Fields with commuting automorphisms

An example of a non-elementary approach to model theory

Kaisa Kangas

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- How about a semantic approach instead of a syntactic one?

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- (Downward Löwenheim-Skolem) If $\mathcal{A} \in \mathcal{K}$ and $B \subseteq \mathcal{A}$, then there is some $\mathcal{A}' \in \mathcal{K}$ such that $B \subseteq \mathcal{A}' \preccurlyeq \mathcal{A}$ and $|\mathcal{A}'| = |B| + \omega$.

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(6) There is a Löwenheim-Skolem number $LS(\mathcal{K})$ such that if $\mathcal{A} \in \mathcal{K}$ and $B \subseteq \mathcal{A}$, then there is some structure $\mathcal{A}' \in \mathcal{K}$ such that $B \subseteq \mathcal{A}' \preccurlyeq \mathcal{A}$ and $|\mathcal{A}'| = |B| + LS(\mathcal{K})$.

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- If $\mathcal{K}$ is an AEC with the amalgamation property (AP) and joint embedding property (JEP), then there is a monster model $\mathbb{M} \in \mathcal{K}$ that is universal (i.e. if $\mathcal{A} \in \mathcal{K}$ and $|\mathcal{A}| < |\mathbb{M}|$, then $\mathcal{A}$ embeds into $\mathbb{M}$) and model homogeneous (i.e. any isomorphism between strong submodels of $\mathbb{M}$ extends to an automorphism of $\mathbb{M}$).

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- Then, Galois types are defined as orbits of automorphisms of the monster model:
$t^g(a/B) = t^g(b/B)$ if there is some automorphism σ of the monster so that σ fixes B pointwise and $\sigma(a) = b$.

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  - Hrushovski: Mordell-Lang, Manin-Mumford

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Vector spaces; geometry of subspaces;

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$$V = \{(x, y) \mid y^2 = x^3 - x - 1\};$$

$$W = \{(x, y, z) \mid x^2 - y^2 z^2 + z^3 = 0\}.$$

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- Cannot be studied in the first order framework.

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$$\sigma \restriction_{\mathbb{C}} = id, \qquad \sigma(t) = t + 1.$$

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- Difference algebra, a geometry of difference varieties defined by difference equations; e.g. $x + x^3 + \sigma(x) + \sigma^2(y) = 0$.

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- An application: Hrushovski's proof for Manin-Mumford

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However, they form an AEC.

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- e.g. (K, σ, τ); σ and τ always have lifts $\tilde{\sigma}$ and $\tilde{\tau}$ to K^{alg} but it might be impossible to find lifts that commute.

Example

There is a number field (i.e. finite algebraic extension of $\mathbb{Q}$) K such that $Gal(K/\mathbb{Q}) \cong Q_8$, where Q_8 is the quaternion group:

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Possible lifts of $[j]$ to $Gal(K/\mathbb{Q})$: $j, \bar{j}$.
No way these lifts commute:

$$\begin{aligned} ij &= k \neq \bar{k} = ji; \\ i\bar{j} &= \bar{k} \neq k = \bar{j}i; \\ \bar{i}j &= \bar{k} \neq k = \neq j\bar{i}; \\ \bar{i}\bar{j} &= k \neq \bar{k} = \bar{j}\bar{i}. \end{aligned}$$

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Let T be the theory of fields with commuting automorphisms.

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Let $\mathcal{A} \models T$. We say $\mathcal{A}$ is relatively algebraically closed if the following holds:

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Let $x_1, \ldots, x_n$ be tuples of variables, and let W be a set of proper subsets of $\{1, \ldots, n\}$ closed under intersection. Assume that for each $w \in W$ we are given a Galois type $p_w(x_w)$ over $E = acl_\sigma(E)$, in the variables $x_w = \{x_i \mid i \in w\}$, which can be realised by some $(a_i \mid i \in w)$ such that the elements a_i, $i \in w$, are independent over E. Assume moreover that if $v \subset w$ are in W, then $p_v(x_v) \subset p_w(x_w)$. Then,

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- Can prove using this: simple in the sense of Buechler & Lessman.

Thank you for your attention!