Lecture notes on the LQG metric

Ewain Gwynne

These are lecture notes for a mini course given at Aalto University in August 2023. They are in a rather rough form: I have not proofread them carefully or optimized the exposition. Note also that the points mentioned in the actual lectures may not exactly match the points mentioned in the notes.

Contents

Main references:

- Introductory articles on LQG [Gwy20b, She22].
- Book in progress on LQG [BP].
- Survey article on LQG metric [DDG21].
- Original papers on the construction of the LQG metric [DDDF20, GM20b, DFG+20, GM20a, GM21b]. See also the supercritical case, [DG20, Pfe21, DG23].
- I have a list of exercises on LQG and related topics which I can provide on request.

1 Introduction

- **Liouville quantum gravity (LQG)**: one-parameter family of models of random surfaces (2d Riemannian manifolds).
- Can define LQG surfaces with the topology of any Riemann surface, and they all have the same local behavior.
- In most of these lectures, we will focus on the whole-plane case.
- Several possible parameters: **(Liouville) central charge** $\mathbf{c}_{\mathrm{L}} > 1$, **matter central charge** $\mathbf{c}_{\mathrm{M}} < 25$, **background charge** $Q > 0$, **coupling constant** $\gamma \in (0, 2] \cup \{z \in \mathbb{C} : |z| = 2\}$.

$$\mathbf{c}_{\mathrm{L}} = 26 - \mathbf{c}_{\mathrm{M}} = 1 + 6Q^2 = 1 + 6\left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2.$$

Definition 1. An **LQG surface** with central charge $\mathbf{c}_{\mathrm{L}} > 1$ parametrized by $U \subset \mathbb{C}$ is the random surface with Riemannian metric tensor g on U, where g is sampled from "uniform measure on Riemannian metric tensors, weighted by $(\det \Delta_g)^{-(26-\mathbf{c}_{\mathrm{L}})/2}$".

- The above definition is not rigorous, but can be made sense of in various ways, see below.
- $\mathbf{c}_{\mathrm{L}} = 0$ corresponds to "uniform measure on surfaces".
- The Kirkhoff matrix-tree theorem says that if G is a graph, then the discrete Laplacian determinant $\det \Delta_G$ counts the number of spanning trees on G. Heuristically, larger (resp. smaller) values of $\mathbf{c}_{\mathrm{L}}$ mean that the surface is biased to have more (resp. less) possible spanning trees, so is more Euclidean-like (resp. tree-like).

Phases:

- **Subcritical (weakly coupled):** $\mathbf{c}_{\mathrm{L}} > 25$, $Q > 2$, $\gamma \in (0, 2)$.
- **Critical:** $\mathbf{c}_{\mathrm{L}} = 25$, $Q = \gamma = 2$.
- **Supercritical (strongly coupled):** $\mathbf{c}_{\mathrm{L}} \in (1, 25)$, $Q \in (0, 2)$, $\gamma \in \mathbb{C}$ with $|\gamma| = 2$.

Most of these lectures will focus on the subcritical case, but much of what we do extends to the critical and supercritical cases with a bit more work.

String theory motivation (Polyakov, 1980's):

- A **string** is a path in $\mathbb{R}^d$ which evolves in time.

- Two parameters: parametrization of string + time.
- Traces out a "surface" in $\mathbb{R}^d$: **worldsheet**
- To analyze this, Polyakov wanted to develop a notion of "sums over surfaces" analogous to Feynman path integral (which can be thought of as a "sum over paths").
- Need a notion of "random surfaces weighted by the number of possible embeddings into $\mathbb{R}^d$".
- Polyakov argued that this should correspond to LQG with $\mathbf{c}_{\mathrm{L}} = 26 - d$: $(\det \Delta_g)^{-(26-\mathbf{c}_{\mathrm{L}})/2}$ counts "number of possible embeddings (partition function of GFF).

Conformal field theory motivation:

- For $\mathbf{c}_{\mathrm{M}} \leq 1$, LQG is equivalent to Liouville conformal field theory: simplest CFT with a continuous spectrum.
- Model of "gravity in two dimensions".
- Rigorous work on LQG from CFT perspective by Kupiannen, Rhodes, Vargas, et. al.

How to define LQG rigorously? One option is to discretize.

Definition 2. A planar map is a graph embedded in the plane, viewed modulo orientation-preserving homeomorphisms.

- Discrete surface: give each face the Riemannian metric of a polygon with unit side length, identify the polygons along the edges in a length-preserving way.
- For $n \in \mathbb{N}$, let M_n be sampled from the uniform measure on planar maps (or a triangulation, quadrangulation, etc.) weighted by $(\det \Delta_n)^{-(26-\mathbf{c}_{\mathrm{L}})/2}$.
  - Uniform planar maps correspond to $\mathbf{c}_{\mathrm{L}} = 26$, equivalently $\gamma = \sqrt{8/3}$.
- Should converge to LQG with central charge $\mathbf{c}_{\mathrm{L}}$.
  - Gromov-Hausdorff (with graph distance).
  - Convergence when embedded into $\mathbb{C}$ (e.g., via circle packing).
- Can also consider other weightings (partition function of statistical mechanics model, number of spanning trees, etc.) which have similar asymptotic behavior to powers of $\det \Delta_n$.
- Proving scaling limits of random planar maps is hard. Requires some "exact solvability", and has only been done in a handful of cases.

Alternative way to construct LQG: exponential of Gaussian free field.

- DDK ansatz: for $\mathbf{c}_{\mathrm{L}} \geq 25$ ($\gamma \in (0, 2]$), LQG metric tensor is given by

  $$g = e^{\gamma h}(dx^2 + dy^2)$$

  where h is a variant of the Gaussian free field (GFF).

- More precisely, h should be sampled from $\exp(-\mathcal{S}_{\mathrm{L}}(\phi))\,d\phi$, where $\mathcal{S}_{\mathrm{L}}$ is the so-called Liouville action and $d\phi$ is the "uniform measure on functions".
- One can make rigorous sense of this ("quantum sphere"), and the field h has the same local behavior as the GFF. Since we are only interested in local properties of LQG (not exact formulas), we will just work with the GFF.

2 LQG area measure

- We want to define area measure and metric associated with LQG.
- Let $\{h_\varepsilon\}_{\varepsilon>0}$ be a family of continuous functions which approximate h, define objects with h_ε instead of h, take a limit as $\varepsilon \to 0$.
- For concreteness, let $p_t(z) := \frac{1}{2\pi t} e^{-|z|^2/2t}$ and define

$$h_\varepsilon^*(z) := (h * p_{\varepsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\varepsilon^2/2}(z-w)\, d^2 w.$$

- $\operatorname{Var} h_\varepsilon^*(z) \sim \log \varepsilon^{-1}$.
- $h_\varepsilon^* \to h$ as $\varepsilon \to 0$.

Theorem 3 (Kahane [Kah85], Duplantier-Sheffield [DS11], et. al.)**.** *The random measures $\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon^*(z)}\, d^2 z$ a.s. converge weakly to a limiting measure μ_h, called the **LQG area measure**.*

- Special case of Gaussian multiplicative chaos.
- Also makes sense if h is a **GFF plus a continuous function**, i.e., $h = h^0 + f$ where h^0 is the GFF on U and $f : U \to \mathbb{R}$ is continuous.
- $\mu_h(\text{open}) > 0$, $\mu_h(\text{point}) = 0$, mutually singular with respect to Lebesgue measure.
- Should be scaling limit of counting measure on embedded random planar maps.
- **LQG coordinate change:** [DS11] suppose $\phi : V \to U$ is a conformal map. Then $\phi_* \mu_{\widetilde{h}} = \mu_h$, where

$$\widetilde{h} = h \circ \phi + Q \log |\phi'|, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

- "Two different parametrizations of the same LQG surface".

3 LQG metric

- We want to use a similar procedure to construct the LQG metric.
- Let $\xi > 0$ to be chosen later (depending on γ).
- For $\varepsilon > 0$, let

$$D_h^\varepsilon(z, w) = \inf_{P : z \to w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt,$$

where the infimum is over piecewise C^1 paths from z to w.

