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Modal Logics with Team Semantics

A Survey

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Basic modal logic $\mathcal{ML}$: Syntax

Let Φ be a set of proposition symbols. The set of $\mathcal{ML}(\Phi)$ -formulas is defined by the following grammar:

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Box\varphi \mid \Diamond\varphi,$$

where $p \in \Phi$.

Note that formulas are assumed to be in *negation normal form*: negations may occur only in front of atomic formulas.

Basic modal logic $\mathcal{ML}$: Semantics

A *Kripke-model* for Φ is a triple $M = (W, R, V)$, where

- $W \neq \emptyset$ is the set of *states* (or possible worlds),
- $R \subseteq W \times W$ is the *accessibility relation*, and
- $V : \Phi \rightarrow \mathcal{P}(W)$ is the *valuation*.

A *team* on M is a subset $T \subseteq W$.

Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics for $\mathcal{ML}$:

- $M, w \models p \quad \Leftrightarrow \quad w \in V(p)$
- $M, w \models \neg p \quad \Leftrightarrow \quad w \notin V(p)$
- $M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$
- $M, w \models \varphi \vee \psi \Leftrightarrow M, w \models \varphi$ or $M, w \models \psi$
- $M, w \models \Box\varphi \quad \Leftrightarrow \quad M, v \models \varphi$ for all v s.t. wRv
- $M, w \models \Diamond\varphi \quad \Leftrightarrow \quad M, v \models \varphi$ for some v s.t. wRv

Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics/team semantics for $\mathcal{ML}$:

- $M, w \models p \quad \Leftrightarrow \quad w \in V(p)$
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- $M, w \models \varphi \wedge \psi \quad \Leftrightarrow \quad M, w \models \varphi$ and $M, w \models \psi$
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Basic modal logic $\mathcal{ML}$: Semantics

Kripke-semantics/team semantics for $\mathcal{ML}$:

- $M, T \models p \quad \Leftrightarrow \quad T \subseteq V(p)$
- $M, T \models \neg p \quad \Leftrightarrow \quad T \cap V(p) = \emptyset$
- $M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$
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- $M, T \models \varphi \vee \psi \Leftrightarrow M, S \models \varphi$ and $M, S' \models \psi$ for some S, S' such that $S \cup S' = T$
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- $M, T \models \Box\varphi \quad \Leftrightarrow \quad M, S \models \varphi$ for $S = \{v \mid \exists w \in T : wRv\}$
- $M, w \models \Diamond\varphi \quad \Leftrightarrow \quad M, v \models \varphi$ for some v s.t. wRv

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- $M, T \models \Diamond\varphi \quad \Leftrightarrow \quad M, S \models \varphi$ for some S such that $\forall u \in T \exists v \in S : uRv$

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- $M, T \models \Diamond\varphi \quad \Leftrightarrow \quad M, S \models \varphi$ for some S such that $\forall u \in T \exists v \in S : uRv$ and $\forall v \in S \exists u \in T : uRv$

The idea behind team semantics is that a team T satisfies an $\mathcal{ML}$-formula φ iff all states $w \in T$ satisfy φ:

Theorem (Flatness property of $\mathcal{ML}$)

For all $\varphi \in \mathcal{ML}$,

$$M, T \models \varphi \;\Leftrightarrow\; M, w \models \varphi \text{ for all } w \in T.$$

In particular $M, \{w\} \models \varphi \;\Leftrightarrow\; M, w \models \varphi$.

Closure properties of $\mathcal{ML}$

Corollary (Downwards closure)

Every formula φ of $\mathcal{ML}$ is downwards closed:

If $M, T \models \varphi$, then $M, S \models \varphi$ for all $S \subseteq T$.

Theorem (Union closure)

Every formula φ of $\mathcal{ML}$ is closed under unions:

If $M, T_i \models \varphi$ for all $i \in I$, then $M, \bigcup_{i \in I} T_i \models \varphi$.

Theorem (Empty team property)

For all $\varphi \in \mathcal{ML}$ and every Kripke models M, $M, \emptyset \models \varphi$.

Modal dependence logic $\mathcal{MDL}$

Let Φ be a set of proposition symbols. The set of $\mathcal{MDL}(\Phi)$ -formulas is defined by the following grammar:

$$\begin{aligned} \varphi \quad ::= \quad & p \mid \neg\, p \mid {=}(p_1, \ldots, p_n, q) \mid \\ & (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi, \end{aligned}$$

where $p, p_1, \ldots, p_n, q \in \Phi$.

