Peano Arithmetic and Square Principles

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Arithmetic

The Peano Axioms, consisting of the axioms for a discretely ordered ring, plus induction, have many models.

In fact they have $2^{\aleph_0}$ non-isomorphic countable models.

Also, the nonstandard countable models are not recursive, the sense that if the domain of the model is identified with the natural numbers, then $+^M$ and x^M, regarded as ternary relations on the natural numbers, are not recursive.

Arithmetic

Theorem (Tennenbaum) Let M be a countable Diophantine correct model of PA^-. Then M can be embedded in $\mathcal{N} = \mathbb{N}^\omega/\mathcal{F}$, where $\mathcal{F}$ is the Frechet filter on $\mathbb{N}$.[1]

(Already follows from the $\aleph_1$- saturation of $\mathcal{N}$, but Tennenbaum constructs the embedding directly.)

Tennenbaum saw this as an antidote to the above two theorems.

[1] PA^- is the theory PA without induction. M is Diophantine correct if whenever a polynomial equation has a solution in the model, it has a solution in the natural numbers.

Proof

Enumerate the elements of M as $m_1, m_2, m_3, \ldots$.

Enumerate the polynomial equations $P(v_1, v_2, \ldots)$ satisfied by $\langle m_1, m_2, \ldots \rangle$ in M.

	m_1	m_2	$\dots$	m_n	$\dots$
P_1	$v_1(1)$	$v_2(1)$	$\dots$	$v_n(1)$	$\dots$
$P_1 \wedge P_2$	$v_1(2)$	$v_2(2)$	$\dots$	$v_n(2)$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
$\bigwedge_{i=1}^{n} P_i$	$v_1(n)$	$v_2(n)$	$\dots$	$v_n(n)$	$\dots$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Mapping is $m_i \to [\langle v_i(n) \rangle]$

The non-Diophantine correct case

Theorem (Tennenbaum). Let M be a countable model of PA^-. Then M can be embedded in $\mathcal{A} = \mathbf{A}^\omega/\mathcal{F}$.

Cohesiveness

A set $X \subseteq \mathbb{N}$ is r-**cohesive (cohesive)**, if for all recursive (r.e.) sets A of natural numbers, either $X \subseteq^* A$ or $X \subseteq^* -A$.

A function $f : \mathbb{N} \to \mathbb{N}$ is r-**cohesive (cohesive)** if its range is.

Which $f \in \mathcal{N}$ occur in a model of arithmetic?

Theorem. Let f be a function $\mathbb{N} \to \mathbb{N}$. Then f is contained in some substructure of $\mathcal{N}$ satisfying $\Pi_2 - \mathrm{Th}(\mathbb{N})$ iff f is r-cohesive. So the identity function cannot belong to a model of True Arithmetic inside $\mathcal{N}$, unlike the ultrafilter case.

Other Cardinalities: Regular filters

- A filter D on I is **regular** if
  $(\exists\{A_\alpha : \alpha < |I|\} \subseteq D)(\forall i \in I)(|\{\alpha < \lambda : i \in A_\alpha\}| < \omega)$
- Generalizes the cofinite filter (Frechet filter) over ω.
- The meaning: there is a "regular" family of $|I|$ sets in the filter such that the intersection of any infinite subfamily is empty.
- On every cardinal there is a regular filter and therefore also a regular ultrafilter.
- If there is a non-regular ultrafilter on ω_1, then $0^{\#}$ exists (J. Ketonen)

Embedding Models of Cardinality $\aleph_1$

Theorem. (K, Shelah) Let M be a Diophantine correct model of PA^- of cardinality $\aleph_1$. Let D be a regular filter on ω. Then M can be embedded in $\mathbb{N}^\omega / D$.

Lemma: There exists a family of sets u_n^α, with $\alpha < \omega_1$, and $n \in \mathbb{N}$, such that for each n, α

(i) $|u_n^\alpha| < n + 1$

(ii) $\alpha \in u_n^\alpha \subseteq u_{n+1}^\alpha$

(iii) $\bigcup_n u_n^\alpha = \alpha + 1$

(iv) $\beta \in u_n^\alpha \Rightarrow u_n^\beta = u_n^\alpha \cap (\beta + 1)$

Other cardinalites

Not provable...

Definitions

- M λ-**universal**: If $|N| < \lambda$ and $N \equiv M$, then N is elementary embeddable into M.
- A way to understand the theory of N (and M).

