UNIVERSITY OF HELSINKI

Analyzing Arrow's Theorem Through Dependence and Independence Logic

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Partly based on joint work with Eric Pacuit (Maryland)

Outline

1. Arrow's Impossiblity Theorem
2. Formalizing Arrow's Theorem in dependence and independence logic
3. Arrow's Theorem as a dependency strengthening theorem

Preference Aggregation

- A set of alternatives: $A = \{a, b, c, d, \dots\}$
- A finite set of voters: $\{v_1, \dots, v_n\}$
- A ranking $R \subseteq A \times A$ is a transitive and complete relation on A.
- A linear ranking is a ranking that is a linear relation.
  Denote by $L(A)$ the set of all linear rankings of A.

year	voter 1	voter 2	$\cdots$	voter n	group decision
2000	abc	cab	$\cdots$	acb	?
2001	bac	cba	$\cdots$	cba	?
2002	cba	bca	$\cdots$	acb	?
2003	cab	cba	$\cdots$	acb	?

Condorcet Paradox (18th century)

Problem with the Majority Rule:

voter 1	voter 2	voter 3	group decision
a	c	b	
b	a	c	?
c	b	a	

- Does the group prefer a over b? Yes.
- Does the group prefer b over c? Yes.
- Does the group prefer a over c? No.

$$a > b > c > a$$

Condorcet Paradox (with arbitrary rankings)

Problem with the Majority Rule:

voter 1	voter 2	voter 3	voter 4	voter 5	group decision
a b∼c	a b c	c a b	b c a	b c a	?

- Does the group prefer a strictly over b? Yes.
- Does the group prefer b strictly over c? Yes.
- Does the group prefer a strictly over c? No.

$$a > b > c > a$$

Condorcet Paradox (with arbitrary rankings)

Problem with the Majority Rule:

voter 1	voter 2	voter 3	voter 4	voter 5	group decision
a b$\sim$c	a b c	c a b	b c a	b c a	?

- Does the group prefer a strictly over b? Yes.
- Does the group prefer b strictly over c? Yes.
- Does the group prefer a strictly over c? No.

$$a > b > c > a$$

May's Theorem (1952): When $|A| \leq 2$, the majority rule is the only "fair" aggregation rule.

Desiderata

1. The voters' votes should *completely determine* the group decision.
2. The voters' votes are *not constrained* in any way.
3. The group decision should depend *in the right way* on the voters' votes.

→ Dictatorship

Theorem (Arrow 1963)

If $|A| \geq 3$, then any preference aggregation rule $F : D \to L(A)$ ($D \subseteq L(A)^n$) satisfying Universal Domain, Independence of Irrelevant Alternatives and Unanimity is a Dictatorship.

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Nobel Prize in Economics, 1972
Kenneth Arrow

Nobel Prize in Economics, 1998
Amartya Kumar Sen
"Liberal paradox"

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voter 1	voter 2	$\cdots$	voter n	group decision
abc	cab	$\cdots$	acb	abc
bac	cba	$\cdots$	cba	cba
cba	bca	$\cdots$	abc	bca
cab	cba	$\cdots$	acb	cab

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a profile **R**

v_1	v_2	$\cdots$	v_n	u
abc	cab	$\cdots$	acb	abc
cab	bca	$\cdots$	cba	cba
cba	cab	$\cdots$	abc	bca
cab	bca	...	acb	cab

$= F(\mathbf{R})$

Theorem (Arrow 1963)

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$$dom(F) = L(A)^n$$

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v_1	v_2	v_3	u
abc	*bac*	*bca*	*cba*
cab	*bca*	*bac*	*bca*
abc	*bac*	*bca*	*bac*

Theorem (Arrow 1963)

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v_1	v_2	v_3	u
abc	*cab*	*acb*	*abc*
bca	*abc*	*bac*	*abc*
cab	*acb*	*abc*	*cab*

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v_1	$\cdots$	v_i	$\cdots$	v_n	u
abc	$\cdots$	acb	$\cdots$	acb	acb
bca	$\cdots$	abc	$\cdots$	bac	abc
cba	$\cdots$	cab	$\cdots$	bca	cab

Theorem (Arrow 1963)

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Goal of the talk:

Formalize & analyze Arrow's Theorem in

Dependence and Independence Logic

Previous formalizations :

- in propositional logic: [Tang, Lin 2009]
- in first-order logic: [Grandi, Endriss 2013]
- in modal logic: [Ågotnes, van der Hoek, Wooldridge 2011], [Cinà, Endriss 2015]
- in higher order logic: [Nipkow 2009], [Wiedijk 2007]

Formalizing Arrow's Theorem in dependence and independence logic

Logics for expressing dependencies

$$\forall u \, \exists v \, \forall x \, \exists y \, \phi$$

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Independence-Friendly Logic (Hintikka and Sandu, 1989):

$$\forall u \exists v \forall x \exists y / \{u\} \phi$$

Thm. (Enderton, Walkoe, Hintikka)
FO+Henkin quantifiers $\equiv$ IF-logic$\equiv \Sigma_1^1$

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Dependence Logic (Väänänen 2007):

$$\forall u \exists v \forall x \exists y (=(x; y) \wedge \phi)$$

$$\exists f$$

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Independence Logic (Grädel, Väänänen 2013):

$$\forall u \exists v \forall x \exists y (u \perp y \land \phi)$$

Thm. (Enderton, Walkoe, Hintikka, Väänänen, Grädel)
FO+Henkin quantifiers $\equiv$ IF-logic $\equiv$ (In)dependence logic$\equiv \Sigma^1_1 \equiv_{finite}$ NP

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Independence Logic (Grädel, Väänänen 2013):

$$\forall u \exists v \forall x \exists y (u \perp y \wedge \phi)$$

Inclusion Logic (Galliani 2012):

$$\forall u \exists x (u \subseteq x \wedge \phi)$$

Thm. (Enderton, Walkoe, Hintikka, Väänänen, Grädel)
FO+Henkin quantifiers $\equiv$ IF-logic $\equiv$ (In)dependence logic $\equiv \Sigma^1_1 \equiv_{finite}$ NP
Thm. (Galliani and Hella 2013) Inclusion logic $\equiv$ GFP$^+$ $\equiv_{finite,<}$ PTIME

Functional dependency expressed in team semantics

Team semantics (Hodges 1997)

	v_1	v_2	v_3	u
s_1	bac	cab	abc	abc
s_2	bac	cab	abc	abc
s_3	cab	bca	acb	cba
s_4	cab	bca	acb	cba

$M \models_{s_2} =\!(\vec{v}; u)$?