The propositional dependence atom ${=}(p_1, \ldots, p_n, q)$ says that the truth value of q is determined by the truth values of $p_1, \ldots, p_n$:

- $M, T \models {=}(p_1, \ldots, p_n, q) \quad \Leftrightarrow$

$$\forall\, v, w \in T : \ \textstyle\bigwedge_{1 \le i \le n}(M, v \models p_i \Leftrightarrow M, w \models p_i) \Rightarrow \ (M, v \models q \Leftrightarrow M, w \models q)$$

Basic properties of $\mathcal{MDL}$

Theorem (Downwards closure for $\mathcal{MDL}$)

Let φ be a formula of $\mathcal{MDL}$.

If $M, T \models \varphi$, then $M, S \models \varphi$ for all $S \subseteq T$.

Theorem (Empty team property for $\mathcal{MDL}$)

If $\varphi \in \mathcal{MDL}$ and M is a Kripke model, then $M, \emptyset \models \varphi$.

However, $\mathcal{MDL}$ is not closed under unions.

Complexity of $\mathcal{MDL}$

Theorem (Sevenster 09)
The satisfiability problem for $\mathcal{MDL}$ is $\mathrm{NEXPTIME}$-complete.

Theorem (Hannula 17)
The validity problem for $\mathcal{MDL}$ is $\mathrm{NEXPTIME}$-complete.

Theorem (Ebbing-Lohmann 12)
The model checking problem for $\mathcal{MDL}$ is NP-complete.

Propositional dependence logic $\mathcal{PDL}$: Syntax

To understand the expressive power of $\mathcal{MDL}$, we study first its restriction to propositional formulas.

Let Φ be a set of proposition symbols. The set of $\mathcal{PDL}(\Phi)$-formulas is defined by the following grammar:

$$\begin{aligned}\varphi \quad ::= \quad & p \mid \neg\, p \mid {=}(p_1, \ldots, p_n, q) \mid \\ & (\varphi \vee \varphi) \mid (\varphi \wedge \varphi),\end{aligned}$$

where $p, p_1, \ldots, p_n, q \in \Phi$.

Propositional dependence logic $\mathcal{PDL}$: Semantics

A *(truth value) assignment* for Φ is a function $s : \Phi \to \{\bot, \top\}$.
The semantics of $\mathcal{PDL}$ is defined on teams, that are just sets of assigments for Φ.

- $X \models p \quad \Leftrightarrow \quad s(p) = \top$ for all $s \in X$
- $X \models \neg p \quad \Leftrightarrow \quad s(p) = \bot$ for all $s \in X$
- $X \models \varphi \wedge \psi \Leftrightarrow X \models \varphi$ and $X \models \psi$
- $X \models \varphi \vee \psi \Leftrightarrow Y \models \varphi$ and $Z \models \psi$ for some $Y \cup Z = X$
- $X \models \ =(p_1, \ldots, p_n, q) \Leftrightarrow \bigwedge_i (s(p_i) = t(p_i)) \Rightarrow s(q) = t(q)$ holds for all $s, t \in X$

Note that there are 2^n assignments, and 2^{2^n} teams, where $n = |\Phi|$.

Intuitionistic disjunction

Add a different version of disjunction $\varovee$ to propositional (dependence) logic with the semantics:

- $X \models \varphi \varovee \psi \;\Leftrightarrow\; X \models \varphi$ or $X \models \psi$

Let $\mathcal{PL}(\varovee)$ be the the logic obtained from $\mathcal{PDL}$ by removing dependence atoms and adding $\varovee$.

Dependence atoms are definable in $\mathcal{PL}(\varovee)$ (Väänänen 09):

$$\models \; =\!(p_1, \ldots, p_n, q) \;\Leftrightarrow\; \bigvee_{s \in F} (\theta_s \wedge (q \varovee \neg q)),$$

where F is the team of all $\{p_1, \ldots, p_n\}$-assignments, and θ_s is the formula $\bigwedge_i p_i^{s(p_i)}$, where $p_i^{\bot} = \neg p_i$ and $p_i^{\top} = p_i$.

