

Differential Chow Forms and Differential Chow Varieties: An Overview

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Differential Algebraic Geometry

*Three possible languages to study differential equations: the classical language, the geometric language and the language of **differential algebra**.—(Yuri Manin, 1979)*

- ▶ A subject, founded by Ritt and Kolchin (1950-70s), aims to study **algebraic differential equations** based on algebraic geometry and commutative algebra.
- ▶ Basic object: **differential variety** (i.e., the solution set of a system of algebraic differential equations)
- ▶ Main Problems: algebraic theory and algorithms around differential equation solving.

Relationship with Model Theory

The origins of model theory and differential algebra may be starkly different in character, but in recent decades large parts of these subjects have developed symbiotically. (Scanlon 2002)

- ▶ Robinson's insight (1950s): model theory provides differential algebra with the notion of **differentially closed fields**.

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Effectiveness of stability-theoretic tools in differential algebra:

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- ▶ Differential closed fields serve as proving and testing grounds for pure model theory.

“the least misleading example for many model-theoretic phenomena” (Sacks, 1972).

Outline

- ▶ Motivation: Algebraic Chow forms and Chow varieties
- ▶ Ordinary differential Chow forms and Chow varieties
- ▶ Partial differential Chow forms and a type of partial differential Chow varieties
- ▶ Summary and Problems

Motivation

Chow-van der Waerden (Math. Ann., 1937) pointed out:

It is principally important to represent geometrical objects by coordinates. Once this has been done for a specific kind of objects G , then it makes sense to speak of an algebraic manifold or an algebraic system of objects G , and to apply the whole theory of algebraic manifolds (decomposition into irreducible components, notion of dimension, notion of general elements in an irreducible manifold). It is desirable to provide the set of objects G with the structure of an algebraic variety (eventually, after a certain compactification), thus to characterise G by algebraic equations in the coordinates.

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Simple examples:

- ▶ Points: Cartesian coordinates
- ▶ Lines in \mathbb{P}^3 : Plücker coordinates
- ▶ Linear spaces: Grassmann coordinates

Example: Plücker coordinates

Let V be a line in \mathbb{P}^3 defined by

$$\begin{cases} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 = 0 \\ a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0. \end{cases}$$

- ▶ **Plücker coordinates** of V : $(p^{01}, p^{02}, p^{03}, p^{12}, p^{13}, p^{23}) \in \mathbb{P}^5$,
where

$$p^{ij} = a_{0i}a_{1j} - a_{0j}a_{1i}.$$

- ▶ **The variety of lines**: $p^{01}p^{23} + p^{02}p^{31} + p^{03}p^{12} = 0$.

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Algebraic varieties: Chow coordinates and Chow varieties
(was developed based on the theory of **algebraic Chow forms**)

Algebraic Chow forms (Chow, 1937)

- ▶ Let $\mathbf{V} = \sum_{i=1}^l s_i V_i$ be a d -cycle in \mathbb{P}^n ($\dim(V_i) = d$).

Let $L_i = u_{i0}y_0 + u_{i1}y_1 + \cdots + u_{in}y_n$ ($i = 0, \dots, d$): hyperplanes

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- ▶ **Chow coordinates** of \mathbf{V} : coefficients $(c_{\omega}) \in \mathbb{P}^N$.

- ▶ Degree of \mathbf{V} : $\deg(\mathbf{V}) = \deg(F, (u_{ij})_{j=0}^n) = g$.

Poisson-product formula: $F = A \prod_{\tau=1}^g (u_{00}\xi_{\tau 0} + \cdots + u_{0n}\xi_{\tau n})$.

Chow varieties

Theorem(Chow, 1937) The set of Chow coordinates of all d -cycles of degree g is a variety in \mathbb{P}^N , called **Chow variety**.

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Examples of applications:

- ▶ Chern classes on algebraic varieties with arbitrary singularities (Wu, 1984);
- ▶ Counting problems in geometry;
- ▶ Arakelov theory and diophantine applications.