Functional dependency expressed in team semantics

Team semantics (Hodges 1997)

a team:
a set T of assignments

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s_1	bac	cab	abc	abc
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s_3	cab	bca	acb	cba
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$M \models_T \; =(\vec{v}; u)$ iff for any $s, s' \in T$,

$$s(\vec{v}) = s'(\vec{v}) \implies s(u) = s'(u).$$

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The functionality of aggregation rule:

$$\theta_F := \; =(v_1, \ldots, v_n; u)$$

Define $\Gamma_{DM} =$ "$dom(M) = L(A)$."

The signature $\mathscr{L}_A$ consists of unary function symbols P_{ab} for each $(a, b) \in A \times A$.

- for all $> \in L(A)$, $\quad P^M_{ab}(>) = 1 \;$ iff $\; a > b$; $\quad P^M_{ab}(>) = 0 \;$ iff $\; a \not> b$.
- Define $\Gamma_{Ord} := \{\forall x\big((P_{ab}(x)=1 \wedge P_{bc}(x) = 1) \rightarrow P_{ac}(x)=1\big) \mid a, b, c \in A\} \cup \ldots$

IIA: The group decision on the relative preference between two alternatives *a*, *b* depends only on how the individual voters rank these two alternatives. It is independent of their rankings with respect to other alternatives.

	v_1	v_2	v_3	u
s_1	*abc*	*bac*	*bca*	*cba*
s_2	*cab*	*bca*	*bac*	*bca*
s_3	*abc*	*bac*	*bca*	*bac*

$$=\big(P_{ab}(v_1),\dots,P_{ab}(v_n);P_{ab}(u)\big)$$

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Independence of irrelevant alternatives (IIA)

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	v_1	v_2	v_3	u
s_1	abc	bac	bca	cba
s_2	cab	bca	bac	bca
s_3	abc	bac	bca	bac

$$=\big(P_{ab}(v_1),\ldots,P_{ab}(v_n);P_{ab}(u)\big)$$

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s_1	abc	bac	bca	cba
s_2	cab	bca	bac	bca
s_3	abc	bac	bca	bac

$$- \theta_{IIA} := \bigwedge\{ =\big(P_{ab}(v_1), \dots, P_{ab}(v_n); P_{ab}(u)\big) \mid a, b \in A\}$$

Independence

Universal domain: $dom(F) = L(A)^n$

	v_1	v_2	v_3	u
s_1	*abc*	*bac*	*cab*	*bca*
s_2	*acb*	*acb*	*bac*	*cba*
s_3	*acb*	*bac*	*cab*	*acb*
s_4	*abc*	*acb*	*bac*	*abc*
$\vdots$				

- $M \models_T$ iff for all $s, s' \in T$, there exists $s'' \in T$ s.t.

$$s''() = s() \text{ and } s''() = s'().$$

Universal domain: $dom(F) = L(A)^n$

	v_1	v_2	v_3	u
s_1	*abc*	*bac*	*cab*	*bca*
s_2	*acb*	*acb*	*bac*	*cba*
s_3	*acb*	*bac*	*cab*	*acb*
s_4	*abc*	*acb*	*bac*	*abc*
$\vdots$				

- $M \models_T v_1 \perp v_2 v_3$ iff for all $s, s' \in T$, there exists $s'' \in T$ s.t.

$$s''(v_1) = s(v_1) \text{ and } s''(v_2 v_3) = s'(v_2 v_3).$$

Independence: $\theta_I := \bigwedge \{v_i \perp \langle v_j \rangle_{j \neq i} \mid 1 \leq i \leq n\}$

Universal domain: $dom(F) = L(A)^n$

		v_1	v_2	v_3	u
s	s_1	*abc*	*bac*	*cab*	*bca*
s'	s_2	*acb*	*acb*	*bac*	*cba*
	s_3	*acb*	*bac*	*cab*	*acb*
	s_4	*abc*	*acb*	*bac*	*abc*
	$\vdots$				

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s'	s_2	*acb*	*acb*	*bac*	*cba*
	s_3	*acb*	*bac*	*cab*	*acb*
s''	s_4	*abc*	*acb*	*bac*	*abc*
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		v_1	v_2	v_3	u
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	s_3	*acb*	*bac*	*cab*	*acb*
s''	s_4	*abc*	*acb*	*bac*	*abc*
	$\vdots$				

- $M \models_T \vec{t} \perp \vec{t'}$ iff for all $s, s' \in T$, there exists $s'' \in T$ s.t.

$$s''(\vec{t}) = s(\vec{t}) \text{ and } s''(\vec{t'}) = s'(\vec{t'}).$$

Independence: $\theta_I := \bigwedge\{v_i \perp \langle v_j \rangle_{j \neq i} \mid 1 \leq i \leq n\}$

Inclusion

Universal domain: $dom(F) = L(A)^n$.

	x	v_1	v_2	v_3	u
s_1	*abc*	*acb*	*cab*	*bac*	*bca*
s_2	*acb*	*bac*	*bac*	*abc*	*cba*
s_3	*bac*	*abc*	*cab*	*acb*	*acb*
s_4	*bca*	*cba*	*acb*	*abc*	*abc*
s_5	*cab*	*bca*	*acb*	*abc*	*abc*
s_6	*cba*	*bac*	*acb*	*bac*	*bac*

$\vdots$

- $M \models_T x \subseteq v_i$ iff for all $s \in T$, there exists $s' \in T$ such that $s(x) = s'(v_i)$.

All ranking: $\theta_{AR} := \quad \forall x(x \subseteq v_i)$

(Recall: $dom(M) = L(A)$.)

Inclusion

Universal domain: $dom(F) = L(A)^n$.

	x	v_1	v_2	v_3	u
s_1	*abc*	*acb*	*cab*	*bac*	*bca*
s_2	*acb*	*bac*	*bac*	*abc*	*cba*
s_3	*bac*	*abc*	*cab*	*acb*	*acb*
s_4	*bca*	*cba*	*acb*	*abc*	*abc*
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$\vdots$

- $M \models_T x \subseteq v_i$ iff for all $s \in T$, there exists $s' \in T$ such that $s(x) = s'(v_i)$.

All ranking: $\theta_{AR} := \bigwedge\{\forall x(x \subseteq v_i) : 1 \leq i \leq n\}$

(Recall: $dom(M) = L(A)$.)

Universal domain: $dom(F) = L(A)^n$.

	x	v_1	v_2	v_3	u
s_1	*abc*	*acb*	*cab*	*bac*	*bca*
s_2	*acb*	*bac*	*bac*	*abc*	*cba*
s_3	*bac*	*abc*	*cab*	*acb*	*acb*
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(Recall: $dom(M) = L(A)$.)

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s_3	*bac*	*abc*	*cab*	*acb*	*acb*
s_4	*bca*	*cba*	*acb*	*abc*	*abc*
s_5	*cab*	*bca*	*acb*	*abc*	*abc*
s_6	*cba*	*bac*	*acb*	*bac*	*bac*

$\vdots$

- $M \models_T x \subseteq v_i$ iff for all $s \in T$, there exists $s' \in T$ such that $s(x) = s'(v_i)$.