It is easy to prove by induction that for every $\mathcal{PDL}$-formula there is an equivalent $\mathcal{PL}(\varovee)$-formula. Thus, $\mathcal{PDL} \leq \mathcal{PL}(\varovee)$.

Intuitionistic disjunction: Completeness

A much stronger result is true: $\mathcal{PL}(\varovee)$ is *complete* with respect to downwards closed properties of teams.

Definition

- A *property* of teams is any set $\mathcal{P}$ of teams (for a fixed Φ).
- $\mathcal{P}$ is *downwards closed* if $X \in \mathcal{P}$ and $Y \subseteq X$ implies $Y \in \mathcal{P}$.

Lemma

For every team X there is $\psi_X \in \mathcal{PL}$ such that $Y \models \psi_X$ iff $Y \subseteq X$.

Proof.

For each Φ-assignment s, let θ_s be the formula $\bigwedge_{p \in \Phi} p^{s(p)}$.
Clearly $Y \models \theta_s$ iff $Y \subseteq \{s\}$.

For each Φ-team $X \neq \emptyset$, let ψ_X be the formula $\bigvee_{s \in X} \theta_s$; for $X = \emptyset$, let $\psi_X := p \wedge \neg p$. Then we have $Y \models \psi_X$ iff $Y \subseteq X$. □

Intuitionistic disjunction: Completeness

Theorem (Yang 14)

Every nonempty downwards closed property $\mathcal{P}$ of teams is definable in $\mathcal{PL}(\varovee)$.

Proof.

Let $\varphi_{\mathcal{P}}$ be the formula $\bigovee_{X \in \mathcal{P}} \psi_X$. Then we have

$$Y \models \varphi_{\mathcal{P}} \iff \exists X \in \mathcal{P} : Y \subseteq X \iff Y \in \mathcal{P}.$$

□

Remark: $\mathcal{PL}(\varovee)$ is actually equivalent with *inquisitive logic* (Ciardelli, Groenendijk). The completeness of inquisitive logic was proved by Ciardelli 09.

$\mathcal{PDL}$ and intuitionistic disjunction

Since $\mathcal{PDL}$ is downwards closed, we have another proof for the fact $\mathcal{PDL} \leq \mathcal{PL}(\varovee)$.

Note that both methods lead to an exponential blow-up in the size of formulas. This not accidental:

Theorem (H, Luosto, Sano, Virtema 14)

If $\varphi \in \mathcal{PL}(\varovee)$ is equivalent with $=(p_1, \ldots, p_n, q)$, then φ contains at least 2^n occurrences of $\varovee$.

On the other hand, there is also a translation in the opposite direction:

Theorem (Huuskonen, Yang 14)

Every nonempty downwards closed property of teams is definable in $\mathcal{PDL}$. Hence $\mathcal{PL}(\varovee) \leq \mathcal{PDL}$.

Proof. Consider the formula $\gamma_\Phi := \bigwedge_{p \in \Phi} {=}(p)$. It says that every $p \in \Phi$ has constant truth value, whence $X \models \gamma_\Phi$ iff $|X| \leq 1$.

Define recursively

$$\gamma_\Phi^1 := \gamma_\Phi, \quad \gamma_\Phi^{k+1} := (\gamma_\Phi^k \vee \gamma_\Phi).$$

Then we have for all k, $X \models \gamma_\Phi^k$ iff $|X| \leq k$.

$\mathcal{PDL}$ and intuitionistic disjunction

If Y is a team such that $|Y| = k+1$, we let $\chi_Y := \psi_Z \vee \gamma^k_\Phi$, where Z is the complement of Y. Now

$$\begin{aligned} X \models \chi_Y \quad &\Leftrightarrow \quad X \cap Z \neq \emptyset \text{ or } (X \cap Z = \emptyset \text{ and } |X| \leq k) \\ &\Leftrightarrow \quad Y \not\subseteq X. \end{aligned}$$

Finally, if $\mathcal{P}$ is a nonempty downwards closed property of teams, then the formula $\eta_\mathcal{P} := \bigwedge_{Y \notin \mathcal{P}} \chi_Y$ defines it. □

Yang 14 also gave a complete axiomatization for $\mathcal{PDL}$.

