and let h be a zero-boundary GFF on U. Also let $V \subset U$ be another simply connected open set and let $z \in V$. Let $\mathfrak{h}^V$ be the harmonic part of $h|_V$. Show that $\mathfrak{h}^V(z)$ is a centered Gaussian random variable with variance $\log \mathrm{CR}(z;U) - \log \mathrm{CR}(z;V)$, where CR denotes the conformal radius.

Problem 5.10. Let $U \subset \mathbb{C}$ be a simply connected open domain and let $G_U(z,w)$ be the zero-boundary Green's function on U. Show that for each $z \in U$,

$$\lim_{w \to z} \left(G_U(z,w) - \log \frac{1}{|z-w|} \right) = -\log \mathrm{CR}(z;U)$$

where $\mathrm{CR}(z;U)$ denotes the conformal radius.

Problem 5.11. Show that the functions

$$\phi_n(z) := (2/n)^{1/2} \operatorname{Re} z^n \quad \text{and} \quad \psi_n(z) := (2/n)^{1/2} \operatorname{Im} z^n$$

for $n \in \mathbb{N}$ give an orthonormal basis for the space of harmonic functions on $\mathbb{D}$ (viewed modulo additive constant) with respect to the Dirichlet inner product.

Problem 5.12. Let h be a free-boundary GFF on the unit disk $\mathbb{D}$. Let $\mathfrak{h}$ be its harmonic part, with the additive constant chosen so that $\mathfrak{h}(0) = 0$. Find $\mathrm{Cov}(\mathfrak{h}(z), \mathfrak{h}(w))$ for each $z, w \in \mathbb{D}$. ***Hint:*** *use problem 5.11 and the orthonormal basis decomposition of the GFF.*

Problem 5.13. Let $\mathring{h}$ (resp. h) be a zero-boundary (resp. free-boundary) GFF on $\mathbb{H}$. For concreteness, assume that h is normalized so that $h_1(0) = 0$.

a. Show that for any $r > 0$, the laws of the restrictions of h and $\mathring{h}$ to the semidisk $B_r(0) \cap \mathbb{H}$ are mutually singular.

b. Show that for any $z \in \mathbb{H}$ and any $r < \operatorname{Im} z$, the laws of the restrictions of $\mathring{h}$ and h to $B_r(z)$ are mutually absolutely continuous.

Problem 5.14. Let h be the whole-plane GFF. Show that for each $r > 0$ and each $z \in \mathbb{C}$, the field $(h - h_r(z))|_{B_r(z)}$ is independent from $\{h_s(z) - h_r(z) : s \geq r\}$. ***Hint:*** *first reduce to the case when $r = 1$ and $z = 0$, then use the decomposition of the whole-plane GFF into the radially symmetric part and the mean-zero part as in the definition of the quantum cone.*

6 LQG measure

Problem 6.1. Let $U \subset \mathbb{C}$ be an open set, let h be the zero-boundary GFF on U and let μ_h be the γ-LQG area measure. Show that almost surely $\mu_h(V) > 0$ for each open set $V \subset U$.

Problem 6.2. Let h be a whole-plane GFF, let μ_h be the γ-LQG area measure, and let $\{h_r\}_{r>0}$ be the circle average process. Show that for each $r > 0$ and $z \in \mathbb{C}$,

$$r^{-(2+\gamma^2/2)} e^{-\gamma h_r(z)} \mu_h(B_r(z)) \stackrel{d}{=} \mu_h(B_1(0)).$$

Hint: *use the LQG coordinate change formula and the scale and translation invariance of the law of h, modulo additive constant.*

Problem 6.3. Let h be a whole-plane GFF and let μ_h be the γ-LQG area measure. Find the limit (in probability) of

$$\frac{\log \mu_h(B_\varepsilon(0))}{\log \varepsilon}$$

as $\varepsilon \to 0$. ***Hint:*** *use Problem 6.2.*

Problem 6.4. Let h be a whole-plane GFF and let μ_h be the γ-LQG area measure. For each $p \in (-\infty, 4/\gamma^2)$, compute

$$\lim_{\varepsilon \to 0} \frac{\log \mathbb{E}[(\mu_h(B_\varepsilon(0)))^p]}{\log \varepsilon}.$$

You may use that $\mu_h(B_1(0))$ has a finite pth moment for all $p \in (-\infty, 4/\gamma^2)$ [RV14, Theorems 2.11 and 2.12]. ***Hint:*** *use Problems 5.14 and 6.2.*