All ranking: $\theta_{AR} := \bigwedge\{\forall x(x \subseteq v_i) : 1 \leq i \leq n\}$

(Recall: $dom(M) = L(A)$.)

Universal domain: $dom(F) = L(A)^n$.

	x	v_1	v_2	v_3	u
s_1	*abc*	*acb*	*cab*	*bac*	*bca*
s_2	*acb*	*bac*	*bac*	*abc*	*cba*
s_3	*bac*	*abc*	*cab*	*acb*	*acb*
s_4	*bca*	*cba*	*acb*	*abc*	*abc*
s_5	*cab*	*bca*	*acb*	*abc*	*abc*
s_6	*cba*	*bac*	*acb*	*bac*	*bac*

$\vdots$

- $M \models_T \vec{t} \subseteq \vec{t'}$ iff for all $s \in T$, there exists $s' \in T$ such that $s(\vec{t}) = s'(\vec{t'})$.

All ranking: $\theta_{AR} := \bigwedge\{\forall x(x \subseteq v_i) : 1 \leq i \leq n\}$

(Recall: $dom(M) = L(A)$.)

Independence Logic $=$ first-order logic $+ =(\bar{t}; \bar{t}') + \bar{t} \perp \bar{t}' + \bar{t} \subseteq \bar{t}'$

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v_1	v_2	v_3	u
abc	*cab*	*acb*	*abc*
bca	*abc*	*bac*	*abc*
cab	*acb*	*abc*	*cab*

- Unanimity:

$$\big(P_{ab}(v_1) = 1 \wedge \cdots \wedge P_{ab}(v_n) = 1\big) \rightarrow P_{ab}(u) = 1$$

Independence Logic = first-order logic + $=(\bar{t};\bar{t}') + \bar{t} \perp \bar{t}' + \bar{t} \subseteq \bar{t}'$

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v_1	v_2	v_3	u
abc	*cab*	*acb*	*abc*
bca	*abc*	*bac*	*abc*
cab	*acb*	*abc*	*cab*

- Unanimity:

$$\theta_{UN} := \bigwedge_{a,b \in A} \Big(\big(P_{ab}(v_1) = 1 \wedge \cdots \wedge P_{ab}(v_n) = 1\big) \rightarrow P_{ab}(u) = 1 \Big)$$

v_1	v_2	v_3	u
abc	cab	acb	abc
bca	abc	bac	abc
cab	acb	abc	cab

- Unanimity:

$$\theta_{UN} := \bigwedge_{a,b \in A} \Big(\big(P_{ab}(v_1) = 1 \wedge \cdots \wedge P_{ab}(v_n) = 1 \big) \rightarrow P_{ab}(u) = 1 \Big)$$

- The set $\{v_{i_1}, \ldots, v_{i_k}\}$ is decisive:

$$\delta(v_{i_1}, \ldots, v_{i_k}) := \bigwedge_{a,b \in A} \Big(\big(P_{ab}(v_{i_1}) = 1 \wedge \cdots \wedge P_{ab}(v_{i_k}) = 1 \big) \rightarrow P_{ab}(u) = 1 \Big)$$

- $\theta_{UN} = \delta(v_1, \ldots, v_n)$

- Dictatorship: $\theta_D := \delta(v_i)$

Decisive sets and dictatorship

v_1	v_2	v_3	u
cba	cab	acb	abc
bca	abc	bac	abc
cab	acb	abc	cab

- Unanimity:

$$\theta_{UN} := \bigwedge_{a,b \in A} \Big(\big(P_{ab}(v_1) = 1 \wedge \cdots \wedge P_{ab}(v_n) = 1 \big) \rightarrow P_{ab}(u) = 1 \Big)$$

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v_1	v_2	v_3	u
cba	cab	acb	abc
bca	abc	bac	abc
cab	acb	abc	cab

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- $\theta_{UN} = \delta(v_1, \ldots, v_n)$
- Dictatorship: $\theta_D := \delta(v_i)$

Decisive sets and dictatorship

v_1	$\cdots$	v_i	$\cdots$	v_n	u
abc	$\cdots$	acb	$\cdots$	acb	acb
bca	$\cdots$	bac	$\cdots$	bac	bac
cba	$\cdots$	cba	$\cdots$	bca	cba

- Unanimity:

$$\theta_{UN} := \bigwedge_{a,b \in A} \Big(\big(P_{ab}(v_1) = 1 \wedge \cdots \wedge P_{ab}(v_n) = 1\big) \to P_{ab}(u) = 1 \Big)$$

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- $\theta_{UN} = \delta(v_1, \ldots, v_n)$
- Dictatorship: $\theta_D := \bigvee_{i=1}^{n} \delta(v_i)$

v_1	$\cdots$	v_i	$\cdots$	v_n	u
abc	$\cdots$	*acb*	$\cdots$	*acb*	*acb*
bca	$\cdots$	*bac*	$\cdots$	*bac*	*bac*
cba	$\cdots$	*cba*	$\cdots$	*bca*	*cba*

- Unanimity:
  $$\theta_{UN} := \bigwedge_{a,b \in A} \Big(\big(P_{ab}(v_1) = 1 \wedge \cdots \wedge P_{ab}(v_n) = 1\big) \rightarrow P_{ab}(u) = 1 \Big)$$
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v_1	$\cdots$	v_i	$\cdots$	v_n	u
abc	$\cdots$	*acb*	$\cdots$	*acb*	*acb*
bca	$\cdots$	*bac*	$\cdots$	*bac*	*bac*
cba	$\cdots$	*cba*	$\cdots$	*bca*	*cba*

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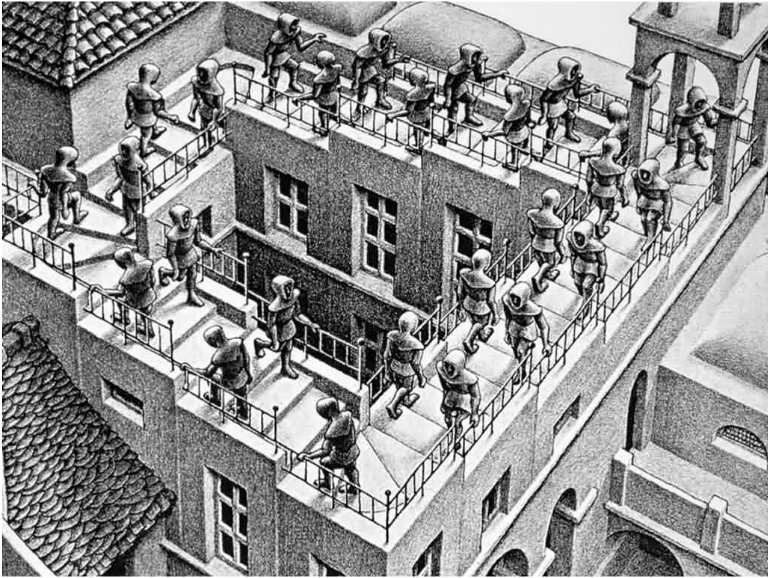
- $\theta_{UN} = \delta(v_1, \ldots, v_n)$
- Dictatorship: $\theta_D := \bigvee_{i=1}^{n} \delta(v_i)$

