Basis problem for regular spaces and compact convex sets

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October 31, 2018

Metric spaces

A metric d on a set X is a function $d : X \times X \to [0, \infty)$ such that

- $d(x, y) = 0$ iff $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

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Every metric space is T_6: $F = \bigcap\{B(F, 2^{-n}) : n \in \mathbb{N}\}$.

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Example

Alexandrov double arrow space $[0, 1] \times 0, 1$ is compact T_6, not metrizable and contains $\mathbb{S}$.

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Is it true that a compact T_6 space is metrizable iff it contains no Sorgenfrey subsets?

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Fact (PFA)

If an uncountable T_3 space X is a continuous image of a separable metric space, then X contains an uncountable subset of $\mathbb{R}$.

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Question (PFA)

Is it true that every uncountable T_3 space contains an uncountable subspace of $\mathbb{R}$, $\mathbb{S}$, or $\mathbb{D}$?

Topological basis problem

For regular uncountable spaces, is there a finite collection $\mathcal{B}$ such that every other regular uncountable space X contains a subspace homeomorphic to one space from $\mathcal{B}$?

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To answer this question we are willing to use standard forcing axioms (MA, PFA,...), and/or restrict ourselves to some appropriate subclass of well-behaved spaces.

The real line and the Sorgenfrey line

Theorem (Baumgartner 1973)

PFA implies that every set of reals of cardinality $\aleph_1$ *embeds homomorphically into any uncountable separable metric space and that*

every subset of the Sorgenfrey line $(\mathbb{R}, \to)$ *of cardinality* $\aleph_1$ *embeds homomorphically into any uncountable subspace of* $(\mathbb{R}, \to)$*.*

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every subset of the Sorgenfrey line $(\mathbb{R}, \rightarrow)$ of cardinality $\aleph_1$ embeds homomorphically into any uncountable subspace of $(\mathbb{R}, \rightarrow)$.

Note that even in the class of first countable spaces the list $\mathcal{B}$ must have at least three elements.

HS and HL

Hereditary Lindelöfness and hereditary separability play important roles in the basis problem.

A regular space is Lindelöf if every open cover has a countable subcover.

An S space is a regular hereditarily separable (HS) space which is not Lindelöf.

An L space is a regular hereditarily Lindelöf (HL) space which is not separable.

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Fact

- *HL implies* T_6*.*
- *For compact/Lindelöf spaces,* T_6 *implies HL.*

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Theorem (Moore, 2005)

There is an L space.

L groups

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Or restrict ourselves to the class of first countable spaces.

Theorem (Szentmiklossy, 1980)

PFA implies that there are no first countable L spaces.

One restriction

A topological space X is cometrizable if it has a weaker metrizable topology and a neighbourhood assignment consisting of closed sets in this weaker topology.

Example: The Sorgenfrey line is a cometrizable space.

Theorem (Gruenhage 1987)

Assume PFA. A cometrizable space is a continuous image of a separable metric space if it contains no $\mathbb{S}$ *or* $\mathbb{D}$*.*

Inner topology

For a topological space (X, τ) and a collection $\mathcal{C} \subset P(X)$, the inner topology $(X, \tau^{I,\mathcal{C}})$ induced by $\mathcal{C}$ is the topology with base $\{\{x\} \cup O^{I,\mathcal{C}} : x \in O, O \text{ is open}\}$ where $O^{I,\mathcal{C}} = \bigcup\{C \in \mathcal{C} : C \subset O\}$.

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X has HL inner topology for some countable $\mathcal{C}$ if for any open set O, $O \setminus \{C \in \mathcal{C} : C \subset O\}$ is at most countable.

Theorem (P-Todorcevic)

Assume PFA. If (X, τ) is regular and $(X, \tau^{I,\mathcal{C}})$ is HL for some countable $\mathcal{C}$, then (X, τ) either is a continuous image of a separable metric space or contains an uncountable Sorgenfrey subset.

Apply to compact convex sets

Suppose X is a T_6 compact convex set.

Proposition (PFA)

X can be embedded into $[0,1]^{\omega_1}$. In particular, $w(X) \leq \omega_1$ and there is an increasing union $X = \bigcup_{\alpha<\omega_1} X_\alpha$ such that each X_α is metrizable and G_δ.

Applications to other problems

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A similar question in perfect normal spaces has drawn people's attention for a long time.

Question (Fremlin)

Is it consistent that every perfectly normal compact space admits a 2-to-1 continuous map to a metric space?

Connection with perfectly normal compact spaces

Gruenhage also pointed out that consistency of basis problem should provide consistency to the following:

Question

Is it consistent that every T_6 locally connected compact space is metrizable?

Question

If X and Y are compact and $X \times Y$ is T_6, must one of X and Y be metrizable?

First countability

Theorem (P-Todorcevic)

Assume PFA. Let X be a first countable HL space. Then either,

1. *for any countable $\mathcal{C}$, the space $(X, \tau^{I,\mathcal{C}})$ is σ-discrete or*
2. *for some countable $\mathcal{C}$ and uncountable $Y \subset X$ the space $(Y, \tau^{I,\mathcal{C}})$ is HL.*

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Theorem (P-Todorcevic)

Assume PFA. If X is a first countable space with HL inner topology and has size $\aleph_1$, then there is a partition $X = \bigcup_{n<\omega} X_n$ such that each X_n is either metrizable or Sorgenfrey.

Outer "topology"

For a topological space (X, τ) and a collection $\mathcal{C} \subset P(X)$, the outer "topology" $(X, \tau^{O,\mathcal{C}})$ induced by $\mathcal{C}$ is the collection $\{O^{O,\mathcal{C}} : O$ is open$\}$ where $O^{O,\mathcal{C}} = \bigcap\{C \in \mathcal{C} : O \subset C\}$.

Proposition (PFA)

Suppose X is a regular, HL space. Any outer topology induced by a countable collection either is a continuous image of a separable metric space or contains $\mathbb{S}$.

If the outer topology guesses almost correctly, then the original topology will either is a continuous image of a separable metric space or contains $\mathbb{S}$.

Example. Cometrizable spaces.

Thank you!