Problem 6.5. Let h be the zero-boundary GFF on a domain $U \subset \mathbb{C}$, let $K \subset U$ be compact, and let μ_h be the γ-LQG area measure. Show that for each $\zeta > 0$, it holds with probability tending to 1 as $\varepsilon \to 0$ that

$$\varepsilon^{2+\gamma^2/2+2\gamma+\zeta} \le \mu_h(B_\varepsilon(z)) \le \varepsilon^{2+\gamma^2/2-2\gamma-\zeta}, \quad \forall z \in K. \tag{6.1}$$

Hint: *use Problem 6.4, Chebyshev's inequality, and a union bound argument.*

Problem 6.6. Let $U \subset \mathbb{C}$ be a connected open set, let h be the zero-boundary GFF on U and let μ_h be the γ-LQG area measure. Show that for any disjoint bounded open sets $V_1, V_2 \subset U$ and any $\varepsilon > 0$,

$$\mathbb{P}[\mu_h(V_1) \le \varepsilon \mu_h(V_2)] > 0.$$

Hint: *if f is a deterministic smooth compactly supported function on U, then the laws of h and $h + f$ are mutually absolutely continuous.*

Problem 6.7. Let $U \subset \mathbb{C}$ be a connected open set, let h be the zero-boundary GFF on U and let μ_h be the γ-LQG area measure. Show that if $V \subset U$ is open and bounded with $\overline{V} \subset U$, then the law of $\mu_h(V)$ is mutually absolutely continuous with respect to Lebesgue measure on $(0, \infty)$.

Problem 6.8. Let h be the whole-plane GFF and let μ_h be the γ-LQG area measure. For $\varepsilon > 0$ and $z, w \in \mathbb{C}$, let $D_h^\varepsilon(z, w)$ be the minimal number of Euclidean balls of μ_h-mass ε whose union contains a path from z to w. Show that for each fixed $z, w \in \mathbb{C}$ and each $\zeta > 0$, it holds with probability tending to 1 as $\varepsilon \to 0$ that

$$D_h^\varepsilon(z, w) \le \varepsilon^{-\frac{4+\gamma^2-\sqrt{16+\gamma^4}}{2\gamma^2}-\zeta}.$$

Hint: *use the KPZ formula.* ***Remark:*** *the metric D_h^ε is called* Liouville graph distance. *It is known that $D_h^\varepsilon(z, w) \approx \varepsilon^{1/d_\gamma}$, where d_γ is the Hausdorff dimension of the γ-LQG metric [DG18]. Hence the estimate from this problem gives a lower bound for d_γ.*

Problem 6.9. Let h be a zero-boundary GFF on a bounded open set $U \subset \mathbb{C}$. For $\gamma \in (0,2)$, let μ_h^γ be the γ-LQG area measure. Show that $\gamma \mapsto \mu_h^\gamma$ is continuous in probability with respect to the Prokhorov distance for measures on U.

7 LQG metric

Problem 7.1. Let h be a whole-plane GFF, let D_h be the γ-LQG metric, and let $\{h_r\}_{r>0}$ be the circle average process. Show that

$$\{r^{-(2+\gamma^2/2)} e^{-\gamma h_r(z)} D_h(rz, rw)\}_{z,w\in\mathbb{C}} \overset{d}{=} \{D_h(z,w)\}_{z,w\in\mathbb{C}}.$$

Problem 7.2. Let h be a whole-plane GFF, let μ_h be the γ-LQG area measure, and let D_h be the γ-LQG metric. Conditional on h, let z be sampled from $\mu_h|_{B_1(0)}$, normalized to be a probability measure. Find the limit (in probability) of

$$\frac{\log D_h(z, \partial B_\varepsilon(z))}{\log \varepsilon}$$

as $\varepsilon \to 0$. ***Hint:*** *What kind of log singularity does one have at z? Also use [DFG$^+$20, Proposition 3.14].*

Problem 7.3. Let h be a whole-plane GFF and let D_h be the γ-LQG metric. Let $K_1, K_2 \subset \mathbb{C}$ be disjoint compact sets.

a. Show that for each $\varepsilon > 0$,

$$\mathbb{P}[D_h(K_1, K_2) < \varepsilon] > 0.$$

b. Show that the law of $D_h(K_1, K_2)$ is mutually absolutely continuous with respect to Lebesgue measure on $(0, \infty)$.