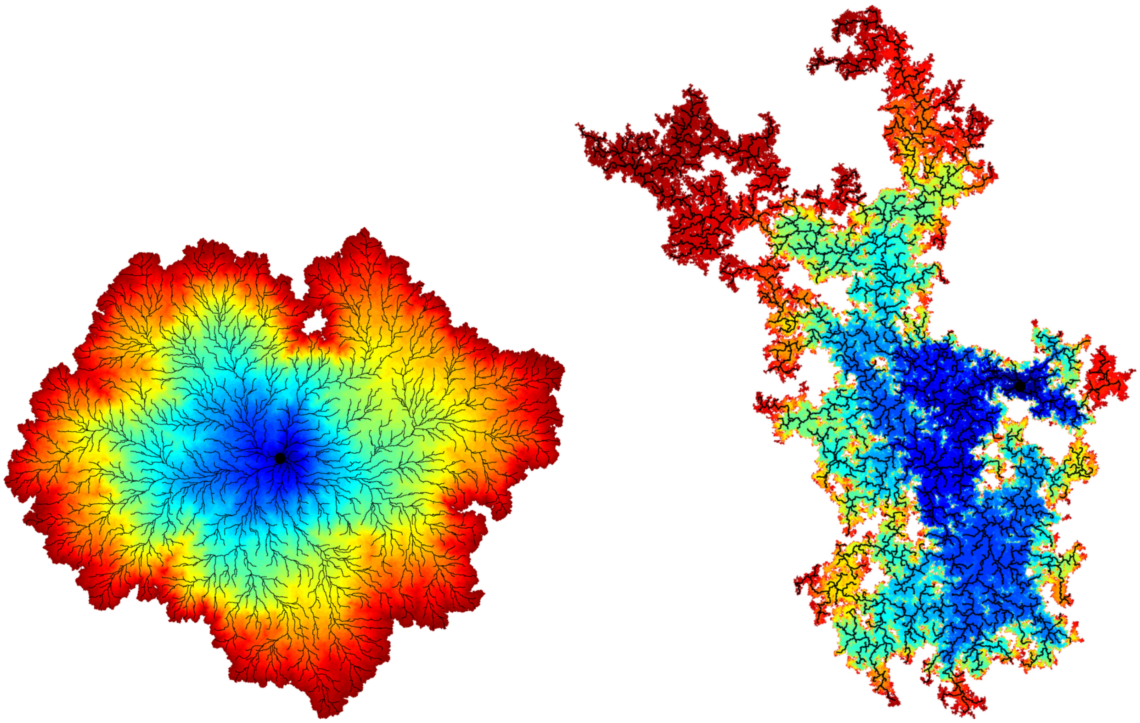

Figure 1: **Left.** Simulation of an LQG metric ball for $\gamma = 1.75$. Colors indicate the distance to the center point and the black curves are geodesics from the center point to other points in the ball. **Right.** Simulation of a supercritical LQG metric ball for $\xi = 2$. Both simulations were made by A. Bou-Rabee.

- We want to take a limit of D_h^ε as $\varepsilon \to 0$ to get the LQG metric.
- What should ξ be?
- Scaling areas by $C \Leftrightarrow$ adding $\frac{1}{\gamma} \log C$ to $h \Leftrightarrow$ scaling distances by $C^{\xi/\gamma}$.
- γ/ξ should be the "dimension" of LQG.
- $\exists\, d_\gamma > 2$ such that for random planar maps in the γ-LQG universality class,

  $$\#B_r(\text{typical vertex}) \approx r^{d_\gamma}$$

  when r is large [DZZ19, DG18].
- d_γ is the "dimension" of the random planar map.
- Not known explicitly except that $d_{\sqrt{8/3}} = 4$ (comes from results for uniform random planar maps).
- Watabiki [Wat93] prediction:

  $$d_\gamma^{\text{Wat}} = 1 + \frac{\gamma^2}{4} + \frac{1}{4}\sqrt{(4+\gamma^2)^2 + 16\gamma^2},$$

  disproven in [DG19], but from numerical simulations is "close" to the actual value of d_γ [AB14, BB19].

- Alternative guess due to Ding-Gwynne [DG18]:
$$d_\gamma^{\mathrm{DG}} = 2 + \frac{\gamma^2}{2} + \frac{\gamma}{\sqrt{6}}.$$
Not disproven rigorously, but believed to be false (see, e.g., [DGS21]).

- We want $\gamma/\xi = d_\gamma$, i.e.,
$$\xi = \frac{\gamma}{d_\gamma}.$$

- A posteriori, can show that d_γ is the Hausdorff dimension of the LQG metric space [GP22].

- Note: relationship between ξ and γ is not known explicitly.

- How to scale D_h^ε to get a non-trivial limit?

- For $\varepsilon > 0$, let
$$\mathfrak{a}_\varepsilon = \mathfrak{a}_\varepsilon(\xi) = \text{median of } D_h^\varepsilon\text{-distance across } [0,1]^2.$$

Proposition 4 (Ding-Zeitouni-Zhang, Ding-Gwynne [DZZ19,DG18,DG20])**.** *For each $\xi > 0$, there exists $Q > 0$ such that*
$$\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o(1)} \quad \text{as } \varepsilon \to 0.$$
For $\gamma \in (0,2)$ and $\xi = \gamma/d_\gamma$, we have $Q = 2/\gamma + \gamma/2$.

Theorem 5 (Ding-Dubédat-Dunlap-Falconet [DDDF20])**.** *The random metrics $\{\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon\}_{\varepsilon>0}$ are tight with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times \mathbb{C}$. Every subsequential limit is a random metric on $\mathbb{C}$ (not a pseudometric) which induces the same topology as the Euclidean metric.*

Theorem 6 (Gwynne-Miller [GM21b])**.** *The subsequential limit is uniquely characterized by a list of axioms, and one has $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \to D_h$ in probability as $\varepsilon \to 0$.*

- The limiting object is defined to be the **Liouville quantum gravity metric**.

- Convergence is *much* harder than for the measure since the minimizing path depends on ε.

- Proofs of tightness and uniqueness are quite involved, but use *only basic properties of the GFF*: nothing about LQG measure, relationship to SLE, relationship to random planar maps, exact formulas, special LQG surfaces, etc.

- Euclidean topology, but very different geometry.

- Hausdorff dimension $d_\gamma > 2$ [GP22].

- $\exists$ LQG geodesic (length-minimizing path) between any two points (take limit of D_h^ε-geodesic).

- Confluence of geodesics [GM20a].

- Metric ball boundary is fractal, infinitely many connected components, Euclidean Hausdorff dimension $2 - \xi Q + \xi^2/2$ [Gwy20a, GPS22].

- If $U \subset \mathbb{C}$ and h is a GFF (or a GFF plus a continuous function) on U, we can define D_h by local absolute continuity.

- **LQG coordinate change:** [GM21a] Let $\phi : V \to U$ be a conformal map. Then a.s.

$$D_{h\circ\phi+Q\log|\phi'|}(z,w) = D_h(\phi(z),\phi(w)), \quad \forall z,w \in V.$$

- Same coordinate change rule as for μ_h.

4 Related objects

4.1 Miller-Sheffield construction and convergence of uniform random planar maps

- Miller-Sheffield [MS20,MS21a,MS21b]: Earlier construction of the LQG metric for $\gamma = \sqrt{8/3}$.
- Use a process called **quantum Loewner evolution** to build a candidate for LQG metric balls.
- Show that there is a unique metric with these metric balls.
- Relies on special symmetries for $\gamma = \sqrt{8/3}$, does not generalize to other values of γ.

Theorem 7 (Le Gall [Le 13], Miermont [Mie13])**.** *Let M_n be a uniform quadrangulation with n edges, μ^n = counting measure on vertices, D^n = graph distance. Then $(M_n, n^{-1/4}D^n, n^{-1}\mu^n)$ converges in law to a random metric measure space called the **Brownian map**, w.r.t. the Gromov-Hausdorff-Prokhorov topology.*

- Also works for other uniform-type random planar maps, e.g., triangulations, unconstrained face degree.

Theorem 8 (Miller-Sheffield [MS21a])**.** *For a special variant of the GFF called the **quantum sphere**, the $\sqrt{8/3}$-LQG metric measure space constructed via quantum Loewner evolution is isometric to the Brownian map.*

Theorem 9 (Gwynne-Miller [GM21b])**.** *The Miller-Sheffield $\sqrt{8/3}$-LQG metric coincides the the limit of $\mathfrak{a}_\varepsilon^{-1}D_h^\varepsilon$ for $\gamma = \sqrt{8/3}$.*

- Hence, uniform random planar maps converge to $\sqrt{8/3}$-LQG in the Gromov-Hausdorff-Prokhorov topology.
- Building on this, Holden and Sun showed that one also has convergence under the so-called **Cardy embedding** [HS23].
- We don't know how to see that $\gamma = \sqrt{8/3}$ is special directly from the properties of $\mathfrak{a}_\varepsilon^{-1}D_h^\varepsilon$. Connection to uniform random planar maps has to go through Miller-Sheffield construction.
- Random planar map convergence is still conjectural for $\gamma \neq \sqrt{8/3}$.

4.2 The supercritical case

- The quantity $\xi = \gamma/d_\gamma$ is increasing in γ [DG18].
- So, $\gamma/d_\gamma \leq 2/d_2 \approx 0.41$.
- What happens when $\xi > 2/d_2$?
- Can still define the approximating metrics D_h^ε and the normalizing factors $\mathfrak{a}_\varepsilon$.