Theorem (Arrow 1963)

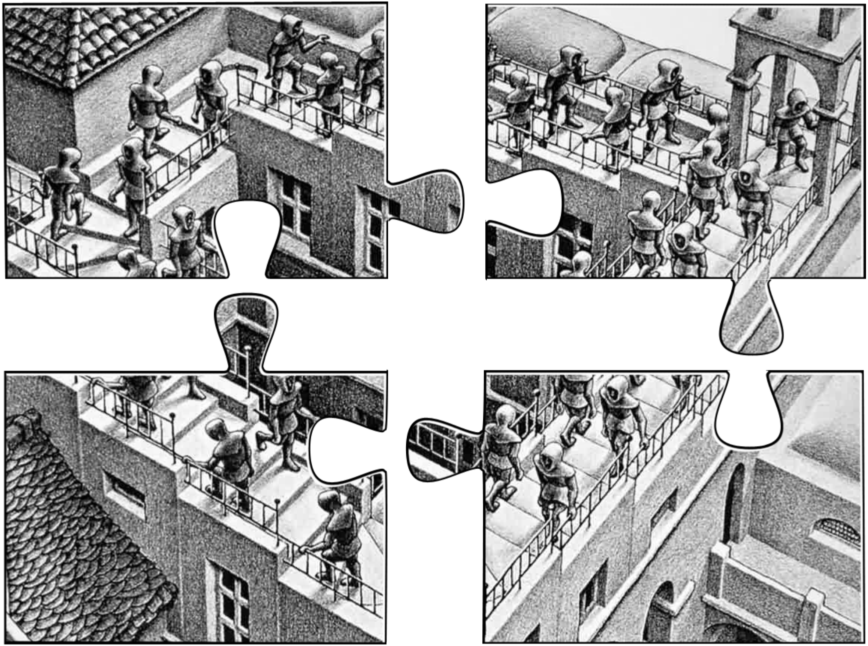
If $|A| \geq 3$, then any preference aggregation rule $F : D \to L(A)$ ($D \subseteq L(A)^n$) satisfying Universal Domain, Independence of Irrelevant Alternatives and Unanimity is a Dictatorship.

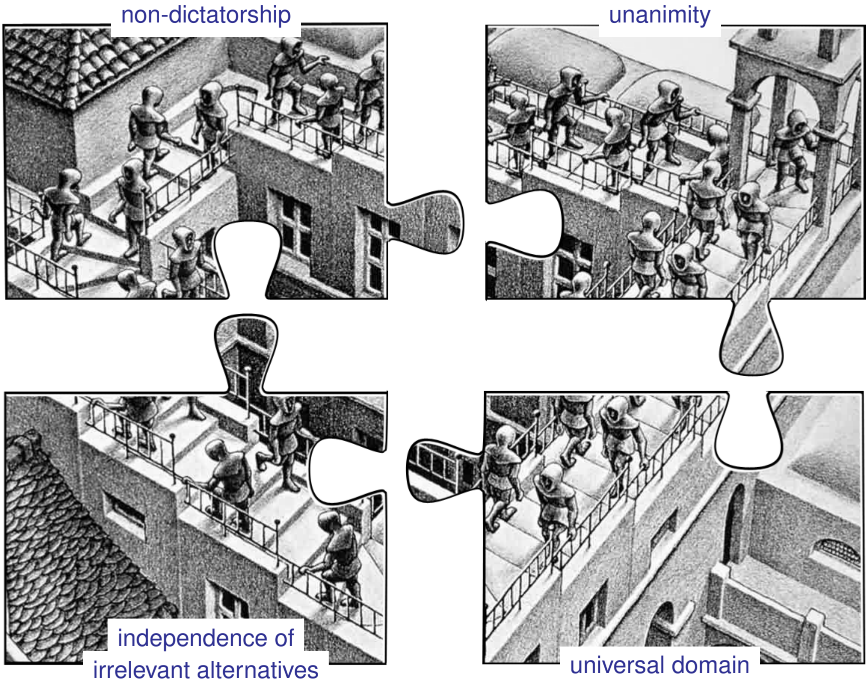
Let $\Gamma_{Arrow} = \Gamma_{Ord} \cup \{\theta_F, \theta_I, \theta_{AR}, \theta_{IIA}, \theta_{UN}\}$.

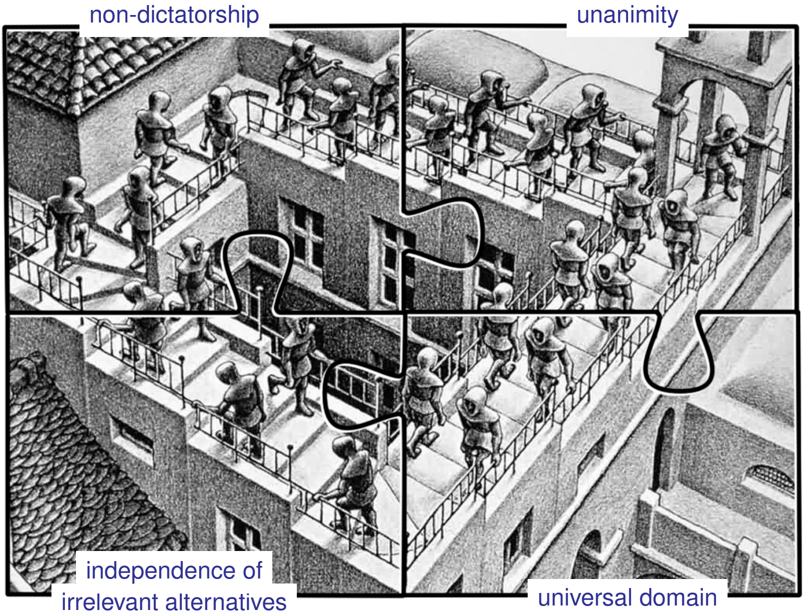
- $\Gamma_{Arrow} \models \theta_D$
- or $\Gamma_{Arrow}, \sim \theta_D \models \bot$



M. C. Escher, 1960




non-dictatorship
unanimity
independence of irrelevant alternatives
universal domain



Local consistency vs. global inconsistency

Let $\Gamma_{Arrow} = \{\Gamma_{Ord}, \theta_F, \theta_{AR}, \theta_I, \theta_{IIA}, \theta_U\}$.