Problem 7.4. Let h be a whole-plane GFF and let D_h be the γ-LQG metric. Show that for any Euclidean-compact set $K \subset \mathbb{C}$, one has

$$\lim_{R\to\infty} D_h(K, \partial B_R(0)) = \infty,$$

where $B_R(0)$ is the Euclidean ball of radius R centered at 0. ***Hint:*** *Use a scaling argument to estimate $D_h(\partial B_{2^k}(0), \partial B_{2^{k+1}}(0))$ for each $k \in \mathbb{N}$.*

Problem 7.5. Let h be a whole-plane GFF and let D_h be the γ-LQG metric. For $s > 0$, we define the *filled LQG metric ball* $B_s^\bullet(0; D_h)$ to be the union of the LQG metric ball $B_s^\bullet(0; D_h)$ of radius s centered at 0 and the points which it disconnects from ∞. Show that for every $s > 0$,

$$\mathbb{P}[B_s^\bullet(0; D_h) \neq B_s(0; D_h)] > 0.$$

Remark: *One can in fact show that a.s. $B_s^\bullet(0; D_h) \neq B_s(0; D_h)$, see [GPS22, Theorem 1.14] for a much stronger statement.*

Problem 7.6. Let h be a whole-plane GFF and let D_h be the γ-LQG metric. We define the *metric net*

$$\mathcal{N}(0; D_h) := \bigcup_{s>0} \partial \mathcal{B}_s^\bullet(0; D_h),$$

where $\mathcal{B}_s^\bullet(0; D_h)$ is as in Problem 7.5. Show that for each fixed $z \in \mathbb{C} \setminus \{0\}$,

$$\mathbb{P}[z \in \mathcal{N}(0; D_h)] = 0.$$

Problem 7.7. Let h be a whole-plane GFF and let D_h be the γ-LQG metric. Let $K \subset \mathbb{C}$ be compact. Show that for each $\zeta > 0$, it holds with probability tending to 1 as $\varepsilon \to 0$ that

$$D_h(\text{around } B_{2\varepsilon}(z) \setminus B_\varepsilon(z)) \leq \varepsilon^{-\zeta} D_h(\text{across } B_{2\varepsilon}(z) \setminus B_\varepsilon(z)), \quad \forall z \in K.$$

Problem 7.8. Let $(\mathbb{C}, h, 0, \infty)$ be an α-quantum cone for $\alpha \in (-\infty, Q)$. Let μ_h be the γ-LQG area measure and let D_h be the γ-LQG metric. Let $B_s(0; D_h)$ denote the LQG metric ball of radius s centered at 0. Show that for each $p \in (0,1)$, there exists $C = C(p, \alpha, \gamma) > 1$ such that for each $s > 0$,

$$\mathbb{P}\left[C^{-1} s^{d_\gamma} \leq \mu_h(B_s(0; D_h)) \leq C s^{d_\gamma}\right] \geq p.$$

Problem 7.9. Let h be the whole-plane GFF and let D_h be the γ-LQG metric. For $s > 0$, let $B_s(0; D_h)$ be the LQG metric ball of radius s centered at 0.

a. Show that there is an $\alpha = \alpha(\gamma) > 0$ such that with probability at least $1 - O(\varepsilon^\alpha)$, the LQG ball $B_{D_h(0,\partial\mathbb{D})}(0; D_h)$ contains the Euclidean ball of radius ε centered at 0.

b. Show that for every $p > 0$, it holds with probability at least $1 - O(\varepsilon^p)$ that $B_{D_h(0,\partial\mathbb{D})}(0; D_h)$ contains a Euclidean ball of radius at least ε (not necessarily centered at 0).

Hint: *for part b, use the "near independence across concentric annuli" lemma.*

Problem 7.10. Let h be the whole-plane GFF and let D_h be the γ-LQG metric. Show that there exists a deterministic constant $C = C(\gamma) > 0$ such that for each compact set $K \subset \mathbb{C}$, it is a.s. the case that for each small enough $\varepsilon > 0$ and each $z \in K$, each D_h-geodesic (between any two points on $\mathbb{C}$) crosses between the inner and outer boundaries of the annulus $B_{\varepsilon^{1/2}}(z) \setminus B_\varepsilon(z)$ at most C times. ***Hint:*** *use the "near independence across concentric annuli" lemma and a union bound over possible center points z.* ***Remark:*** *this property can be used to show that LQG geodesics do not locally look like SLE$_\kappa$ curves for any κ, see [MQ20].*

Problem 7.11. Show that the locality axiom in the definition of the LQG metric is redundant. That is, let $h \mapsto D_h$ be a measurable function from generalized functions on $\mathbb{C}$ to metrics on $\mathbb{C}$ such that when h is a GFF plus a continuous function, a.s. D_h induces the Euclidean topology on $\mathbb{C}$, is a length metric, and satisfies the Weyl scaling and LQG coordinate change axioms. Show that for any deterministic open set $U \subset \mathbb{C}$, a.s. $D_h(\cdot, \cdot; U)$ is a measurable function of $h|_U$. ***Remark:*** *The locality axiom is nevertheless included as an axiom since it is frequently useful when studying the LQG metric and it is not an obvious consequence of the other axioms. Moreover, the locality axiom does not follow from the other axioms if one does not assume a priori that the metric is a measurable function of the GFF h, so one has to prove a version locality* before *proving measurability, see [GM20].*

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