An extension of a theorem of Zermelo

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100th anniversary of the death of Cantor



Georg Cantor 1845 (Saint Petersburg) — 1918 (Halle).

- Second order logic has great power in characterizing categorically mathematical structures: natural numbers, real numbers, cumulative hierarchy of sets up to the first inaccessible, etc.
- Which structures are second order characterizable?
- For which A is there a second order θ such that for all B:

$$\mathcal{A} \cong \mathcal{B} \iff \mathcal{B} \models \theta$$

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Sometimes infinitary second order logic can characterize "all" models.

Theorem (Hyttinen-Kangas-V. 2013)

Let T be a countable complete first order theory and κ an uncountable cardinal with certain not too uncommon properties¹. Then the following are equivalent:

- 1. Every model of T of size κ is $L^2_{\kappa\omega}$ -characterizable.
- 2. T is superstable, shallow, without DOP or OTOP.

¹A regular cardinal such that $\kappa = \aleph_{\alpha}, \exists_{\omega_1}(|\alpha| + \omega) \leq \kappa \text{ and } 2^{\lambda} < 2^{\kappa} \text{ for all } \lambda < \kappa.$

Theorem (V. 2011)

- 1. If a model is second order characterizable, its isomorphism class is Δ_2 -definable in set theory.
- 2. A model class is second order definable² if and only if it is Δ_2 -definable in set theory.

²More exactly, second order Δ -definable.

A hierarchy of second order characterizable models



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Theorem (V. 2011)

- 1. Second order validity is Π_2 -complete in set theory.
- 2. The second order theory of a second order characterizable structure is always Δ_2 in set theory.

Corollary

Second order validity cannot be second order defined in any second order characterizable structure.

This is a critique of structuralism as I understand that view.

Definition (Zermelo 1930)

- The Second Order Zermelo-Fraenkel axioms ZFC² are as the first order one except that the Separation and Replacement Schemas are replaced by single second order axioms.
- 2. The Second Order Peano axioms P^2 are as the first order Peano axioms except that the Induction Schema is replaced by a single second order axiom.

Theorem (Zermelo 1930)

(M, E) satisfies the second order Zermelo-Fraenkel axioms ZFC^2 if and only if $(M, E) \cong (V_{\kappa}, \in)$ for some strongly inaccessible κ .

Proof.

Zermelo shows that if $(M, E) \models ZFC^2$ and κ is the smallest ordinal not "represented" in (M, E), then $(M, E) \cong (V_{\kappa}, \in)$.

Corollary

If (M, \in_1) and (M, \in_2) both satisfy the second order Zermelo-Fraenkel axioms ZFC², then $(M, \in_1) \cong (M, \in_2)$.

Theorem (Dedekind 1888)

If $(M_1, +_1, \times_1)$ and $(M_2, +_2, \times_2)$ both satisfy the second order Peano axioms P^2 , then $(M_1, +_1, \times_1) \cong (M_2, +_2, \times_2)$.

Proof.

Dedekind essentially argues that if $(M, +, \times) \models P^2$, then $(M, +, \times)$ is isomorphic with the standard model $(\mathbb{N}, +, \times)$.

- Let us consider the vocabulary {∈1, ∈2}, where both ∈1 and ∈2 are binary predicate symbols.
- ZFC(∈1) is the first order Zermelo-Fraenkel axioms of set theory when ∈1 is the membership relation and formulas are allowed to contain ∈2, too.
- ZFC(∈₂) is the first order Zermelo-Fraenkel axioms of set theory when ∈₂ is the membership relation and formulas are allowed to contain ∈₁, too.

Theorem (V. 2018) If $(M, \in_1, \in_2) \models ZFC(\in_1) \cup ZFC(\in_2)$, then $(M, \in_1) \cong (M, \in_2)$.

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- We work in $ZFC(\in_1) \cup ZFC(\in_2)$.
- We alternate between \in_1 -set theory and \in_2 -set theory.

- Let tr_i(x) be the formula ∀t ∈_i x∀w ∈_i t(w ∈_i x). It says that x is transitive in ∈_i-set theory.
- Let $\operatorname{TC}_i(x)$ be the unique u such that $\operatorname{tr}_i(u) \land x \in_i u \land \forall v((\operatorname{tr}_i(v) \land x \in_i v) \to \forall w \in_i u(w \in_i v)))$ (i.e. "u is the \in_i -transitive closure of x").
- Let φ(x, y) be the formula ∃fψ(x, y, f), where ψ(x, y, f) is the conjunction of the following formulas (where f(t) and f(w) are understood in the sense of ∈₁):

 $\psi(\mathbf{x}, \mathbf{y}, \mathbf{f})$:

- (1) In the sense of \in_1 , the set *f* is a function with $TC_1(x)$ as its domain.
- (2) $\forall t \in \operatorname{TC}_1(x)(f(t) \in \operatorname{TC}_2(y))$
- (3) $\forall t \in_2 \operatorname{TC}_2(y) \exists w \in_1 \operatorname{TC}_1(x) (t = f(w))$
- (4) $\forall t \in TC_1(x) \forall w \in TC_1(x) (t \in W \leftrightarrow f(t) \in f(w))$ (5) f(x) = y

Lemma If $\psi(x, y, f)$ and $\psi(x, y, f')$, then f = f'. Proof:



Lemma If $\psi(x, y, f)$ and $x' \in_1 x$, then $\varphi(x', f(x'))$. Proof:



Lemma If $\psi(x, y, f)$ and $y' \in_2 y$, then there is $x' \in_1 x$ such that f(x') = y' and $\varphi(x', y')$. Proof:



Lemma If $\varphi(x, y)$ and $\varphi(x, y')$, then y = y'. Proof:



Lemma If $\varphi(x, y)$ and $\varphi(x', y)$, then x = x'. Proof:



Lemma If $\varphi(x, y)$ and $\varphi(x', y')$, then $x' \in x \leftrightarrow y' \in y$.