- $\Gamma_{Arrow} \models \theta_D$
- or $\Gamma_{Arrow}, \sim \theta_D \models \bot$

$\Longrightarrow \Gamma \vdash \theta_D$ (by Completeness Theorem, and by (Pacuit, Y. 2017))

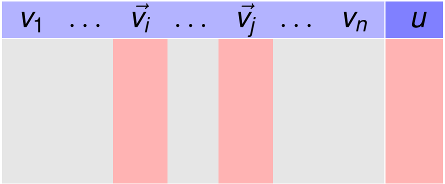
Theorem ((Kontinen, Väänänen, 2012), (Hannula 2015), (Y. 2016))

For any set $\Gamma \cup \{\theta\}$ of formulas of independence logic with θ (essentially) first-order,

$$\Gamma \vdash \theta \iff \Gamma \models \theta.$$

Lemma. If $\{v_{i_1}, \ldots, v_{i_k}\} \cap \{v_{j_1}, \ldots, v_{j_m}\} = \emptyset$, then

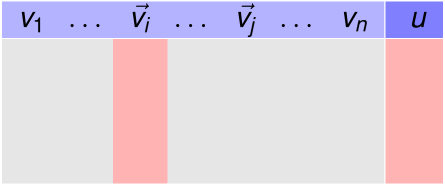
$$\Gamma_{Arrow}, \delta(v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_m}) \vdash \delta(v_{i_1}, \ldots, v_{i_k}) \vee \delta(v_{j_1}, \ldots, v_{j_m})$$



- $\delta(v_{i_1}, \ldots, v_{i_k}) \rightsquigarrow =(v_{i_1}, \ldots, v_{i_k}; u)$

Lemma. If $\{v_{i_1}, \ldots, v_{i_k}\} \cap \{v_{j_1}, \ldots, v_{j_m}\} = \emptyset$, then

$$\Gamma_{Arrow}, \delta(v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_m}) \vdash \delta(v_{i_1}, \ldots, v_{i_k}) \vee \delta(v_{j_1}, \ldots, v_{j_m})$$



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Lemma. If $\{v_{i_1}, \ldots, v_{i_k}\} \cap \{v_{j_1}, \ldots, v_{j_m}\} = \emptyset$, then

$$\Gamma_{Arrow}, \delta(v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_m}) \vdash \delta(v_{i_1}, \ldots, v_{i_k}) \vee \delta(v_{j_1}, \ldots, v_{j_m})$$

Therefore,

$$\begin{aligned} \Gamma_{Arrow}, \delta(v_1, \ldots, v_n) &\vdash \delta(v_1) \vee \delta(v_2, \ldots, v_n) \\ &\vdash \delta(v_1) \vee \delta(v_2) \vee \delta(v_3, \ldots, v_n) \\ &\vdots \\ &\vdash \delta(v_1) \vee \delta(v_2) \vee \cdots \vee \delta(v_n). \end{aligned}$$

- $\delta(v_{i_1}, \ldots, v_{i_k}) \quad \rightsquigarrow \quad =(v_{i_1}, \ldots, v_{i_k}; u)$

Proof idea

Lemma. If $\{v_{i_1}, \ldots, v_{i_k}\} \cap \{v_{j_1}, \ldots, v_{j_m}\} = \emptyset$, then

$$\Gamma_{Arrow}, \delta(v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_m}) \vdash \delta(v_{i_1}, \ldots, v_{i_k}) \vee \delta(v_{j_1}, \ldots, v_{j_m})$$

Therefore,

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- $\delta(v_{i_1}, \ldots, v_{i_k}) \quad \rightsquigarrow \quad {=}(v_{i_1}, \ldots, v_{i_k}; u)$

$$\delta(v_{i_1}, \ldots, v_{i_k}) = \bigwedge_{a,b \in A} \Big((P_{ab}(v_{i_1}) = 1 \wedge \cdots \wedge P_{ab}(v_{i_k}) = 1) \to P_{ab}(u) = 1 \Big)$$

Lemma. If $\{v_{i_1}, \ldots, v_{i_k}\} \cap \{v_{j_1}, \ldots, v_{j_m}\} = \emptyset$, then

$$\Gamma_{Arrow}, =\!(v_{i_1}, \ldots, v_{i_k}, v_{j_1}, \ldots, v_{j_m}; u) \vdash =\!(v_{i_1}, \ldots, v_{i_k}; u) \vee =\!(v_{j_1}, \ldots, v_{j_m}; u)$$

Therefore,

$$\begin{aligned} \Gamma_{Arrow}, =\!(v_1, \ldots, v_n; u) &\vdash =\!(v_1; u) \vee =\!(v_2, \ldots, v_n; u) \\ &\vdash =\!(v_1; u) \vee =\!(v_2; u) \vee =\!(v_3, \ldots, v_n; u) \\ &\vdots \\ &\vdash =\!(v_1; u) \vee =\!(v_2; u) \vee \cdots \vee =\!(v_n; u). \end{aligned}$$

- $\delta(v_{i_1}, \ldots, v_{i_k}) \quad \rightsquigarrow \quad =\!(v_{i_1}, \ldots, v_{i_k}; u)$

$$\delta(v_{i_1}, \ldots, v_{i_k}) = \bigwedge_{a,b \in A} \Big((P_{ab}(v_{i_1}) = 1 \wedge \cdots \wedge P_{ab}(v_{i_k}) = 1) \to P_{ab}(u) = 1 \Big)$$

Arrow's Theorem as a dependency strengthening theorem (work in progress)

A “dependency strengthening" theorem

$=\!(v_1, \ldots, v_n; u)$ i.e., $u = F(v_1, \ldots, v_n)$

v_1	$\ldots\ \ldots$	v_n	u

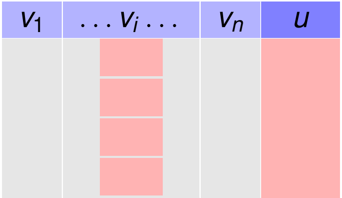
A “dependency strengthening" theorem

$$\Gamma$$

$$=\!(v_1, \ldots, v_n; u) \qquad \text{i.e., } u = F(v_1, \ldots, v_n)$$

$$\Downarrow$$

$$=\!(v_1; u) \vee \cdots \vee =\!(v_n; u) \qquad \text{i.e., } u = G(v_i)$$



A “dependency strengthening" theorem

Γ

$$=\!(v_1, \ldots, v_n; u) \qquad \text{i.e., } u = F(v_1, \ldots, v_n)$$

$$\Downarrow$$

$$=\!(v_1; u) \vee \cdots \vee =\!(v_n; u) \qquad \text{i.e., } u = G(v_i)$$

$$\Downarrow$$

$$\delta(v_1) \vee \cdots \vee \delta(v_n) \qquad \text{i.e., } v_i \text{ is a dictator}$$