Expressive power in modal case

Let $\mathcal{ML}(\varovee)$ be the extension of $\mathcal{ML}$ with intuitionistic disjunction. We lift the completeness result from $\mathcal{PL}(\varovee)$ to $\mathcal{ML}(\varovee)$.

First we recall a characterization for the expressive power of $\mathcal{ML}$ with respect to Kripke-semantics.

Definition (k-equivalence, k-bisimilarity)

Let (M, w) and $(M,' w')$ be pointed Kripke models.

(a) We write $M, w \equiv_k M', w'$ if

$$M, w \models \varphi \;\Leftrightarrow\; M,' w' \models \varphi$$

for all $\varphi \in \mathcal{ML}$ with modal depth at most k.

(b) We write $M, w \rightleftarrows_k M', w'$ if (M, w) and $(M,' w')$ are k-bisimilar.

Definition (Hintikka-formulas)

Assume that Φ is a finite set of proposition symbols. Let (M, w) be a pointed Φ-model. The *k-th Hintikka-formula* $\chi^k_{M,w}$ of (M, w) is defined recursively as follows:

- $\chi^0_{M,w} := \bigwedge\{p \mid p \in \Phi, w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \Phi, w \notin V(p)\}$
- $\chi^{k+1}_{M,w} := \chi^k_{M,w} \wedge \bigwedge_{(w,v)\in R} \Diamond\chi^k_{M,v} \wedge \Box \bigvee_{(w,v)\in R} \chi^k_{M,v}$.

It is easy to see that $\mathrm{md}(\chi^k_{M,w}) = k$, and $M, w \models \chi^k_{M,w}$ for every pointed Φ-model (M, w).

The Hintikka-formula $\chi^k_{M,w}$ captures the essence of k-bisimulation:

Theorem

Let Φ be a finite set of proposition symbols, and (M, w) and $(M,' w')$ pointed Φ-models. Then the following holds:

$$M, w \equiv_k M', w' \;\Leftrightarrow\; M, w \rightleftarrows_k M', w' \;\Leftrightarrow\; M', w' \models \chi^k_{M,w}.$$

The expressive power of basic modal logic can be characterized Using bisimulation.

Theorem (van Benthem)

A class $\mathcal{K}$ of pointed Kripke models (K, w) is definable in $\mathcal{ML}$ iff $\mathcal{K}$ is closed under k-bisimulation for some $k \in \mathbb{N}$.

Expressive power of $\mathcal{ML}(\varovee)$

We lift the definition of k-bisimulation to the context of team semantics as follows:

Definition

Let (M, T) and (M', T') be models with teams. We say that (M, T) and (M', T') are *team k-bisimilar*, $M, T \, [\rightleftarrows_k] \, M', T'$ if the following conditions hold:

- D_k for every $w \in T$ there exists some $w' \in T'$ such that $M, w \rightleftarrows_k M', w'$
- R_k for every $w' \in T'$ there exists some $w \in T$ such that $M, w \rightleftarrows_k M', w'$

Definition

- A property of Kripke models with teams is a class $\mathcal{K}$ of pairs (M, T), where T is a team of M for a fixed Φ.
- $\mathcal{K}$ is downwards closed if $(M, T) \in \mathcal{K}$ and $S \subseteq T$ implies $(M, S) \in \mathcal{K}$.
- $\mathcal{K}$ has the empty team property if $(M, \emptyset) \in \mathcal{K}$ for every M.
- $\mathcal{K}$ is closed under k-bisimulation if $(M, T) \in \mathcal{K}$ and $M, T\, [\rightleftarrows_k]\, M', T'$ imply that $(M', T') \in \mathcal{K}$.

Theorem (H, Luosto, Sano, Virtema 14)

Assume that Φ is finite. A property $\mathcal{K}$ of Kripke models with teams is definable in $\mathcal{ML}(\varovee)$ iff $\mathcal{K}$ is downwards closed, closed under k-bisimulation for some k, and has the empty team property.

The proof this characterization is based on the following lemma:

Lemma

For every pair (M, T) *there is a formula* $\psi_{M,T} \in \mathcal{ML}$ *such that* $M', T' \models \psi_{M,T}$ *iff* $M, S\,[\rightleftarrows_k]\,M', T'$ *for some* $S \subseteq T$*.*

Proof.