Theorem 10 (Ding-Gwynne [DG20, DG23])**.** *For all $\xi > 0$, the random metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability with respect to the topology on lower semicontinuous functions $\mathbb{C} \times \mathbb{C} \to \mathbb{R} \cup \{\infty\}$ (weaker than local uniform topology).*

- For $\xi > 2/d_2$, the metric D_h satisfies the LQG coordinate change rule with $Q \in (0, 2)$.
- Central charge $\mathbf{c}_{\mathrm{L}} = 1 + 6Q^2 \in (1, 25)$: supercritical phase.
- We say that $z \in \mathbb{C}$ is a **singular point** if

$$D_h(z, w) = \infty, \ \forall w \neq z.$$

- For each fixed $z \in \mathbb{C}$, a.s. z is not a singular point (singular points have zero Lebesgue measure).
- A.s., for any two non-singular points z, w, we have $D_h(z, w) < \infty$ (typical points lie at finite distance).
- For $\xi > 2/d_2$, the set of singular points is uncountable and Euclidean dense.
- Non-Euclidean topology.
- Metric balls have positive Lebesgue measure but empty Euclidean interior.
- For $\alpha > 0$, an **thick point** of h is a point z such that $\limsup_{\varepsilon \to 0} h_\varepsilon(z)/\log \varepsilon^{-1} \geq \alpha$.
- Singular points are (almost) the same as Q-thick points [Pfe21].
- In the critical case $\gamma = 2$, $\xi = 2/d_2$, there are no singular points and the metric induces the Euclidean topology [DG21b].
- Most results about the subcritical LQG metrics can be extended to the critical and supercritical LQG metrics, but I will focus on just the subcritical case for this talk.
- Existence of singular points is consistent with predictions from LCFT: "tachyon operators" (negative mass) which "tear the surface apart" [Sei90].
- Other features of supercritical LQG, besides the metric:
  - Coupling with CLE_4 (analogous to SLE/LQG relationship in the subcritical case, but the CLE_4 is not independent from or determined by the LQG) [AG23].
  - Conjectural scaling limit of certain infinite random planar maps, with infinitely many ends [AG23].
  - No measure which is locally determined by h and compatible with supercritical LQG coordinate change formula. But, there exists a one-parameter family of finite, non-local measures compatible with LQG coordinate change [BGS23].

4.3 Other random fractal metrics

There has been a number of other recent works, besides ones about the LQG metric, which study random fractal-type metrics, e.g.:

- Directed landscape [DOV18] (Dauvergne, Ortmann, Virag): random directed metric related to KPZ universality class. Also satisfies "confluence of geodesics" property, some similar tools are applicable but the construction is quite different.
- Metric on CLE gasket for $\kappa \in (8/3, 4)$ (tightness proven by Miller [Mil21]).
- Limiting metric for critical long-range percolation on $\mathbb{Z}^d$ [DFH23] (Ding-Fan-Huang), proven by adapting the uniqueness argument for the LQG metric.

5 Axiomatic definition

- Suppose we are given a random metric on the plane, coupled with the GFF. What properties would it need to satisfy for us to say that it is the LQG metric?
- Let us formalize the problem. Let $h \mapsto D_h$ be a measurable function

$$\{\text{generalized functions on } \mathbb{C}\} \to \{\text{metrics on } \mathbb{C}\}.$$

- We require that whenever h is a GFF or a GFF plus a (possibly random) continuous function, the following is true.
  1. **Euclidean topology.** Same topology as Euclidean metric.
  2. **Length metric.** $D_h(z, w)$ is the infimum of the D_h-lengths of paths from z to w.
  3. **Locality.** For $U \subset \mathbb{C}$, define the **internal metric** by

     $$D_h(z, w; U) = \inf\{D_h\text{-length of } P : P \text{ is a path in } U \text{ from } z \text{ to } w\}.$$

     Then $D_h(z, w; U)$ is a measurable function of $h|_U$.
  4. **Weyl scaling.** Almost surely, for each continuous function $f : \mathbb{C} \to \mathbb{R}$,

     $$D_{h+f}(z, w) = \inf_{P: z \to w} \int_0^{D_h(z,w)} e^{\xi f(P(t))}\, dt,$$

     where the infimum is over paths parametrized by D_h-length.
  5. **LQG coordinate change.** Let $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$. Almost surely,

     $$D_{h(a\cdot+b)+Q\log|a|}\left(\frac{z-b}{a}, \frac{w-b}{a}\right) = D_h(z, w), \quad \forall z, w \in \mathbb{C}.$$

- At first glance, there seems to be a two-parameter family (ξ and Q), but one can show that in fact ξ and Q must be related by $\xi = \gamma/d_\gamma$, $Q = 2/\gamma + \gamma/2$ (rough comparison to D_h^ε).

Theorem 11 (Gwynne-Miller [GM21b]). *Let D and $\widetilde{D}$ be two metrics satisfying the above axioms. There is a deterministic constant $C > 0$ such that a.s. $D_h = \widetilde{D}_h$ whenever h is a GFF or a GFF plus a continuous function.*

- Does this imply the uniqueness of the subsequential limit of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$?
- Euclidean topology proven by DDDF, length metric, locality, Weyl scaling easy to check.
- LQG coordinate change is a problem since if we scale space by C, we replace D_h^ε by $D_h^{C\varepsilon}$.
- This might give us a different subsequence.
- To get around this, we prove a stronger characterization theorem with LQG coordinate change replaced by **tightness across scales**. Roughly speaking, this condition says that we can get up-to-constants comparisons between $D_{h(a\cdot)+Q\log|a|}(z/a, w/a)$ and $D_h(z,w)$ with high probability.
- Most of the proofs are only slightly harder when we replace LQG coordinate change by tightness across scales.
- Once tightness is proven, every proof about the LQG metric uses only the axioms (we don't need to go back to the definition of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$).
- Existence of the metric can be taken as a black box.

6 Adding a bump function

- Assume that the additive constant for the whole-plane GFF is chosen so that the circle average $h_1(0) = 0$.
- The following Cameron-Martin type lemma is one of the most useful tools for studying the LQG metric (see, e.g., [BP, Proposition 1.29]).

Lemma 12. *Let h be the whole-plane GFF. Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function whose Dirichlet energy $(f,f)_\nabla = \int_{\mathbb{C}} |\nabla f(z)|^2 \, d^2 z$ is finite such that $f_1(0) = 0$. Then the laws of $h+f$ and h are mutually absolutely continuous, and the Radon-Nikodym derivative of the law of $h+f$ with respect to the law of h is*

$$\exp\left((h,f)_\nabla - \frac{1}{2}(f,f)_\nabla \right).$$

- If we want to show that D_h does something with positive probability, we just need to find a suitable bump function f such that D_{h+f} has the desired behavior with positive probability.

Lemma 13. *Fix $z, w \in \mathbb{C}$ and let $U \subset \mathbb{C}$ be a deterministic open set which contains a path from z to w. With positive probability, every D_h-geodesic from z to w is contained in U.*

Proof. Choose a deterministic smooth bump function f which is supported on a compact subset of U and which is equal to 1 on a neighborhood of a path in U from z to w. By Weyl scaling, if C is large then with high probability there is a path from z to w which is contained $\operatorname{supp} f$ and whose D_{h-Cf}-length is much smaller than the D_{h-Cf}-distance from $\operatorname{supp} f$ to ∂U. Thus every D_{h-Cf}-geodesic from z to w is contained in U. By absolute continuity, it holds with positive probability that every D_h-geodesic from z to w is contained in U. $\square$

7 Independence of the GFF across disjoint concentric annuli

- One of the most important tools for studying the LQG metric is the following lemma.

Lemma 14. *Let h be a whole-plane GFF. For $k \in \mathbb{N}$, let E_k be an event which is determined by the restriction of h to the annulus $B_{2^{-k}}(0) \setminus B_{2^{-k-1}}(0)$, viewed modulo additive constant.*

1. *For each $p \in (0,1)$, there exists $q = q(p) \in (0,1)$ such that if $\mathbb{P}[E_k] \geq p$ for each k, then for each $K \in \mathbb{N}$,*
$$\mathbb{P}[E_k \text{ occurs for at least one } k \in \{1,\dots,K\}] \geq 1 - q^K. \tag{7.1}$$
2. *For each $q \in (0,1)$, there exists $p = p(q) \in (0,1)$ such that if $\mathbb{P}[E_k] \geq p$ for each k, then for each $K \in \mathbb{N}$, (7.1) holds.*

- Show that $h|_{B_{2^{-k}}(0)\setminus B_{2^{-k-1}}(0)}$ are approximately independent, apply concentration for binomial(q,K) distribution.
- Idea originally due to Miller-Qian [MQ20], formulated precisely by Gwynne-Miller [GM20b].
- Various improvements are possible.
  - Replace $B_{2^{-k}}(0)\setminus B_{2^{-k-1}}(0)$ by disjoint concentric annuli with uniformly bounded aspect ratios.
  - If p is close enough to 1, then E_k has to occur for "most" $k \in \{1,\dots,K\}$.

Lemma 15. *For each $\gamma \in (0,2)$, there exists $\alpha = \alpha(\gamma) > 0$ and $c = c(\gamma) > 0$ such that the following is true. For each $z \in \mathbb{C}$ and each $\varepsilon > 0$, the probability that there is a D_h-geodesic between two points in $\mathbb{C} \setminus B_{\varepsilon^{1/2}}(z)$ which enters $B_\varepsilon(z)$ is at most $c\varepsilon^\alpha$.*

Roughly speaking, Lemma 15 says that "most" points in $\mathbb{C}$ are not hit by D_h-geodesics except at their endpoints. Lemma 15 immediately implies that the Hausdorff dimension of the union of all of the LQG geodesics w.r.t. the Euclidean metric is strictly less than 2. See [GP22] for an explicit upper bound for the Hausdorff dimension of a single LQG geodesic. Similar (but more complicated) ideas to the ones in the proof of Lemma 15 are used in the proof of confluence of geodesics in [GM20a, DG21a].

Definition 16. For a Euclidean annulus $A \subset \mathbb{C}$, we define $D_h(\text{across } A)$ to be the D_h-distance between the inner and outer boundaries of A. We define $D_h(\text{around } A)$ to be the infimum of the D_h-lengths of paths in A which separate the inner and outer boundaries of A.

Both $D_h(\text{across } A)$ and $D_h(\text{around } A)$ are determined by the internal metric of D_h on A, so by locality these quantities are a.s. determined by $h|_A$.