v_1	$\ldots v_i \ldots$	v_n	u
	cab		cab
	cba		cba
	bca		bca
	cba		cba

A “dependency strengthening" theorem

$$\Gamma$$

$$=\!(v_1, \ldots, v_n; u) \qquad \text{i.e., } u = F(v_1, \ldots, v_n)$$

$$\Downarrow$$

$$=\!(v_1; u) \vee \cdots \vee =\!(v_n; u) \qquad \text{i.e., } u = G(v_i)$$

$$\Downarrow$$

$$(u = v_1) \vee \cdots \vee (u = v_n) \qquad \text{i.e., } u = v_i$$

v_1	$\ldots v_i \ldots$	v_n	u
	cab		cab
	cba		cba
	bca		bca
	cba		cba

A “dependency strengthening" theorem

$$\Gamma$$

$$=\!(v_1, \ldots, v_n; u) \qquad \text{i.e., } u = F(v_1, \ldots, v_n)$$

$$\Downarrow$$

$$=\!(v_1; u) \vee \cdots \vee =\!(v_n; u) \qquad \text{i.e., } u = G(v_i)$$

$$\Downarrow$$

$$(u = v_1) \vee \cdots \vee (u = v_n) \qquad \text{i.e., } u = v_i$$

v_1	$\ldots v_i \ldots$	v_n	u
	cab		cab
	cba		cba
	bca		bca
	cba		cba

Work in progress: Find in general Γ and $\alpha(x)$ s.t.

$$\Gamma, =\!(\alpha(v_1), \ldots, \alpha(v_n); \alpha(u)) \vdash \bigvee_{i=1}^{n} (\alpha(v_i) \leftrightarrow \alpha(u)).$$

Arrow’s Thm and its generalizations (e.g., Kalai-Muller-Satterthwaite Thm) are special cases.

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

Analyzing Arrow's Theorem

v_1					
$P_{ab}(v_1)$	$P_{bc}(v_1)$	$P_{ac}(v_1)$			
1	0	1	v_2	v_3	u
		acb	*bac*	*cab*	*bca*
		abc	*acb*	*bac*	*abc*
		bac	*bac*	*cab*	*acb*

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$x_1y_1z_1$	$x_2y_2z_2$	$x_3y_3z_3$	xyz
101	011	100	010
111	101	010	111
011	011	100	101

Judgement aggregation: (Dietrich, List 2007), etc.

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$x_1 y_1 z_1$	$x_2 y_2 z_2$	$x_3 y_3 z_3$	xyz
101	011	100	010
111	101	010	111
011	011	100	101

Functionality: $\theta_F := =\!(x_1 y_1 z_1, \ldots, x_n y_n z_n; xyz)$

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$x_1y_1z_1$	$x_2y_2z_2$	$x_3y_3z_3$	xyz
101	011	100	010
111	101	010	111
011	011	100	101

$$xyz = F(x_1y_1z_1, \ldots, x_ny_nz_n)$$

Functionality: $\theta_F := {=}(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$x_1y_1z_1$	$x_2y_2z_2$	$x_3y_3z_3$	xyz
101	011	100	010
111	101	010	111
011	011	100	101

$$\begin{aligned} xyz &= F(x_1y_1z_1, \ldots, x_ny_nz_n) \\ &= f(\bar{x})g(\bar{y})h(\bar{z}) \end{aligned}$$

Functionality: $\theta_F := {=}(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := {=}(x_1, \ldots, x_n; x) \wedge {=}(y_1, \ldots, y_n; y) \wedge {=}(z_1, \ldots, z_n; z)$

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$x_1y_1z_1$	$x_2y_2z_2$	$x_3y_3z_3$	xyz
101	011	100	010
111	101	010	111
011	011	100	101

$$\begin{aligned} xyz &= F(x_1y_1z_1, \ldots, x_ny_nz_n) \\ &= f(\bar{x})g(\bar{y})h(\bar{z}) \\ &= x_iy_iz_i \end{aligned}$$

Functionality: $\theta_F := =\!(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := =\!(x_1, \ldots, x_n; x) \wedge =\!(y_1, \ldots, y_n; y) \wedge =\!(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D := \bigvee_{i=1}^{n} xyz = x_iy_iz_i$

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$$xyz = F(x_1y_1z_1, \ldots, x_ny_nz_n) \\ = f(\bar{x})g(\bar{y})h(\bar{z}) \\ = x_iy_iz_i$$

Functionality: $\theta_F := =\!(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := =\!(x_1, \ldots, x_n; x) \wedge =\!(y_1, \ldots, y_n; y) \wedge =\!(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D := \bigvee_{i=1}^{n} xyz = x_iy_iz_i$

Unanimity: $\theta_{UN} := (\bar{x} = \bar{1} \rightarrow x = 1) \wedge (\bar{x} = \bar{0} \rightarrow x = 0)$
$\wedge(\bar{y} = \bar{1} \rightarrow y = 1) \wedge (\bar{y} = \bar{0} \rightarrow y = 0)$
$\wedge(\bar{z} = \bar{1} \rightarrow z = 1) \wedge (\bar{z} = \bar{0} \rightarrow z = 0)$

Fixed values

Analyzing Arrow's Theorem

v_1	v_2	v_3	u
acb	*bac*	*cab*	*bca*
abc	*acb*	*bac*	*abc*
bac	*bac*	*cab*	*acb*

$$\begin{aligned} xyz &= F(x_1y_1z_1, \ldots, x_ny_nz_n) \\ &= f(\bar{x})g(\bar{y})h(\bar{z}) \\ &= x_iy_iz_i \end{aligned}$$

Functionality: $\theta_F := {=}(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := {=}(x_1, \ldots, x_n; x) \wedge {=}(y_1, \ldots, y_n; y) \wedge {=}(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D := \bigvee_{i=1}^{n} xyz = x_iy_iz_i$

Unanimity: $\theta_{UN} := (\bar{x} = \bar{1} \to x = 1) \wedge (\bar{x} = \bar{0} \to x = 0) \wedge(\bar{y} = \bar{1} \to y = 1) \wedge (\bar{y} = \bar{0} \to y = 0) \wedge(\bar{z} = \bar{1} \to z = 1) \wedge (\bar{z} = \bar{0} \to z = 0)$

Fixed values

Universal Domain: $\theta_{UD} := \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \to \flat_3 = \flat_1\} \wedge \bigwedge\{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i\rangle \mid 1 \leq i \leq n\}$

Collective Rationality: $\theta_{CR} := (x = 1 \wedge y = 1 \to z = 1) \wedge (x = 0 \wedge y = 0 \to z = 0)$

Excluded values and interactions

Dependency strengthening theorems

v_1	v_2	v_3	u
abc	*bac*	*cab*	*bca*
acb	*acb*	*bac*	*abc*
acb	*bac*	*cab*	*acb*

$$\begin{aligned} xyz &= F(x_1y_1z_1, \ldots, x_ny_nz_n) \\ &= f(\bar{x})g(\bar{y})h(\bar{z}) \end{aligned}$$