If $T \neq \emptyset$, let $\psi_{M,T}$ be the formula $\bigvee_{w \in T} \chi^k_{M,w}$; for $T = \emptyset$, let $\psi_{M,T} := p \wedge \neg p$. □

Now it is easy to see that if $\mathcal{K}$ is downwards closed, closed under k-bisimulation and has the empty team property, then the formula

$$\bigovee_{(M,T) \in \mathcal{K}} \psi_{M,T}$$

defines it.

Expressive power $\mathcal{MDL}$

Since $\mathcal{PDL} \equiv \mathcal{PL}(\varovee)$, it is natural to ask, whether the modal counterpart of this equivalence holds.
It is not difficult to prove the first direction:

Theorem (Ebbing, H, Meier, Müller, Virtema, Vollmer 13)

$\mathcal{MDL} \leq \mathcal{ML}(\varovee)$.

However, the converse is not true:

Example

There is no formula in $\mathcal{MDL}$ that is equivalent to $\theta = \Diamond p \varovee \Box \neg p$.

Proof: $\mathcal{MDL} \equiv \mathcal{ML}$, on $M = (\{a, b\}, \{(b, b)\}, p \mapsto \{a, b\})$, whence $\mathcal{MDL}$-formulas are flat on it. However, θ is not flat on M, as $M, \{a\}, M, \{b\} \models \theta$, but $M, \{a, b\} \not\models \theta$.

Extended modal dependence logic $\mathcal{EMDL}$

What is missing from $\mathcal{MDL}$? The counterexample gives a clue: the formula $\Diamond p \varovee \Box\neg p$ says that the truth value of $\Diamond p$ is constant. That is, $\Diamond p \varovee \Box\neg p$ is equivalent to $=(\Diamond p)$.

The set of $\mathcal{EMDL}(\Phi)$-formulas is defined as follows:

$$\begin{aligned}\varphi \quad ::= \quad & p \mid \neg\, p \mid =(\alpha_1, \ldots, \alpha_n, \beta) \mid \\ & (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi,\end{aligned}$$

where $p \in \Phi$ and $\alpha_1, \ldots, \alpha_n, \beta \in \mathcal{ML}$.

The semantics of $=(\alpha_1, \ldots, \alpha_n, \beta)$ is defined in the same way as for $=(p_1, \ldots, p_n, q)$.

Remark: We do not allow nested dependence atoms!

Expressive power of $\mathcal{EMDL}$

Since $=(\Diamond p)$ is not expressible in $\mathcal{MDL}$, $\mathcal{EMDL}$ is a proper extension of $\mathcal{MDL}$. It is straightforward to prove that $\mathcal{EMDL}$ is still contained in $\mathcal{ML}(\varovee)$:

Theorem (Ebbing, H, Meier, Müller, Virtema, Vollmer 13)

$\mathcal{MDL} < \mathcal{EMDL} \leq \mathcal{ML}(\varovee)$.

Using the method of Huuskonen, we can now prove that $\mathcal{ML}(\varovee)$ is contained $\mathcal{EMDL}$:

Theorem (H, Luosto, Sano, Virtema 14)

A property $\mathcal{K}$ of Kripke models with teams is definable in $\mathcal{EMDL}$ iff $\mathcal{K}$ is downwards closed, closed under k-bisimulation for some k, and has the empty team property. Thus, $\mathcal{EMDL} \equiv \mathcal{ML}(\varovee)$.

Complexity of $\mathcal{EMDL}$

Although $\mathcal{EMDL}$ is a proper extension of $\mathcal{MDL}$, it has the same complexity:

Theorem (Ebbing, H, Meier, Müller, Virtema, Vollmer 13)

The satisfiability problem for $\mathcal{EMDL}$ is $\mathrm{NEXPTIME}$-complete.

On the other hand, $\mathcal{ML}(\varovee)$ is less complex than $\mathcal{EMDL}$:

Theorem (Lohmann, Vollmer 10)

The satisfiability problem for $\mathcal{ML}(\varovee)$ is PSPACE-complete.

This is explained by the fact that there is an exponential blow-up in translating from $\mathcal{EMDL}$ to $\mathcal{ML}(\varovee)$.

Modal inclusion logic $\mathcal{MINC}$

Modal inclusion logic is the extension of $\mathcal{ML}$ with modal inclusion atoms $\alpha_1 \ldots \alpha_n \subseteq \beta_1 \ldots \beta_n$.