- Let On₁(x) be the ∈₁-formula saying that x is an ordinal i.e. a transitive set of transitive sets, and similarly On₂(x).
- For On₁(α) let V¹_α be the αth level of the cumulative hierarchy in the sense of ∈₁, and similarly V²_a.

Lemma

- 1. If $\varphi(\alpha, y)$, then $On_1(\alpha)$ if and only if $On_2(y)$.
- 2. If α is a limit ordinal then so is y i.e. if $\forall u \in_1 \alpha \exists v \in_1 \alpha (u \in_1 v)$, then $\forall u \in_2 y \exists v \in_2 y (u \in_2 v)$.

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3. Also vice versa.

Lemma

Suppose $\psi(\alpha, y, f)$. If $On_1(\alpha)$ (or equivalently $On_2(y)$), then there is $\overline{f} \supseteq f$ such that $\psi(V_{\alpha}^1, V_y^2, \overline{f})$.



Lemma $\forall x \exists y \varphi(x, y) \text{ and } \forall y \exists x \varphi(x, y).$

Proof: Consider

$$\forall \alpha (\mathrm{On}_{1}(\alpha) \to \exists y \varphi(\alpha, y)) \tag{1}$$

$$\forall \mathbf{y}(\mathrm{On}_{2}(\mathbf{y}) \to \exists \alpha \varphi(\alpha, \mathbf{y})).$$
⁽²⁾

Case 1: $(1) \land (2)$. The claim can be proved.

Case 2: \neg (1) $\land \neg$ (2). Impossible!

Case 3: (1) $\land \neg$ (2). Impossible!

Case 4: \neg (1) \land (2). Impossible!



Lemma

The class defined by $\varphi(x, y)$ is an isomorphism between the \in_1 -reduct and the \in_2 -reduct.

Proof.

By the previous Lemmas.

The Theorem is proved.

Theorem

If $(M, +_1, \times_1, +_2, \times_2) \models P(+_1, \times_1) \cup P(+_2, \times_2)$, then $(M, +_1, \times_1) \cong (M, +_2, \times_2)$.

Proof.

(Sketch) Let 0_1 and 1_1 be the first elements of $(M, +_1, \times_1)$, and respectively 0_2 , 1_2 . Let $\psi(x, u, v)$ say that x codes, using $+_1$ and \times_1 , an initial segment I with the last element u, of $(M, +_1, \times_1)$, an initial segment I' with the last element v, of $(M, +_2, \times_2)$, and a function $f : I \to I'$ such that $f(0_1) = 0_2$, $f(y +_1 1_1) = f(y) +_2 1_2$ for all $y \in I \setminus \{u\}$, and f(u) = v. Let $\varphi(u, v)$ be the formula $\exists x \psi(x, u, v)$. This formula defines the desired isomorphism.

- D. Martin 2018 (unpublished) argues informally that if (*M*₁, ∈₁) and (*M*₂, ∈₂) both satisfy the axioms of set theory with informal Full Comprehension and their classes of ordinals are isomorphic, then (*M*, ∈₁) ≅ (*M*, ∈₂).
- Our result is a formalization of Martin's informal result. We need not assume that the ordinals are isomorphic.

- Early researchers (Dedekind, Frege, Russell, Hilbert, Zermelo, Gödel, Mostowski) axiomatized mathematics using second order logic or its extension simple theory of types.
- Then ZFC emerged as a first order theory.
- Later philosophers (e.g. S. Shapiro) claimed second order logic would be better (can characterize mathematical structures) and first order logic is flawed (cannot characterize mathematical structures).
- I suggest: Zermelo's and Dedekind's second order categoricity results are actually first order at heart.
- The difference between second order logic or first order set theory is not as clear as what was previously thought.

- Second order logic is praised for its categoricity results, i.e. its ability to characterize structures.
- But what is universal second order truth a problem!
- Best understood in terms of provability i.e. truth in all Henkin (rather than "full") models.
- But Henkin models seem to ruin the categoricity results.
- However, categoricity can be proved for Henkin models, too, in the form of internal categoricity, which implies full categoricity in full models.

- Note that (M, ∈₁) and (M, ∈₂) above can be models of V = L, V ≠ L, CH, ¬CH, even of ¬Con(ZF).
- It is easy to construct such pairs of models using classical methods of Gödel and Cohen.
- Not all of them can be models of second order set theory.

Continuum Hypothesis (CH)

- What if $(M, \in_1) \models CH$ and $(M, \in_2) \models \neg CH$?
- Then either (*M*, ∈₁) or (*M*, ∈₂) does not satisfy the Separation Schema or the Replacement Schema if formulas are allowed to mention the other membership-relation.

- An internal categoricity result.
- A strong robustness result for set theory.
- The model cannot be changed "internally".
- To get non-isomorphic models one has to go "outside" the model.
- But going "outside" raises the potential of an infinite regress of meta theories.

- Should we think of second order logic or first order set theory as the foundation of classical mathematics?
- The answer: We need a new understanding of the difference between the two. The difference is not as clear as what was previously thought.
- The nice categoricity results of second order logic can be seen already on the first order level, revealing their inherent limitations.

Thank you!