For $z \in \mathbb{C}$ and $r > 0$, let
$$E_r(z) := \{D_h(\text{around } B_{3r}(z) \setminus B_{2r}(z)) < D_h(\text{across } B_{2r}(z) \setminus B_r(z))\}. \tag{7.2}$$

As noted above, $E_r(z)$ is a.s. determined by $h|_{B_{3r}(z)\setminus B_r(z)}$. In fact, adding a constant to h results in scaling D_h-distances by a constant (Weyl scaling), so adding a constant to h does not affect whether $E_r(z)$ occurs. Hence $E_r(z)$ is a.s. determined by $h|_{B_{3r}(z)\setminus B_r(z)}$ modulo additive constant.

Lemma 17. *There exists $\alpha = \alpha(\gamma) > 0$ and $c = c(\gamma) > 0$ such that for each $z \in \mathbb{C}$ and each $\varepsilon > 0$,*
$$\mathbb{P}\left[\exists r \in \left[\varepsilon, \frac{1}{4}\varepsilon^{1/2}\right] \text{ such that } E_r(z) \text{ occurs}\right] \geq 1 - c\varepsilon^\alpha.$$

Proof. By the scale and translation invariance of the law of h, modulo additive constant, $\mathbb{P}[E_r(z)]$ does not depend on z or r. Using a "subtracting a bump function" argument, one can show that $p := \mathbb{P}[E_1(0)] > 0$. Hence $\mathbb{P}[E_r(z)] = p$ for each $z \in \mathbb{C}$ and $r > 0$. We now apply Lemma 14 with $K \asymp \log \varepsilon^{-1}$ to get

$$\mathbb{P}\left[\exists r \in [\varepsilon, \varepsilon^{1/2}] \text{ such that } E_r(z) \text{ occurs}\right] \geq 1 - q^{\log \varepsilon^{-1}}$$

for $q = q(p) \in (0,1)$. This last quantity is at least $1 - c\varepsilon^{\alpha}$ for an appropriate $c, \alpha > 0$. □

Proof of Lemma 15. By Lemma 17, it suffices to show that if there is an $r \in [\varepsilon, \frac{1}{4}\varepsilon^{1/2}]$ such that $E_r(z)$ occurs, then no D_h-geodesic between two points in $\mathbb{C} \setminus B_{\varepsilon^{1/2}}(z)$ can enter $B_\varepsilon(z)$. Indeed, assume that $E_r(z)$ occurs, let $u, v \in \mathbb{C} \setminus B_{\varepsilon^{1/2}}(z)$, and let P be a path from u to v which hits $B_r(z) \supset B_\varepsilon(z)$. We will show that P is not a D_h-geodesic. By the definition (7.2) of $E_r(z)$, there is a path π in $B_{3r}(z) \setminus B_{2r}(z)$ which disconnects the inner and outer boundaries of this annulus and has D_h-length strictly less than $D_h(\text{across } B_{2r}(z) \setminus B_r(z))$. Let σ (resp. τ) be the first (resp. last) time that P hits π. Since P hits $B_r(z)$ and $u, v \notin B_{3r}(z)$, the path P crosses between the inner and outer boundaries of $B_{2r}(z) \setminus B_r(z)$ between times σ and τ. Hence

$$\left(D_h\text{-length of } P|_{[\sigma,\tau]}\right) \geq D_h(\text{across } B_{2r}(z) \setminus B_r(z)). \tag{7.3}$$

But, since $P(\tau), P(\sigma) \in \pi$,

$$\begin{aligned} D_h(P(\sigma), P(\tau)) \leq (D_h\text{-length of } \pi) &< D_h(\text{across } B_{2r}(z) \setminus B_r(z)) \\ &\leq \left(D_h\text{-length of } P|_{[\sigma,\tau]}\right). \end{aligned} \tag{7.4}$$

This implies that P is not a D_h-geodesic since it is not the D_h-shortest path from $P(\sigma)$ to $P(\tau)$. □

8 Bi-Lipschitz equivalence

This corresponds to Section 4 of [GM20b].

- Recall the uniqueness theorem: if D_h and $\widetilde{D}_h$ are two metrics satisfying the list of axioms, we want to show that $\exists$ deterministic $C > 0$ such that a.s. $D_h = C\widetilde{D}_h$.
- The first step is to show that D_h and $\widetilde{D}_h$ are bi-Lipschitz equivalent.

Proposition 18 (Gwynne-Miller [GM20b])**.** *There are deterministic constants $C_* > c_* > 0$ such that a.s.*

$$c_* D_h(z,w) \leq \widetilde{D}_h(z,w) \leq C_* D_h(z,w), \quad \forall z, w \in \mathbb{C}.$$

- Proof is surprisingly easy. Takes about 5 pages to write, I will give it in full.
- Once the proposition is proven, to get uniqueness we only need to show that $c_* = C_*$. This takes several hundred pages and is the main goal of [GM20b, DFG+20, GM20a, GM21b].
- The proof exemplifies many of the most important techniques used to study the LQG metric, including adding a bump function and independence across concentric annuli.
- For $r > 0$, $z \in \mathbb{C}$, and $C > 0$, let

$$E_r(z, C) := \left\{\widetilde{D}_h(\text{around } B_{2r}(z) \setminus B_r(z)) \leq C D_h(\text{across } B_r(z) \setminus B_{r/2}(z))\right\}.$$

- We want to apply the independence across annuli lemma to these events.

Lemma 19. *For each $q \in (0,1)$, there exists $C = C(q) > 0$ such that $\mathbb{P}[E_r(z,C)] \geq q$ for all $z \in \mathbb{C}$ and all $r > 0$.*

Proof. By Weyl scaling, adding a constant to h scales $\widetilde{D}_h$ and D_h in the same way. Hence $E_r(z,C)$ is determined by h, modulo additive constant. The law of h, viewed modulo additive constant, is invariant under scaling and translating space. By this and LQG coordinate change, $\mathbb{P}[E_r(z,C)] = \mathbb{P}[E_1(0,C)]$. Since $\widetilde{D}_h(\text{around } B_2(0) \setminus B_1(0))$ is a.s. finite and $D_h(\text{across } B_2(0) \setminus B_1(0))$ is a.s. positive, we can find $C > 0$ such that $\mathbb{P}[E_1(0,C)] \geq q$. □

Lemma 20. *There exists $C > 0$ such that for each $z \in \mathbb{C}$ and each $\varepsilon > 0$,*

$$\mathbb{P}\Big[E_r(z,C) \text{ occurs for at least one value of } r \in [\varepsilon, \varepsilon^{1/2}]\Big] \geq 1 - \varepsilon^{100}.$$

Proof. By the locality property of the metric, the event $E_r(z,C)$ is determined by $h|_{B_{2r}(z)\setminus B_{r/2}(z)}$, viewed modulo additive constant. Hence we can apply the previous lemma together with the "independence across annuli" lemma with $K \asymp \log \varepsilon^{-1}$. □

Lemma 21. *Let $U \subset \mathbb{C}$ be a deterministic bounded open set and let $C > 0$ be as in the previous lemma. It holds with probability tending to 1 as $\varepsilon \to 0$ that for each $z \in U \cap \frac{\varepsilon}{100}\mathbb{Z}^2$, there exists $r = r(z) \in [\varepsilon, \varepsilon^{1/2}]$ such that $E_r(z,C)$ occurs.*

Proof. Immediate from the previous lemma and a union bound over all $z \in U \cap \frac{\varepsilon}{100}\mathbb{Z}^2$. □

Henceforth assume that the event of the previous lemma occurs. Let $z, w \in U$ such that the D_h-geodesic P from z to w is contained in U. We inductively define times $s_0 \leq t_0 \leq s_1 \leq t_1 \leq \dots$ for P.

Let $s_0 = t_0 = 0$. Inductively, assume that $j \in \mathbb{N}$ and s_j and t_j have been defined. If $t_j = D_h(z,w)$, let $s_{j+1} = t_{j+1} = D_h(z,w)$. Otherwise, choose $z_j \in \frac{\varepsilon}{100}\mathbb{Z}^2$ such that $P(t_j) \in B_{\varepsilon/2}(z_j)$ and let $r_j \in [\varepsilon, \varepsilon^{1/2}]$ be such that $E_{r_j}(z_j, C)$ occurs. Let s_{j+1} (resp. t_{j+1}) be the first time after t_j that P hits $\partial B_{r_j}(z)$ (resp. $\partial B_{r_j/2}(z)$), or $s_{j+1} = D_h(z,w)$ (resp. $t_{j+1} = D_h(z,w)$) if no such time exists.

