A model theory dichotomy in generalized descriptive set theory

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Outline

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The spectrum problem

Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

For first order theory in a countable vocabulary:

- **Löwenheim-Skolem Theorem:** $\exists\alpha \geq \omega \ I(T,\alpha) \neq 0 \Rightarrow \forall\beta \geq \omega \ I(T,\beta) \neq 0$.
- **Morley's categoricity:** $\exists\alpha > \omega \ I(T,\alpha) = 1 \Rightarrow \forall\beta > \omega \ I(T,\beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T,\alpha) = 2^\alpha$, or $\forall\alpha > 0 \ I(T,\aleph_\alpha) < \beth_{\omega_1}(\mid \alpha \mid)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

The generalized Baire space is the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

κ-Borel sets

The collection of κ-Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is κ-*Borel*, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a κ-Borel subset of κ^κ.

Reductions

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a κ-Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B^\kappa E_2$.

Coding structures

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong_T^\kappa$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The Borel-reducibility hierarchy

We can define a partial order on the set of all first-order countable theories

$$T \leqslant^\kappa_\kappa T' \text{ iff } \cong^\kappa_T \leqslant^\kappa_B \cong^\kappa_{T'}$$

The generalized Cantor space

In the subspace 2^κ, we can define the following notions in the same way:

- $E_1 \leqslant^2_B E_2$.
- $f \cong^2_T g$.
- $T \leqslant^2_\kappa T'$.

Since $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

defines the structure $\mathcal{A}_f$ in both cases ($f \in 2^\kappa$ and $f \in \kappa^\kappa$), $T \leqslant^2_\kappa T'$ if and only if $T \leqslant^\kappa_\kappa T'$.

Question

Is there a relation between these two notions of complexity?

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Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Question:

Is there a Borel reducibility counterpart of the Main Gap Theorem?

The countable case

$T = Th(\mathbb{Q}, \leqslant)$.
T', the theory of vector space over the field of rational numbers.

By the Borel-reducibility hierarchy:

$$T \leqslant^{\omega}_{\omega} T'$$

$$T' \not\leqslant^{\omega}_{\omega} T$$

By the stability theory T' is simpler than T.

$E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$

For every regular cardinal $\lambda < \kappa$, the relations $E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$ are defined as follow.

Definition

- *On the space κ^κ, we say that $f, g \in \kappa^\kappa$ are $E^{\kappa}_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set closed under λ-limits.*
- *On the space 2^κ, we say that $f, g \in 2^\kappa$ are $E^{2}_{\lambda\text{-club}}$ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set closed under λ-limits.*

Non-classifiable theories

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

1. *If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-club}} \leqslant^2_B \cong^2_T$.*
2. *If $\lambda \geq 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-club}} \leqslant^2_B \cong^2_T$.*

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable theory. Then $E^2_{\omega\text{-club}} \leqslant^2_B \cong^2_T$.

Classifiable theories

Theorem (Hyttinen, Moreno)

Suppose T is a classifiable theory and $\lambda < \kappa$. Then $\cong^\kappa_T \leqslant^\kappa_B E^\kappa_{\lambda\text{-club}}$.

A Borel reducibility counterpart

Let $H(\kappa)$ be the following property: If T is classifiable and T' is not, then $T \leqslant^2_\kappa T'$ and $T' \not\leqslant^2_\kappa T$.

Theorem (Hyttinen, Kulikov, Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. *If $V = L$, then $H(\kappa)$ holds.*
2. *It is consistent that $H(\kappa)$ holds and there are 2^κ equivalence relations strictly between $\cong^2_T$ and $\cong^2_{T'}$.*

Questions

Question

Is there an uncountable cardinal κ, such that $H(\kappa)$ is a theorem of ZFC?

Question

Have all the non-classifiable theories the same complexity?

Outline

Σ_1^1-completeness

An equivalence relation E on $X \in \{\kappa^\kappa, 2^\kappa\}$ is Σ_1^1 or *analytic*, if E is the projection of a closed set in $X^2 \times \kappa^\kappa$ and it is Σ_1^1-*complete* or *analytic complete* if it is Σ_1^1 (analytic) and every Σ_1^1 (analytic) equivalence relation is Borel reducible to it.