Functionality: $\theta_F := {=\!\!(}x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := {=\!\!(}x_1, \ldots, x_n; x) \wedge {=\!\!(}y_1, \ldots, y_n; y) \wedge {=\!\!(}z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D := \bigvee_{i=1}^{n} xyz = \mathrm{id}(x_i)\mathrm{id}(y_i)\mathrm{id}(z_i)$

Unanimity:
$$\begin{aligned} \theta_{UN} := \; & (\bar{x} = \bar{1} \to x = 1) \wedge (\bar{x} = \bar{0} \to x = 0) \\ & \wedge(\bar{y} = \bar{1} \to y = 1) \wedge (\bar{y} = \bar{0} \to y = 0) \\ & \wedge(\bar{z} = \bar{1} \to z = 1) \wedge (\bar{z} = \bar{0} \to z = 0) \end{aligned}$$

Fixed values

Universal Domain:
$$\begin{aligned} \theta_{UD} := & \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \to \flat_3 = \flat_1\} \\ & \wedge \bigwedge\{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i\rangle \mid 1 \leq i \leq n\} \end{aligned}$$

Collective Rationality: $\theta_{CR} := (x = 1 \wedge y = 1 \to z = 1) \wedge (x = 0 \wedge y = 0 \to z = 0)$

Excluded values and interactions

Dependency strengthening theorems

v_1	v_2	v_3	u
abc	*bac*	*cab*	*bca*
acb	*acb*	*bac*	*abc*
acb	*bac*	*cab*	*acb*

$$xyz = F(x_1y_1z_1, \dots, x_ny_nz_n) = f(\bar{x})g(\bar{y})h(\bar{z})$$

Functionality: $\theta_F := =\!(x_1y_1z_1, \dots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := =\!(x_1, \dots, x_n; x) \wedge =\!(y_1, \dots, y_n; y) \wedge =\!(z_1, \dots, z_n; z)$

Dictatorship: $\theta_D := \bigvee_{i=1}^n xyz = \mathrm{id}(x_i)\mathrm{id}(y_i)\mathrm{id}(z_i)$

Unanimity: $\theta_{UN} := (\bar{x} = \bar{1} \to x = \mathrm{id}(1)) \wedge (\bar{x} = \bar{0} \to x = \mathrm{id}(0))$
$\wedge(\bar{y} = \bar{1} \to y = 1) \wedge (\bar{y} = \bar{0} \to y = 0)$
$\wedge(\bar{z} = \bar{1} \to z = 1) \wedge (\bar{z} = \bar{0} \to z = 0)$

Fixed values

Universal Domain: $\theta_{UD} := \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \to \flat_3 = \flat_1\}$
$\wedge \bigwedge\{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i\rangle \mid 1 \leq i \leq n\}$

Collective Rationality: $\theta_{CR} := (x = 1 \wedge y = 1 \to z = 1) \wedge (x = 0 \wedge y = 0 \to z = 0)$

Excluded values and interactions

Dependency strengthening theorems

v_1	v_2	v_3	u
abc	*bac*	*cab*	*bca*
acb	*acb*	*bac*	*abc*
acb	*bac*	*cab*	*acb*

$$\begin{aligned} xyz &= F(x_1y_1z_1,\ldots,x_ny_nz_n) \\ &= f(\bar{x})g(\bar{y})h(\bar{z}) \end{aligned}$$

Functionality: $\theta_F := {=}(x_1y_1z_1,\ldots,x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := {=}(x_1,\ldots,x_n;x) \wedge {=}(y_1,\ldots,y_n;y) \wedge {=}(z_1,\ldots,z_n;z)$

Dictatorship: $\theta_D^{\mathrm{id}} := \bigvee_{i=1}^n xyz = \mathrm{id}(x_i)\mathrm{id}(y_i)\mathrm{id}(z_i)$

Unanimity:
$$\begin{aligned}\theta_{UN}^{\mathrm{id}} := {} & (\bar{x} = \bar{1} \rightarrow x = \mathrm{id}(1)) \wedge (\bar{x} = \bar{0} \rightarrow x = \mathrm{id}(0)) \\ & \wedge(\bar{y} = \bar{1} \rightarrow y = \mathrm{id}(1)) \wedge (\bar{y} = \bar{0} \rightarrow y = \mathrm{id}(0)) \\ & \wedge(\bar{z} = \bar{1} \rightarrow z = \mathrm{id}(1)) \wedge (\bar{z} = \bar{0} \rightarrow z = \mathrm{id}(0))\end{aligned}$$

Fixed values

Universal Domain:
$$\begin{aligned}\theta_{UD}^{\mathrm{id}} := & \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1,\flat_2,\flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \rightarrow \flat_3 = \mathrm{id}(\flat_1)\} \\ & \wedge \bigwedge\{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i\rangle \mid 1 \leq i \leq n\}\end{aligned}$$

Collective Rationality: $\theta_{CR}^{\mathrm{id}} := (x = 1 \wedge y = 1 \rightarrow z = \mathrm{id}(1)) \wedge (x = 0 \wedge y = 0 \rightarrow z = \mathrm{id}(0))$

Excluded values and interactions

Dependency strengthening theorems

v_1	v_2	v_3	u
abc	bac	cab	bca
acb	acb	bac	abc
acb	bac	cab	acb

$$xyz = F(x_1y_1z_1, \ldots, x_ny_nz_n) = f(\bar{x})g(\bar{y})h(\bar{z})$$

Functionality: $\theta_F := =\!(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := =\!(x_1, \ldots, x_n; x) \wedge =\!(y_1, \ldots, y_n; y) \wedge =\!(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D^{ex} := \bigvee_{i=1}^{n} xyz = \mathrm{ex}(x_i)\mathrm{ex}(y_i)\mathrm{ex}(z_i)$

$ex(1) = 0$
$ex(0) = 1$

Unanimity: $\theta_{UN}^{id} := (\bar{x} = \bar{1} \to x = \mathrm{id}(1)) \wedge (\bar{x} = \bar{0} \to x = \mathrm{id}(0)) \wedge (\bar{y} = \bar{1} \to y = \mathrm{id}(1)) \wedge (\bar{y} = \bar{0} \to y = \mathrm{id}(0)) \wedge (\bar{z} = \bar{1} \to z = \mathrm{id}(1)) \wedge (\bar{z} = \bar{0} \to z = \mathrm{id}(0))$

Fixed values

Universal Domain: $\theta_{UD}^{id} := \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \to \flat_3 = \mathrm{id}(\flat_1)\} \wedge \bigwedge\{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i\rangle \mid 1 \leq i \leq n\}$

Collective Rationality: $\theta_{CR}^{id} := (x = 1 \wedge y = 1 \to z = \mathrm{id}(1)) \wedge (x = 0 \wedge y = 0 \to z = \mathrm{id}(0))$