Let

$$J := \min\{j : t_j = D_h(z,w)\}.$$

Then $t_j \leq s_{j+1} \leq t_{j+1}$ for each $j \in \mathbb{N}$. Furthermore,

$$t_{j+1} - s_{j+1} \geq D_h(\text{across } B_{r_j}(z_j) \setminus B_{r_j/2}(z_j)).$$

By the definition of $E_{r_j}(z_j, C)$, there exists a path π_j in $B_{2r_j}(z_j) \setminus B_{r_j}(z_j)$ with $\widetilde{D}_h$-length at most

$$\widetilde{D}_h(\text{around } B_{2r_j}(z_j) \setminus B_{r_j}(z_j)) \leq C D_h(\text{across } B_{r_j}(z_j) \setminus B_{r_j/2}(z_j)) \leq C(t_{j+1} - s_{j+1}).$$

Hence

$$\sum_{j=1}^{\infty} \Big(\widetilde{D}_h\text{-length of } \pi_j\Big) \leq C \sum_{j=1}^{\infty} t_{j+1} - s_{j+1} \leq C D_h(z,w).$$

Simple topological considerations show that the union of the paths π_j contains a path from $B_{2\varepsilon^{1/2}}(z)$ to $B_{2\varepsilon^{1/2}}(w)$. Hence

$$\widetilde{D}_h(B_{2\varepsilon^{1/2}}(z), B_{2\varepsilon^{1/2}}(w)) \leq C D_h(z,w).$$

Sending $\varepsilon \to 0$ and using that the $\widetilde{D}_h$ induces the Euclidean topology now gives $\widetilde{D}_h \leq C D_h$. By symmetry, we also have an analogous inequality in the opposite direction. □

9 Tightness

We explain the proof of the following theorem, due to [DDDF20].

Theorem 22. *The metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ are tight with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times \mathbb{C}$. Every subsequential limit is a metric which induces the Euclidean topology on $\mathbb{C} \times \mathbb{C}$.*

9.1 Step 0: Setup

We use the white noise decomposition of the GFF.

Let W be a space-time white noise on $\mathbb{C}\times[0,\infty)$, i.e., for every $f \in L^2(\mathbb{C}\times[0,\infty))$, $\int_0^\infty \int_{\mathbb{C}} f(z,t) W(dz,dt)$ is Gaussian with mean zero and variance $\int_0^\infty \int_{\mathbb{C}} f(z,t)^2\, dz\, dt$.

For $n, m \in \mathbb{Z}$ with $m < n$, define the white noise field

$$\phi_{m,n}(z) := \int_{2^{-2n}}^{2^{-2m}} \int_{\mathbb{C}} p_{t/2}(z-w) W(dw,dt).$$

The following lemma relates the field $\phi_{0,n}$ to the GFF mollified by the heat kernel, and allows us to work with $\phi_{0,n}$ instead of h_ε when proving tightness. See [DDDF20, Section 6.1] for a proof.

Lemma 23. *Let $U \subset \mathbb{C}$ be open and bounded. For each $n \in \mathbb{N}$, there is a coupling of $\phi_{0,n}$ with the whole-plane GFF h and constants $c, C > 0$ depending only on U such that the following is true. If $h^*_{2^{-n}}$ is as above with $\varepsilon = 2^{-n}$, then $\phi_{0,n} - h^*_{2^{-n}}$ is a continuous function on U and*

$$\mathbb{P}\left[\sup_{z\in U} |\phi_{0,n}(z) - h_{2^{-n}}(z)| > M\right] \le Ce^{-cM^2}, \quad \forall M > 0.$$

The above lemma is not true it we replace the convolution of h with the heat kernel by a different mollification, e.g., the circle average process. This is the reason why we only prove tightness of the approximating metrics defined using the convolution of h with the heat kernel.

For $z, w \in \mathbb{C}$, let

$$D_n(z,w) := \inf_{P: z\to w} \int_0^1 e^{\phi_{0,n}(P(t))} |P'(t)|\, dt,$$

where the infimum is over piecewise continuously differentiable paths from z to w. By the above lemma, it suffices to show that the metrics $\{\mathfrak{a}_{2^{-n}}^{-1} D_n\}_{n\in\mathbb{N}}$ are tight with respect to the topology of local uniform convergence on $\mathbb{C}\times\mathbb{C}$.

9.2 Step 1: RSW estimate

This corresponds to Section 3 of [DDDF20].

We denote by $L^n_{a,b}$ the infimum of the D_n-lengths of paths in the rectangle $[0,a]\times[0,b]$ joining the left and right sides of $[0,a]\times[0,b]$. For $\varepsilon > 0$, we define the q quantile

$$\ell^n_{a,b}(q) := \inf\{s > 0 : \mathbb{P}[L^n_{a,b} \ge s] \ge q\}.$$

Note that the comparison of $h_{2^{-2n}}$ and $\phi_{0,n}$ implies that

$$\mathfrak{a}_{2^{-n}} \asymp \ell^n_{1,1}(1/2).$$

The following estimate allows us to compare quantiles of rectangle crossings in the "hard" direction (i.e., those of the form $\ell^n_{a,b}(q)$ for $a > b$) and in the "easy" direction (i.e., those of the form $\ell^n_{a,b}(q)$ for $a < b$). For simplicity we fix $a, b \in \{1, 3\}$.

Proposition 24. *There exists a constant $C > 1$, depending only on ξ, such that for each $\delta > 0$ and each large enough $n \in \mathbb{N}$,*

$$\ell^n_{3,1}(\delta/C) \le C\ell^n_{1,3}(\delta)$$

and

$$\ell^n_{3,1}(1-\delta^C) \le C\ell^n_{1,3}(1-\delta)$$

Proof. The proof is via an approximate conformal invariance argument. Let E be an ellipse which is contained in $[0,1]\times\mathbb{R}$ and which disconnects the left and right sides of $[0,1]\times[0,3]$. Let E' be an ellipse which is contained in $\mathbb{R}\times[0,3]$ and which intersects both the left and right sides of $[0,3]\times[0,1]$.

The set $\partial E \cap ([0,1]\times[0,3])$ has two connected components, say A_L and A_R. If $I_L \subset A_L$ and $I_R \subset A_R$ are small enough arcs, then there is a conformal map f going from a neighborhood of $\overline{E}$ to a neighborhood of $\overline{E}'$ which takes I_L and I_R into the two connected components of $\partial E' \setminus ([0,3]\times[0,1])$.

By the definition of $\ell^n_{1,3}(\delta)$, it holds with probability at least δ that there is a path between the left and right sides of $[0,1]\times[0,3]$ of D_n-length at most $\ell^n_{1,3}(\delta)$. This path must cross both A_L and A_R. Hence, we can find $C > 1$ and small deterministic arcs $I_L \subset A_L$, $I_R \subset A_R$ such that with probability at least δ/C, there is a path in E from I_L to I_R of length at most $\ell^n_{1,3}(\delta)$. Let f be a conformal map as above for this choice of I_L, I_R. Then the image under f of a path in E from I_L to I_R has a sub-path which crosses between the left and right sides of $[0,3]\times[0,1]$.

The metric D_n is not conformally invariant (or conformally covariant), but it is "close enough", in the sense that with high probability one can compare the metrics $D_n(f^{-1}(\cdot), f^{-1}(\cdot))$ and $D_n(\cdot,\cdot)$ on E' up to constants. This leads to the bound $\ell^n_{3,1}(\delta/C) \le C\ell^n_{1,3}(\delta)$.

The comparison of high quantiles is similar, with one extra step. Let $\mathcal{I}_L$ and $\mathcal{I}_R$ be collections of small enough subarcs which cover A_L and A_R, respectively. For $I_L \in A_L$ and $I_R \in A_R$, let $F(I_L, I_R)$ be the event that there is a path in E from I_L to I_R of D_n-length at most $\ell^n_{1,3}(1-\delta)$.

As above, it holds with probability at least $1-\delta$ that for some $I_L \in \mathcal{I}_L$ and $I_R \in \mathcal{I}_R$, the event $F(I_L, I_R)$ occurs. By FKG (for positively correlated Gaussian fields), the events $F(I_L, I_R)$ for different choices of I_L, I_R are positively correlated. By the square root trick, at least one of the events $F(I_L, I_R)$ has probability at least $1-\delta^{1/m}$, where $m = \#\mathcal{I}_L \times \#\mathcal{I}_R$. We then use approximate conformal invariance as above. □

9.3 Step 2: percolation argument

This corresponds to Section 4 of [DDDF20]. We use a percolation argument to prove that there is some fixed p_0 such that the left-right crossing distance of a rectangle is extremely unlikely to be smaller than its p_0-quantile or larger than its $1-p_0$-quantile.

Proposition 25. *There exists a small universal constant $p_0 > 0$ and constants $c, C > 0$ depending only on ξ such that for each $n \in \mathbb{N}$ and each $M > 2$,*

$$\mathbb{P}\left[L^n_{3,1} \le M\ell^n_{3,1}(1-p_0)\right] \ge 1 - Ce^{-c(\log M)^2/\log\log M}$$

and

$$\mathbb{P}\left[L^n_{1,3} \ge M^{-1}\ell^n_{1,3}(p_0)\right] \ge 1 - Ce^{-c(\log M)^2/\log\log M}.$$

Proof. Fix $p_0 > 0$ to be chosen later. We first prove the upper bound for $L^n_{3,1}$. Let $\mathcal{S}$ be the set of 1×1 squares with corners in $\mathbb{Z}^2$. We say that $S \in \mathcal{S}$ is *open* if for each of the six 1×3 or 3×1 rectangles R with corners in $\mathbb{Z}^2$ which intersect S, the D_n-distance between the two shorter sides of R is at most $\ell^n_{1,3}(1-p_0)$. By translation invariance and the definition of $\ell^n_{1,3}(1-p_0)$, the probability that each square S is open is at least $1-6p_0$.