$E^2_{\lambda\text{-club}}$ in L

Theorem

$(V = L)$. *For every $\lambda < \kappa$ regular, $E^2_{\lambda\text{-club}}$ is a Σ^1_1-complete equivalence relation.*

Proof

Definition

- *We define a class function* $F_\diamondsuit : On \to L$. *For all* α, $F_\diamondsuit(\alpha)$ *is a pair* (X_α, C_α) *where* $X_\alpha, C_\alpha \subseteq \alpha$, C_α *is a club if* α *is a limit ordinal and* $C_\alpha = \emptyset$ *otherwise. We let* $F_\diamondsuit(\alpha) = (X_\alpha, C_\alpha)$ *be the* $<_L$*-least pair such that for all* $\beta \in C_\alpha$, $X_\beta \neq X_\alpha \cap \beta$ *if* α *is a limit ordinal and such pair exists and otherwise we let* $F_\diamondsuit(\alpha) = (\emptyset, \emptyset)$.
- *We let* $C_\diamondsuit \subseteq On$ *be the class of all limit ordinals* α *such that for all* $\beta < \alpha$, $F_\diamondsuit \restriction \beta \in L_\alpha$. *Notice that for every regular cardinal* α, $C_\diamondsuit \cap \alpha$ *is a club.*

Definition

For all regular cardinal α and set $A \subset \alpha$, we define the sequence $(X_\gamma, C_\gamma)_{\gamma \in A}$ as the sequence $(F_\diamondsuit(\gamma))_{\gamma \in A}$, and the sequence $(X_\gamma)_{\gamma \in A}$ as the sequence of sets X_γ such that $F_\diamondsuit(\gamma) = (X_\gamma, C_\gamma)$ for some C_γ.

By ZF^- we mean $ZFC + (V = L)$ without the power set axiom. By $ZF^\diamondsuit$ we mean ZF^- with the following axiom:
"For all regular ordinals $\mu < \alpha$ if $(S_\gamma, D_\gamma)_{\gamma \in \alpha}$ is such that for all $\gamma < \alpha$, $F_\diamondsuit(\gamma) = (S_\gamma, D_\gamma)$, then $(S_\gamma)_{\gamma \in S^\alpha_\mu}$ is a diamond sequence, where S^α_μ is the set of ordinals smaller than α with cofinality μ."

Lemma

($V = L$) For any Σ_1-formula $\varphi(\eta, x)$ with parameter $x \in 2^\kappa$, a regular cardinal $\mu < \kappa$, the following are equivalent for all $\eta, \xi \in 2^\kappa$:

- $\varphi(\eta, \xi, x)$
- *$S \backslash A$ is non-stationary, where $S = \{\alpha \in S^\kappa_\mu \mid X_\alpha = \eta^{-1}\{1\} \cap \alpha\}$ and*

  $$A = \{\alpha \in C_\diamondsuit \cap \kappa \mid \exists \beta > \alpha (L_\beta \models ZF^\diamond \wedge \varphi(\eta \restriction \alpha, \xi \restriction \alpha, x \restriction \alpha) \wedge r(\alpha))\}$$

  where $r(\alpha)$ is the formula "α is a regular cardinal".

Suppose E is a Σ_1^1 equivalence relation. There is a Σ_1-formula of set theory $\psi(\eta,\xi)=\psi(\eta,\xi,x)=\exists k\varphi(k,\eta,\xi,x)\vee\eta=\xi$ with $x\in 2^\kappa$, such that for all $\eta,\xi\in 2^\kappa$,

$$(\eta,\xi)\in E\Leftrightarrow\psi(\eta,\xi),$$

Let $r(\alpha)$ be the formula "α is a regular cardinal" and $\psi^E(\kappa)$ be the sentence with parameter κ that asserts that $\psi(\eta,\xi)$ defines an equivalence relation on 2^κ. For all $\eta\in 2^\kappa$ and $\alpha<\kappa$, let

$$T_{\eta,\alpha}=\{p\in 2^\alpha\mid\exists\beta>\alpha(L_\beta\models ZF^\diamond\wedge\psi(p,\eta\restriction\alpha,x\restriction\alpha)\wedge r(\alpha)\wedge\psi^E(\alpha))\}.$$

Let $(X_\alpha)_{\alpha\in S_\mu^\kappa}$ be the diamond sequence from $F_\diamondsuit$, and for all $\alpha\in S_\mu^\kappa$, let $\mathcal{X}_\alpha$ be the characteristic function of X_α. Define $\mathcal{F}\colon 2^\kappa\to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha)=\begin{cases}1 & \text{if } \mathcal{X}_\alpha\in T_{\eta,\alpha} \text{ and } \alpha\in S_\mu^\kappa\\ 0 & \text{otherwise}\end{cases}$$

The dichotomy

Theorem

($V = L$) Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. If T is a theory in a countable vocabulary. Then one of the following holds.

- *$\cong^\kappa_T$ is Δ^1_1 (all the complete extensions of T are classifiable).*
- *$\cong^\kappa_T$ is Σ^1_1-complete (T has at least one non-classifiable extension).*

Notice that T is not required to be complete.

Remark

The previous result is not a theorem of ZFC, not even when it is restricted to complete theories.

Thank you