Excluded values and interactions

Dependency strengthening theorems

Functionality: $\theta_F := {=}(x_1 y_1 z_1, \ldots, x_n y_n z_n; xyz)$

IIA: $\theta_{IIA} := {=}(x_1, \ldots, x_n; x) \wedge {=}(y_1, \ldots, y_n; y) \wedge {=}(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D^{ex} := \bigvee_{i=1}^{n} xyz = \mathrm{ex}(x_i)\mathrm{ex}(y_i)\mathrm{ex}(z_i)$

$ex(1) = 0$
$ex(0) = 1$

Unanimity: $\theta_{UN}^{id} := (\bar{x} = \bar{1} \to x = \mathrm{id}(1)) \wedge (\bar{x} = \bar{0} \to x = \mathrm{id}(0))$
$\wedge(\bar{y} = \bar{1} \to y = \mathrm{id}(1)) \wedge (\bar{y} = \bar{0} \to y = \mathrm{id}(0))$
$\wedge(\bar{z} = \bar{1} \to z = \mathrm{id}(1)) \wedge (\bar{z} = \bar{0} \to z = \mathrm{id}(0))$

Fixed values

Universal Domain: $\theta_{UD}^{id} := \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_i y_i z_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \to \flat_3 = \mathrm{id}(\flat_1)\}$
$\wedge \bigwedge\{x_i y_i z_i \perp \langle x_j y_j z_j \mid j \neq i\rangle \mid 1 \leq i \leq n\}$

Collective Rationality: $\theta_{CR}^{id} := (x = 1 \wedge y = 1 \to z = \mathrm{id}(1)) \wedge (x = 0 \wedge y = 0 \to z = \mathrm{id}(0))$

Excluded values and interactions

Thm. For any $\mathrm{f}, \mathrm{g} \in \{\mathrm{id}, \mathrm{ex}\}$, $\theta_F, \theta_{IIA}, \theta_{UD}^{\mathrm{f}}, \theta_{CR}^{\mathrm{f}}, \theta_{UN}^{\mathrm{g}} \vdash \theta_D^{\mathrm{g}}$.

C.f. [Tang, Lin 2009]

Dependency strengthening theorems

Functionality: $\theta_F := =(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := =(x_1, \ldots, x_n; x) \wedge =(y_1, \ldots, y_n; y) \wedge =(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D^{\text{id},\text{id},\text{ex}} := \bigvee_{i=1}^n xyz = \text{id}(x_i)\text{id}(y_i)\text{ex}(z_i)$

$ex(1) = 0$
$ex(0) = 1$

Unanimity: $\theta_{UN}^{\text{id},\text{id},\text{ex}} := (\bar{x} = \bar{1} \rightarrow x = \text{id}(1)) \wedge (\bar{x} = \bar{0} \rightarrow x = \text{id}(0))$
$\wedge(\bar{y} = \bar{1} \rightarrow y = \text{id}(1)) \wedge (\bar{y} = \bar{0} \rightarrow y = \text{id}(0))$
$\wedge(\bar{z} = \bar{1} \rightarrow z = \text{ex}(1)) \wedge (\bar{z} = \bar{0} \rightarrow z = \text{ex}(0))$

Fixed values

Universal Domain: $\theta_{UD}^{\text{id}} := \bigwedge\{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \rightarrow \flat_3 = \text{id}(\flat_1)\}$
$\wedge \bigwedge\{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i\rangle \mid 1 \leq i \leq n\}$

Collective Rationality: $\theta_{CR}^{\text{id}} := (x = 1 \wedge y = 1 \rightarrow z = \text{id}(1)) \wedge (x = 0 \wedge y = 0 \rightarrow z = \text{id}(0))$

Excluded values and interactions

Thm. For any $\text{f}, \text{g} \in \{\text{id}, \text{ex}\}$, $\quad \theta_F, \theta_{IIA}, \theta_{UD}^{\text{f}}, \theta_{CR}^{\text{f}}, \theta_{UN}^{\text{g}} \vdash \theta_D^{\text{g}}$.

C.f. [Tang, Lin 2009]

Dependency strengthening theorems

Functionality: $\theta_F := =\!(x_1y_1z_1, \ldots, x_ny_nz_n; xyz)$

IIA: $\theta_{IIA} := =\!(x_1, \ldots, x_n; x) \wedge =\!(y_1, \ldots, y_n; y) \wedge =\!(z_1, \ldots, z_n; z)$

Dictatorship: $\theta_D^{\text{id},\text{id},\text{ex}} := \bigvee_{i=1}^n xyz = \text{id}(x_i)\text{id}(y_i)\text{ex}(z_i)$

$ex(1) = 0$
$ex(0) = 1$

Unanimity: $\theta_{UN}^{\text{id},\text{id},\text{ex}} := (\bar{x} = \bar{1} \to x = \text{id}(1)) \wedge (\bar{x} = \bar{0} \to x = \text{id}(0))$
$\wedge (\bar{y} = \bar{1} \to y = \text{id}(1)) \wedge (\bar{y} = \bar{0} \to y = \text{id}(0))$
$\wedge (\bar{z} = \bar{1} \to z = \text{ex}(1)) \wedge (\bar{z} = \bar{0} \to z = \text{ex}(0))$

Fixed values

Universal Domain: $\theta_{UD}^{\text{id}} := \bigwedge \{\flat_1\flat_2\flat_3 \subseteq x_iy_iz_i \mid \flat_1, \flat_2, \flat_3 \in \{0,1\},\ \flat_1 = \flat_2 \to \flat_3 = \text{id}(\flat_1)\}$
$\wedge \bigwedge \{x_iy_iz_i \perp \langle x_jy_jz_j \mid j \neq i \rangle \mid 1 \leq i \leq n\}$

Collective Rationality: $\theta_{CR}^{\text{id}} := (x = 1 \wedge y = 1 \to z = \text{id}(1)) \wedge (x = 0 \wedge y = 0 \to z = \text{id}(0))$

Excluded values and interactions

Thm. For any $\text{f}, \text{g} \in \{\text{id}, \text{ex}\}$, $\quad \theta_F, \theta_{IIA}, \theta_{UD}^{\text{f}}, \theta_{CR}^{\text{f}}, \theta_{UN}^{\text{g}} \vdash \theta_D^{\text{g}}$.

Thm. $\theta_F, \theta_{IIA}, \theta_{UD}^{\text{id}}, \theta_{CR}^{\text{ex}}, \theta_{UN}^{\text{id},\text{id},\text{ex}} \vdash \theta_D^{\text{id},\text{id},\text{ex}}$; $\quad \theta_F, \theta_{IIA}, \theta_{UD}^{\text{ex}}, \theta_{CR}^{\text{id}}, \theta_{UN}^{\text{ex},\text{ex},\text{id}} \vdash \theta_D^{\text{ex},\text{ex},\text{id}}$.

谢谢！ Kiitos!