The covariance $\mathrm{Cov}(\phi_{0,n}(z), \phi_{0,n}(w))$ decays like $e^{-|z-w|^2/2}$. Therefore, $\phi_{0,n}$ has very weak long-range dependence (to make this precise we look at a truncated version of $\phi_{0,n}$ which has a finite range of dependence, but is still close to $\phi_{0,n}$). Therefore, the set of open squares in $\mathcal{S}$ looks a lot like a percolation with finite range of dependence. If p_0 is small enough, this percolation is supercritical. So, if k is large it holds with probability at least $1 - Ce^{-ck}$ that there is a left-right crossing of $[0, 3k] \times [0, k]$ by open squares. If such a crossing exists, then $L^n_{k,3k} \leq k^2 \ell^n_{1,3}(1 - p_0)$. Therefore,

$$\mathbb{P}\left[L^n_{k,3k} \leq k^2 \ell^n_{1,3}(1 - p_0)\right] \geq 1 - Ce^{-ck}.$$

To conclude, we choose $k = \sqrt{M}$ and apply a scaling argument.

The lower bound for $L^n_{1,3}$ is proven similarly. We declare that $S \in \mathcal{S}$ is open if the D_n-distance between the two longer sides of R is at least $\ell^n_{1,3}(1-p_0)$ for each of the four 1×3 or 3×1 rectangles with corners in $\mathbb{Z}^2$ which are contained in the annulus between S and the 3×3 square with the same center as S. We use a percolation argument to find a path of open square between the two shorter sides of $[0, k] \times [0, 3k]$. Any path between the two longer sides of $[0, k] \times [0, 3k]$ has to cross one of these open squares, which allows us to lower-bound its length. □

9.4 Step 3: Efron-Stein argument

This corresponds to Section 5.1 in [DDDF20].

Let us review what we have done so far. From Step 2, we can upper-bound D_n-distances with high probability in terms of the $1 - p_0$ quantile $\ell^n_{3,1}(1 - p_0)$ of the crossing distance $L^n_{3,1}$. By Step 1, we can upper-bound $\ell^n_{3,1}(1 - p_0)$ in terms of $\ell^n_{1,3}(1 - p_1)$, for some constant $p_1 \in (0, p_0)$. This, in turn, can be upper bounded by $\ell^n_{1,1}(1 - p_1)$ (cover $[0, 1]$ by three $1/3 \times 1$ rectangles). Hence, we can upper-bound distances with high probability in terms of $\ell^n_{1,1}(1 - p_1)$.

Similarly, we can lower-bound distances with high probability in terms of $\ell^n_{1,1}(p_1)$.

Since $\mathfrak{a}_{2^{-n}} = \ell^n_{1,1}(1/2)$, to show tightness of $\mathfrak{a}^{-1}_{2^{-n}} D_n$, the main remaining obstacle is to establish an up-to-constants comparison of the quantiles $\ell^n_{1,1}(p_1)$ and $\ell^n_{1,1}(1-p_1)$. This is the most technically involved part of the proof.

Proposition 26. *Let $p \in (0, 1/2)$ be small. For $n \in \mathbb{N}$, let*

$$\Lambda_n := \max_{k=1,\ldots,n} \frac{\ell^k_{1,1}(1 - p)}{\ell^k_{1,1}(p)}.$$

There is a constant $C = C(p, \xi) > 0$ such that $\Lambda_n \leq C$ for all $n \in \mathbb{N}$.

Proof. We will bound Λ_n by induction on n. We have the elementary inequality

$$\frac{\ell^n_{1,1}(1 - p)}{\ell^n_{1,1}(p)} \leq \exp\Big(C\sqrt{\mathrm{Var} \log L^n_{1,1}}\Big).$$

So, we need to bound $\mathrm{Var} \log L^n_{1,1}$.

To do this, we use the Efron-Stein inequality.

Lemma 27 (Efron-Stein)**.** *Let $(X_1, \ldots, X_n)$ be independent random variables. Let $(X'_1, \ldots, X'_n)$ be an independent copy of $(X_1, \ldots, X_n)$. Let $F = F(X_1, \ldots, X_n)$ be a real-valued measurable function of $(X_1, \ldots, X_n)$. For $i = 1, \ldots, n$, let F^i be obtained by replacing X_i by X'_i (leaving the other X_js the same), then applying F. We have*

$$\mathrm{Var}\, F \leq \sum_{i=1}^{n} \mathbb{E}\Big[\big(F^i - F\big)^2_+\Big]$$

where $(x)_+ = \max\{x, 0\}$.

Remark 28. The Efron-Stein inequality is often written with $\frac{1}{2}(F^i - F)^2$ instead of $(F^i - F)_+^2$. The two versions of the inequality are equivalent since $(F^i, F) \overset{d}{=} (F, F^i)$, hence

$$\mathbb{E}\Big[\big(F^i - F\big)_+^2\Big] = \frac{1}{2}\mathbb{E}\Big[\big(F^i - F\big)^2\Big]$$

We need to write our observable $L_{1,1}^n$ as a function of many independent random variables. Let $K \in \mathbb{N}$ be large but fixed (independently from n). Let $\mathcal{S}_K$ be the set of $2^{-K} \times 2^{-K}$ squares with corners in $2^{-K}\mathbb{Z}^2$. We decompose the field $\phi_{K,n}$ into pieces which depend only on the white noise in the square S:

$$\phi_{K,n}^S(z) := \int_{2^{-2n}}^{2^{-2K}} \int_S p_{t/2}(z - w)\, W(dw, dt).$$

Then $\phi_{K,n}^S$ for different choices of S are independent from each other and from $\phi_{0,K}$, and

$$\phi_{0,n} = \phi_{0,K} + \sum_{S \in \mathcal{S}_K} \phi_{K,n}^S.$$

Let $L_{1,1}^n(K)$ be defined in the same manner as $L_{1,1}^n$, with $\phi_{0,n}$ replaced by $\phi_{0,n} + (\widetilde{\phi}_{0,K} - \phi_{0,K})$, where $\widetilde{\phi}_{0,K}$ is an independent copy of $\phi_{0,K}$. For $S \in \mathcal{S}_K$, let $L_{1,1}^n(S)$ be defined in the same manner as $L_{1,1}^n$, with $\phi_{0,n}$ replaced by $\phi_{0,n} + (\widetilde{\phi}_{K,n}^S - \phi_{K,n}^S)$, where $\widetilde{\phi}_{K,n}^S$ is an independent copy of $\phi_{K,n}^S$. By the Efron-Stein inequality,

$$\operatorname{Var} \log L_{1,1}^n \le \mathbb{E}\Big[\big(\log L_{1,1}^n(K) - \log L_{1,1}^n\big)_+^2\Big] + \sum_{S \in \mathcal{S}_K} \mathbb{E}\Big[\big(\log L_{1,1}^n(S) - \log L_{1,1}^n\big)_+^2\Big].$$

For the first term, one can use a Gaussian concentration bound to get

$$\mathbb{E}\Big[\big(\log L_{1,1}^n(K) - \log L_{1,1}^n\big)_+^2\Big] \le CK.$$

To bound the second term, for $S \in \mathcal{S}_K$, let π_S be a path of minimal D_n-length which disconnects the inner and outer boundaries of the annulus

$$A_S := B_{K^{1/100}2^{-K+1}}(S) \setminus B_{K^{1/100}2^{-K}}(S).$$

Let P be a path between the left and right boundaries of $[0,1]^2$ of minimal D_n-length. For each square S, the union $P \cup \pi_S$ contains a path between the left and right boundaries of $[0,1]^2$ which does not enter $B_{K^{1/100}2^{-K}}(S)$. This path can just be taken to be P if P itself does not enter $B_{K^{1/100}2^{-K}}(S)$. The difference of the fields $\phi_{0,n}$ and $\phi_{0,n} + (\widetilde{\phi}_{K,n}^S - \phi_{K,n}^S)$ outside of $B_{K^{1/100}2^{-K}}(S)$ is very small, so we get

$$\big(L_{1,1}^n(S) - L_{1,1}^n\big)_+ \le (D_n\text{-length of } \pi_S)\mathbb{1}_{(P \text{ hits } B_{K^{1/100}2^{-K}}(S))}$$

and hence

$$\big(\log L_{1,1}^n(S) - \log L_{1,1}^n\big)_+ \le \frac{1}{L_{1,1}^n}(D_n\text{-length of } \pi_S)\mathbb{1}_{(P \text{ hits } S)}.$$

We can write

$$(D_n\text{-length of } \pi_S) \approx e^{\xi\phi_{K,n}(v_S)}(D_{K,n}\text{-length of } \pi_S).$$

where v_S is the center point of S. Using Step 2 and the scaling properties of the white noise field, we see that $D_{K,n}$-length of π_S can be bounded above in terms of Λ_{n-K} with high probability. Furthermore, the quantity

$$\frac{1}{L_{1,1}^n} e^{\xi \phi_{K,n}(v_S)} \mathbb{1}_{(P \text{ hits } B(S))}$$

corresponds, roughly speaking, to the fraction of D_n-time that the path P spends near $B(S)$. We therefore arrive at the bound

$$\mathbb{E}\Big[\big(\log L_{1,1}^n(S) - \log L_{1,1}^n\big)_+\Big] \qquad \lesssim [\Lambda_{n-K}]^2 \mathbb{E}\left[\sum_{S \in \mathcal{S}_K} (\text{fraction of time that } P \text{ spends near } S)^2\right].$$

The last expectation can be bounded above by $e^{-\alpha K}$ for $\alpha = \alpha(\xi) > 0$ using crude bounds for the maximum of the GFF.