Here $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are arbitrary $\mathcal{ML}$-formulas.
(We adopt the “extended” framework from the beginning.)

The semantics of modal inclusion atoms is defined as follows:

- $M, T \models \vec{\alpha} \subseteq \vec{\beta} \;\Leftrightarrow\; \forall v \in T \exists w \in T :$
$$\bigwedge_i (M, v \models \alpha_i \Leftrightarrow M, w \models \beta_i),$$

  where $\vec{\alpha} = \alpha_1 \ldots \alpha_n$ and $\vec{\beta} = \beta_1 \ldots \beta_n$.

Closure properties of $\mathcal{MINC}$

The formulas of $\mathcal{MINC}$ are not downwards closed. However, $\mathcal{MINC}$ is closed under unions and has the empty team property:

Theorem

Let $\varphi \in \mathcal{MINC}$.

(a) *If* $M, T_i \models \varphi$ *for all* $i \in I$, *then* $M, \bigcup_{i \in I} T_i \models \varphi$.

(b) $M, \emptyset \models \varphi$ *for all Kripke models* M.

Furthermore, $\mathcal{MINC}$ is closed under team bisimulation:

Theorem (H, Stumpf 15)

For every $\varphi \in \mathcal{MINC}$ *there is* k *such that* $M, T \models \varphi$ *and* $M, T\,[\rightleftarrows_k]\,M', T'$ *imply* $M', T' \models \varphi$.

Expressive power of $\mathcal{MINC}$

These three closure properties actually characterize the expressive power of $\mathcal{MINC}$:

Theorem (H, Stumpf 15)

A property $\mathcal{K}$ of Kripke models with teams is definable in $\mathcal{MINC}$ iff $\mathcal{K}$ is closed under unions, closed under k-bisimulation for some k, and has the empty team property.

Here the crucial lemma states that every team can be characterized up to k-bisimilarity and nonemptiness by using inclusion atoms:

Lemma

For every pair (M, T) there is a formula $\theta_{M,T} \in \mathcal{MINC}$ such that $M', T' \models \theta_{M,T}$ iff $M, T\,[\rightleftarrows_k]\,M', T'$ or $T' = \emptyset$.

Proof.

If $T \neq \emptyset$, let $\theta_{M,T} := \big(\bigvee_{w\in T} \chi^k_{M,w}\big) \wedge \big(\bigwedge_{u,v\in T} \chi^k_{M,u} \subseteq \chi^k_{M,v}\big)$;
if $T = \emptyset$, let $\theta_{M,T} := p \wedge \neg p$. □

If $\mathcal{K}$ is closed under unions and k-bisimulation, and has the empty team property, then it is defined by the formula

$$\bigvee_{(M,T)\in\mathcal{K}} \theta_{M,T}.$$

Complexity of $\mathcal{MINC}$

The basic complexity problems for $\mathcal{MINC}$ have been settled during the last three years.

Theorem (H, Kuusisto, Meier, Vollmer 15)

The satisfiability problem for $\mathcal{MINC}$ is $\mathrm{EXPTIME}$-complete.

Theorem (H, Kuusisto, Meier, Virtema 17)

The validity problem for $\mathcal{MINC}$ is $\mathrm{coNEXPTIME}$-complete.

Theorem (H, Kuusisto, Meier, Virtema 17)

The model checking problem for $\mathcal{MINC}$ is PTIME-complete.

Other modal logics with team semantics

Modal independence logic, $\mathcal{MIL}$: add propositional independence atoms $\vec{p} \perp_{\vec{r}} \vec{q}$ to $\mathcal{ML}$.

Kontinen, Müller, Schnoor, Vollmer 14: (1) $\mathcal{MIL}$ is closed under team bisimulation (2) the complexity of satisfiability and model checking for $\mathcal{MIL}$ are the same as for $\mathcal{MDL}$.

Modal team logic, $\mathcal{MTL}$: add strong negation $\sim$ to $\mathcal{ML}$. Here $M, T \models \sim\varphi$ iff $M, T \not\models \varphi$.

KMSV 15: a property of teams is $\mathcal{MTL}$-definable iff it is $\mathcal{FO}$-definable and closed under team bisimulation.

Lück 17: nonelementary complexity for the satisfiability problem of $\mathcal{MTL}$.

Thanks for your attention!