Ordinary Differential Chow Forms

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- ▶ By GDIT, there exists a unique $F(u_{ij}) = \sum_{\omega} c_{\omega} M_{\omega}(u_{ij})$ s.t.

$$\text{sat}(F) = 0 \iff V \cap \mathbb{V}(L_0, \dots, L_d) \neq \emptyset.$$

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Example: $V = \mathbb{V}(y' - y) \subset \mathbb{A}^1$

- ▶ δ -Chow form of V : $F = u'_{00}u_{01} - u_{00}u'_{01} - u_{00}u_{01}$;
- ▶ δ -Chow coordinate of V : $(1, -1, -1, 0, 0, 0, 0, 0, 0) \in \mathbb{P}^9$.

New invariants of differential varieties

Properties of δ -Chow form $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$:

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 $V, L_1 = 0, \dots, L_d = 0, {}^a L_0^{(0)} = 0, \dots, {}^a L_0^{(h-1)} = 0$.
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The above V is called of index (d, h, g, m) .

Ordinary Diff Chow variety

- ▶ δ -cycle: $\mathbf{V} = \sum_i s_i V_i$ ($V_i \subseteq \mathbb{A}^n$ irr. of dim d , order h).
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Existence of δ -Chow varieties:

- ▶ For $g = 1$, δ -Chow varieties exists. (Gao-Li-Yuan, 2013)
A constructive proof.

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- ▶ δ -Chow varieties $\mathcal{C}_{(d,h,g,m)}$ exist. (Frietag-Li-Scanlon, 2017)
A model-theoretical proof.

Ingredients of the proof

- ▶ $\delta\text{-chow}_n(d, h, g, m) = \{\text{diff cycles in } \mathbb{A}^n \text{ of index } (d, h, g, m)\}$.
- ▶ $\text{chow}_n(d, s)$: the Chow variety in \mathbb{A}^n of dim d and degree s .

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$$\delta\text{-chow}_n(d, h, g, m) \ni \sum_{i=1}^{\ell} V_i$$

$$\downarrow \phi$$

$$\bigcup_{e=D_1}^{D_2} \text{chow}_{n(h+1)}(d(h+1) + h, e) \ni \sum_{i=1}^{\ell} B_h(V_i)$$

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Theorem. The image \mathcal{C} is δ -constructible and the map ϕ identifies $\delta\text{-chow}_n(d, h, g, m)$ with \mathcal{C} .

Tools:

- ▶ Definability results in the model theory of ACF_0 and DCF_0 ;
- ▶ Moosa-Scanlon's prolongation theory of jet spaces.

Partial Differential Chow Forms and Chow Varieties

Partial differential Chow form: problem

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- ▶ $S := \{(u_{ij})_{i,j=0}^{d,n} : V \cap \mathbb{V}(L_0, \dots, L_d) \neq \emptyset\}^{\text{cl}}$.

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Problem: There may not exist a single F s.t. $S = \mathbb{V}(\text{sat}(F))$.

Non-Example. Let $m = 2$, $V = \mathbb{V}(\delta_1(y), \delta_2(y)) \subset \mathbb{A}^1$. Here $S = \mathbb{V}(\text{sat}(u_{01}\delta_1(u_{00}) - u_{00}\delta_1(u_{01}), u_{01}\delta_2(u_{00}) - u_{00}\delta_2(u_{01})))$.

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Def. If $S = \mathbb{V}(\text{sat}(F))$, then call F the Δ -Chow form of V .

For which varieties V , the Δ -Chow form of V exists?

A type of Δ -Chow varieties (Li, 2017)

Theorem. If $\omega_V(t) = (d+1)\binom{t+m}{m} - \binom{t+m-s}{m}$, then the Δ -Chow form F of V exists. Moreover, $\text{ord}(F) = s$ and

- ▶ F is Δ -homogenous of degree r in the coefficients of each L_i .
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Theorem. $\Delta\text{-Chow}_n(d, s, r)$ is a Δ -constructible set.

Summary

The theory of ordinary differential Chow forms and differential Chow varieties is established, and limited results are obtained in the partial differential case.

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Further Problems:

- ▶ Constructive proof for the existence of δ -Chow varieties;
- ▶ Conjecture on the the existence of Δ -Chow forms;
- ▶ How to give coordinate representations and parameter space for general Δ -varieties?

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Thanks!