Going back to the Efron-Stein inequality, we therefore get

$$\operatorname{Var} \log L_{1,1}^n \le CK + Ce^{-\alpha K}[\Lambda_{n-K}]^2$$

and so

$$\Lambda_n \le \exp\left(C\sqrt{CK + e^{-\alpha K}[\Lambda_{n-K}]^2}\right).$$

If K is taken to be large enough (but independent from n), this recursive bound gives a constant-order upper bound for Λ_n. □

9.5 Step 4: Conclusion

This corresponds to Sections 5.2-5.4 in [DDDF20].

By combining Steps 2 and 3, we obtain bounds of the form

$$\mathbb{P}\left[\mathfrak{a}_{2^{-n}}^{-1} L_{3,1}^n \le M\right] \ge 1 - Ce^{-c(\log M)^2/\log\log M}.$$

and

$$\mathbb{P}\left[\mathfrak{a}_{2^{-n}}^{-1} L_{1,3}^n \ge M^{-1}\right] \ge 1 - Ce^{-c(\log M)^2/\log\log M}$$

Applying the upper bound at multiple scales, taking a union bound, and stringing together paths appropriately gives us the tightness of the metrics $\mathfrak{a}_{2^{-n}} D_n$. Similarly, the lower bound tells us that every subsequential limit is a metric which induces the Euclidean topology.

10 Uniqueness

10.1 Step 1: bi-Lipschitz equivalence

- Recall the uniqueness theorem: if D_h and $\widetilde{D}_h$ are two metrics satisfying the list of axioms, we want to show that $\exists$ deterministic $C > 0$ such that a.s. $D_h = C\widetilde{D}_h$.
- The first step, which we have already carried out in Section 8. is to show that D_h and $\widetilde{D}_h$ are bi-Lipschitz equivalent.

Proposition 29 (Gwynne-Miller [GM20b])**.** *There are deterministic constants $C_* \ge c_* > 0$ such that a.s.*

$$c_* D_h(z,w) \le \widetilde{D}_h(z,w) \le C_* D_h(z,w), \quad \forall z,w \in \mathbb{C}.$$

Let c_* and C_* be the optimal constants in the previous proposition. Our next goal is to prove that $c_* = C_*$. To do this, we assume by way of contradiction that $c_* < C_*$. We aim to show that $\widetilde{D}_h \le C' D_h$ for $C' < C_*$. This will contradict the optimality of C_*.

10.2 Step 2: existence of shortcuts at many scales

This corresponds to Section 3 of [GM21b].

Let $C' \in (c_*, C_*)$. By the definition of c_* and C_*, it holds with positive probability that there exists $u, v \in \mathbb{C}$ such that $\widetilde{D}_h(u,v) \geq C' D_h(u,v)$. We need a quantitative version of this statement.

Proposition 30. *There exists $p \in (0,1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $C' \in (0, C_*)$ and each sufficiently small $\varepsilon > 0$ (depending on C' and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ such that*

$$\mathbb{P}\Big[\exists \text{ a ``regular'' pair of points } u, v \in B_r(0) \setminus B_{r/2}(0) \text{ s.t. } \widetilde{D}_h(u,v) \geq C' D_h(u,v)\Big] \geq p. \tag{10.1}$$

Here "regular" involves a number of conditions on u, v, including that $|u - v| \geq \text{const} * r$.

Basically, if the above proposition were false, we could use the annulus independence lemma to get the following. For any fixed open set U, it holds with high probability that if $P : [0,T] \to U$ is a D_h-geodesic contained in U, then there exist times $s_1 < t_1 < \cdots < s_N < t_N$ such that

$$\widetilde{D}_h(P(s_j), P(t_j)) \leq C' D_h(P(s_j), P(t_j)) = C'(t_j - s_j)$$

for each j, and the intervals $[s_j, t_j]$ cover a positive fraction β of $[0,T]$. If this is the case, then the $\widetilde{D}_h$-distance between the endpoints of P is at most

$$C' \sum_{j=1}^{N} (t_j - s_j) + C_*(T - \sum_{j=1}^{N}(t_j - s_j)) \leq \big(C_* - \beta(C_* - C')\big)T.$$

This contradicts the optimality of C_*.

10.3 Step 3: counting "good" and "very good" annuli

We follow the argument of [DG23] (which does not use confluence of geodesics) instead of the original argument given in [GM21b].

Fix $c' \in (c_*, C_*)$ and $m \in (0,1)$ (close to 1).

We define for each $r > 0$ and each $z \in \mathbb{C}$ an event $E_{z,r}$ and a deterministic function $f_{z,r}$ satisfying the following properties.

- $E_{z,r}$ is determined by $h|_{B_{4r}(z)\setminus B_r(z)}$, viewed modulo additive constant, and $\mathbb{P}[E_{z,r}] \geq m$.
- $f_{z,r}$ is smooth, non-negative, and supported on the annulus $B_{3r}(z) \setminus B_r(z)$.
- Assume that $E_{z,r}$ occurs and P' is a $D_{h-f_{z,r}}$-geodesic between two points of $\mathbb{C} \setminus B_{4r}(z)$ which spends "enough" time in the support of $f_{z,r}$. Then there are times $s < t$ such that $P'([s,t]) \subset B_{4r}(z)$,
$$\widetilde{D}_{h-f_{z,r}}(P'(s), P'(t)) \leq c'(t-s), \tag{10.2}$$
and $t - s \geq \text{const} \cdot r^{\xi Q} e^{\xi h_r(z)}$.

Roughly speaking, the support of $f_{z,r}$ is a long narrow tube contained in a small neighborhood of $\partial B_{2r}(0)$. On the event $E_{z,r}$, there are many "good" pairs of points u, v in the support of $f_{z,r}$ such that $\widetilde{D}_h(u,v) \leq c_0' D_h(u,v)$ and the $\widetilde{D}_h$-geodesic from u to v is contained in the support of $f_{z,r}$, where $c_0' \in (c_*, c')$ is fixed.

We show that $E_{z,r}$ occurs with high probability using Proposition 30 and a long-range independence statement for the GFF.

The function $f_{z,r}$ will be very large on most of its support. So, by Weyl scaling, a $D_{h-f_{z,r}}$-geodesic which enters the support of $f_{z,r}$ will tend to spend a long time in the support of $f_{z,r}$. This will force the $D_{h-f_{z,r}}$-geodesic to get $D_{h-f_{z,r}}$-close to each of u and v for one of the aforementioned "good" pairs of points u, v. The estimate (10.2) will follow from this and the triangle inequality.

The core part of the proof is the following proposition.

Proposition 31. *Assume $c_* < C_*$. Let $a > 0$ and let $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ with $|\mathbb{z} - \mathbb{w}| \geq a$. As $\delta \to 0$,*

$$\mathbb{P}\Big[\widetilde{D}_h(\mathbb{z}, \mathbb{w}) > (C_* - \delta) D_h(\mathbb{z}, \mathbb{w}), \text{ regularity conditions}\Big] = O_\delta(\delta^\mu), \quad \forall \mu > 0, \tag{10.3}$$

with the $O_\delta(\delta^\mu)$ depending only on a.

We think of a ball $B_{4r}(z)$ as "good" if the event $E_{z,r}$ occurs and "very good" if the event $E_{z,r}(h + f_{z,r})$, which is defined in the same manner as $E_{z,r}$ but with $h + f_{z,r}$ instead of h, occurs. By definition, if $B_{4r}(z)$ is"good" for h, then $B_{4r}(z)$ is "very good" for $h - f_{z,r}$.

Let P be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. Recall that $\mathbb{P}[E_{z,r}] \geq m$, which is close to 1, and $E_{z,r}$ is determined by $h|_{B_{4r}(z) \setminus B_r(z)}$, viewed modulo additive constant. From this, it is easy to show using the near-independence of the restrictions of h to disjoint concentric annuli (Lemma 14) that P has to hit $B_r(z)$ for lots of "good" balls $B_{4r}(z)$.

To prove Proposition 31, we need to show that P also hits $B_r(z)$ for many "very good" balls $B_{4r}(z)$, and spends lots of time in the support of $f_{z,r}$ for such balls. Indeed, the condition (10.2) (with $h + f_{z,r}$ instead of h) will then give us lots of pairs of points s, t such that $\widetilde{D}_h(P(s), P(t)) \leq c'(t - s)$, which in turn will show that $\widetilde{D}_h(\mathbb{z}, \mathbb{w})$ is bounded away from $C_* D_h(\mathbb{z}, \mathbb{w})$.

Fix a large bounded open set U. Let $\mathcal{Z}_r$ be the set of finite subsets Z of $U \cap r\mathbb{Z}^2$ with the property that the balls $B_{4r}(z)$ for $z \in Z$ are disjoint. For $Z \in \mathcal{Z}_r$, let

$$f_{Z,r} = \sum_{z \in Z} f_{z,r}.$$

For $Z \in \mathcal{Z}_r$ and $q > 0$, let $F_{Z,r}(q)$ be the event that the following is true.

- $\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \geq C_* D_h(\mathbb{z}, \mathbb{w}) - q$.
- Each ball $B_{4r}(z)$ for $z \in Z$ is "good".
- The D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ hits $B_r(z)$ for each $z \in Z$.
- The $D_{h-f_{Z,r}}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$ spends "enough" time in the support of $f_{z,r}$ for each $z \in Z$.
- $r^{\xi Q} e^{\xi h_r(z)} \in [q, 2q]$ for each $z \in Z$.

We also let $F'_{Z,r}(q)$ be defined in the same manner as $F_{Z,r}$ but with $h + f_{Z,r}$ in place of h, i.e., $F'_{Z,r}$ is the event that the following is true.

- $\widetilde{D}_{h+f_{Z,r}}(\mathbb{z}, \mathbb{w}) \geq C_* D_{h+f_{Z,r}}(\mathbb{z}, \mathbb{w}) - q$.
- Each $B_{4r}(z)$ for $z \in Z$ is "very good".
- The $D_{h+f_{Z,r}}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$ hits $B_r(z)$ for each $z \in Z$.
- The D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$ spends "enough" time in the support of $f_{z,r}$ for each $z \in Z$.

- $r^{\xi Q}e^{\xi h_r(z)} \in [q, 2q]$ for each $z \in Z$. (Note that the support of $f_{z,r}$ is disjoint from the $B_r(z)$, so adding $f_{z,r}$ does not change $h_r(z)$).

Lemma 32. *There is a constant $A > 1$ depending only on the laws of D_h and $\widetilde{D}_h$ such that for any choice of r, q,*

$$A^{-k}\mathbb{P}[F_{Z,r}(q)] \le \mathbb{P}[F'_{Z,r}(q)] \le A^k \mathbb{P}[F_{Z,r}(q)], \quad \textit{whenever } \#Z \le k. \tag{10.4}$$

Proof. This follows from a basic Radon-Nikodym derivative estimate for the GFF. □

We will eventually take k to be a large constant, independent of $r, \mathbb{z}, \mathbb{w}$, depending on the number μ in (10.3). So, the relation (10.4) suggests that the number of sets Z such that $\#Z \le k$ and $F_{Z,r}$ occurs should be comparable to the number of such sets for which $F'_{Z,r}$ occurs.

Lemma 33. *There exist constants $\alpha, c_1, c_2 > 0$ such that the following is true. On the event*

$$\left\{\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \ge C_* D_h(\mathbb{z}, \mathbb{w}) - \varepsilon^{c_2}\right\}$$

and certain high-probability global regularity events occur. Let $k \in \mathbb{N}$. Then there exists $r \in [\varepsilon^2, \varepsilon] \cap \{2^{-j}\}_{j\in\mathbb{N}}$ and $q \in [\varepsilon^{c_1}, \varepsilon^{c_2}] \cap \{2^{-j}\}_{j\in\mathbb{Z}}$ such that

$$\#\{Z \in \mathcal{Z}_r : \#Z \le k,\ F_{z,r}(q) \text{ occurs}\} \ge \varepsilon^{-\alpha k}.$$

Proof. Using the annulus independence lemma and a union bound, we see that the number of "good" balls $B_r(z)$ for $r \in [\varepsilon^2, \varepsilon] \cap \{2^{-k}\}_{k\in\mathbb{N}}$ and $z \in r\mathbb{Z}^2$ hit by P is typically at least $\varepsilon^{-\beta}$ for some $\beta > 0$. We can choose r and q so that $r^{\xi Q}e^{\xi h_r(z)} \in [q, 2q]$ for at least $\varepsilon^{-\beta}/(\log \varepsilon^{-1})^2$ of these good balls. We then use that

$$\binom{\varepsilon^{-\beta}(\log \varepsilon^{-1})^{-2}}{k} \ge \text{const} \times \varepsilon^{-\alpha k}$$

for some $\alpha > 0$. (Technically, we also need some arguments to discard the good balls $B_r(z)$ such that P hits $B_r(z)$, but the $D_{h+f_{Z,r}}$-geodesic does not spend enough time in the support of $f_{z,r}$). □

Lemma 34. *There is a constant $C > 0$ such that for each $r, q > 0$,*

$$\#\big\{Z \in \mathcal{Z}_r : \#Z \le k,\ F'_{Z,r}(q) \text{ occurs}\big\} \le C^k.$$

Proof. Assume without loss of generality that there exists W with $\#W \le k$ such that $F'_{W,r}(q)$ occurs. By the definition of $F'_{W,r}(q)$, on this event

$$\widetilde{D}_{h+f_{W,r}}(\mathbb{z}, \mathbb{w}) \ge C_* D_{h+f_{W,r}}(\mathbb{z}, \mathbb{w}) - q. \tag{10.5}$$

Now let S be the set of $z \in r\mathbb{Z}^2$ such that $E_{z,r}(h + f_{z,r})$ occurs and the D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ spends enough time in the support of $f_{z,r}$. By our hypotheses on $E_{z,r}$ and $f_{z,r}$, there are times $s_z < t_z$ with $P([s_z, t_z]) \subset B_{4r}(z)$ such that

$$\widetilde{D}_h(P(s_z), P(t_z)) \le C'(t_z - s_z).$$

Using the regularity conditions in the definition of $E_{z,r}$, we can arrange that $t_z - s_z \ge \text{const} \cdot q$. Thus, each $z \in S$ contributes a "shortcut" for $\widetilde{D}_h$ of length of order q. Hence

$$\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \le C_* D_h(\mathbb{z}, \mathbb{w}) - \text{const} \cdot q\#S.$$

Since $\#W \le k$, adding $f_{W,r}$ increases distances by at most a constant times qk. Hence

$$\widetilde{D}_{h+f_{W,r}}(\mathbb{z},\mathbb{w}) \le C_* D_{h+f_{W,r}}(\mathbb{z},\mathbb{w}) + \text{const}\cdot qk - \text{const}\cdot q\#S.$$

By (10.5), this implies that $\#S$ is most a constant times k. To get a Z such that $F'_{Z,r}(q)$ occurs, we need to choose at most k of the $\#S$ elements of S. There are at most C^k ways to do so. $\square$

Proof of Proposition 31. Trivially,

$$(\log \varepsilon^{-1})^2 \succeq \sum_{r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{q\in[\varepsilon^{c_1},\varepsilon^{c_2}]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{\substack{Z\in\mathcal{Z}_r\\ \#Z\le k}} \frac{\mathbb{1}_{F'_{Z,r}(q)}}{\#\{\widetilde{Z}\in\mathcal{Z}_r : \#\widetilde{Z}\le k, F'_{\widetilde{Z},r}(q)\text{ occurs}\}}.$$

Take expectations of both sides to get

$$\begin{aligned}
(\log \varepsilon^{-1})^2 &\succeq \sum_{r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{q\in[\varepsilon^{c_1},\varepsilon^{c_2}]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{\substack{Z\in\mathcal{Z}_r\\ \#Z\le k}} \mathbb{E}\left[\frac{\mathbb{1}_{F'_{Z,r}}}{\#\{\widetilde{Z}\in\mathcal{Z}_r : \#\widetilde{Z}\le k, F_{\widetilde{Z},r}\text{ occurs}\}}\right]\\
&\succeq C^{-k} \sum_{r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{q\in[\varepsilon^{c_1},\varepsilon^{c_2}]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{\substack{Z\in\mathcal{Z}_r\\ \#Z\le k}} \mathbb{P}\left[F'_{Z,r}(q)\right]\\
&\succeq A^{-k}C^{-k} \sum_{r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{q\in[\varepsilon^{c_1},\varepsilon^{c_2}]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{\substack{Z\in\mathcal{Z}_r\\ \#Z\le k}} \mathbb{P}[F_{Z,r}(q)]\\
&\succeq A^{-k}C^{-k}\mathbb{E}\left[\sum_{r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \sum_{q\in[\varepsilon^{c_1},\varepsilon^{c_2}]\cap\{2^{-k}\}_{k\in\mathbb{N}}} \#\{Z\in\mathcal{Z}_r : F_{Z,r}\text{ occurs}\}\right].
\end{aligned}$$

On the event

$$\left\{\widetilde{D}_h(\mathbb{z},\mathbb{w}) \ge C_* D_h(\mathbb{z},\mathbb{w}) - \varepsilon^{c_2}\right\}$$

the last line is at least $A^{-k}C^{-k}\varepsilon^{-\alpha k}$. Therefore, the probability of this event is at most a k-dependent constant times $\varepsilon^{\alpha k}(\log \varepsilon^{-1})^2$. $\square$

Once Proposition 31 is established, one can take a union bound over many pairs of points $\mathbb{z},\mathbb{w}\in B_r(0)\setminus B_{r/2}(0)$ to get, roughly speaking, the following.

Proposition 35. *Assume that $c_* < C_*$. For each $\zeta > 0$, there exists $\delta > 0$ such that for each sufficiently small $\varepsilon > 0$ (depending only on the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r\in[\varepsilon^2,\varepsilon]\cap\{8^{-k}\}_{k\in\mathbb{N}}$ for which*

$$\mathbb{P}\left[\exists \text{ a “regular” pair } \mathbb{z},\mathbb{w}\in B_r(0)\setminus B_{r/2}(0) \text{ s.t. } \widetilde{D}_h(\mathbb{z},\mathbb{w}) \ge (C_*-\delta)D_h(\mathbb{z},\mathbb{w})\right] \ge 1-\zeta. \quad (10.6)$$

Proposition 35 is incompatible with Proposition 30 since the parameter p in Proposition 30 does not depend on C'. We thus obtain a contradiction to the assumption that $c_* < C_*$, so we conclude that $c_* = C_*$.

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