The Main Results

The following statements are independent of ZFC, assuming the consistency of large cardinals:

- If M is a structure in a vocabulary of size $\leq \lambda$ and D a regular ultrafilter on λ, then M^λ/D is λ^{++}-universal. (Keisler & Chang: Open Problem 18)
- Suppose M and N are structures in a vocabulary of size $\leq \lambda$ such that $|M|, |N| \leq \lambda$. If $M \equiv N$, D is a regular ultrafilter on λ, and $2^\lambda = \lambda^+$, then $M^\lambda/D \cong N^\lambda/D$. (Keisler & Chang: Open Problem 19)

A finitary square

$\square^{fin}_{\lambda,D}$: For each $i < \lambda$ there is a natural number n_i, and for each $i < \lambda$ and $\zeta < \lambda^+$ there exists a set u_i^ζ such that:

(i) $|u_i^\zeta| < n_i$

(ii) $u_i^\zeta \subseteq \zeta$

(iii) For all $B \subseteq \lambda^+$, B finite, there exists ϵ such that $\{i : u_i^\epsilon \supseteq B\} \in D$

(iv) Coherency: $\gamma \in u_i^\zeta \Rightarrow u_i^\gamma = u_i^\zeta \cap \gamma$

Universality Theorems

- Assume $\square^{fin}_{\lambda,D}$. For all λ-regular ultrafilters D: $M^\lambda/_D$ is λ^{++}-universal.
- Best possible result assuming GCH, since then $|M^\lambda/_D| \leq \lambda^+$ for $|M| \leq \lambda^+$.

The transfer principle $\langle\aleph_1, \aleph_0\rangle \to \langle\lambda^+, \lambda\rangle$

- Due to C.C.Chang
- Follows from GCH for λ regular (Chang two-cardinal theorem) and from V=L for other λ (Jensen).
- False for $\lambda = \aleph_1$ in the "Mitchell model", which uses an inaccessible cardinal, and for $\lambda = \aleph_\omega$ (with GCH) in the Litman-Shelah model, which uses a supercompact.

$\langle\aleph_1, \aleph_0\rangle \to \langle\lambda^+, \lambda\rangle$ implies $\square^{fin}_{\lambda,D}$ for any regular filter D on λ.

The weak square principle $\square_\lambda^{b^*}$

- Transfer principle is equivalent to the following weak square principle.
- $\square_\lambda^{b^*}$ says: There are a λ^+-like linear order L, increasing (in ζ) sets $C_a^\zeta, a \in L, \zeta < cf(\lambda)$, equivalence relations $\langle E^\zeta : \zeta < cf(\lambda)\rangle$, and functions $\langle f_{a,b}^\zeta : \zeta < \lambda, a \in L, b \in L\rangle$ such that

1. $\bigcup_\zeta C_a^\zeta = \{b : b <_L a\}$
2. If $b \in C_a^\zeta$, then $C_b^\zeta = \{c \in C_a^\zeta : c <_L b\}$ (coherence)
3. E^ζ is an equivalence relation on L with $\leq \lambda$ equivalence classes.
4. If $\zeta < \xi < cf(\lambda)$, then E^ξ refines E^ζ.
5. If $aE^\zeta b$ then $f_{a,b}^\zeta$ is an order-preserving map from C_a^ζ onto C_b^ζ.
6. If $\zeta < \xi < cf(\lambda)$ and $aE^\xi b$, then $f_{a,b}^\zeta \subseteq f_{a,b}^\xi$.
7. If $f_{a,b}^\zeta(a_1) = b_1$, then $f_{a_1,b_1}^\zeta \subseteq f_{a,b}^\zeta$.
8. $a \in C_b^\zeta \Rightarrow \neg(aE^\zeta b)$.

- A transfer of the case $\lambda = \aleph_1$ (where the principle is provable), written in the logic $L(Q_1)$, "there are uncountably many".
- $\square_\lambda^{b^*}$ implies $\square_{\lambda,D}^{fin}$ for any regular filter D on λ.
- Converse true for s.s.l. λ, and D generated by $\leq \lambda$ sets.

Isomorphism Theorem for λ

Assume $\Box^{fin}_{\lambda,D}$. Let L be a language of cardinality $\leq \lambda$ and for each $i < \lambda$ let M_i and N_i be two elementarily equivalent L-structures. If D is a regular filter on λ, then Player II has a winning strategy in the game EFG_{λ^+} on $\prod_i M_i/D$ and $\prod_i N_i/D$.
(Previous result of Shelah: "... EFG_α for any $\alpha < \lambda^+$")

Corollary

Assuming $2^\lambda = \lambda^+$, $\Box^{fin}_{\lambda,D}$, D regular filter on λ, and $|A| \leq \lambda$, $|B| \leq \lambda$. Then $A \equiv B \Rightarrow A^\lambda/D \cong B^\lambda/D$.

Proof

- Use sets u_i^α to define a winning strategy for the isomorphism player.
- Elementary equivalence means isomorphism player has a winning strategy for Ehrenfeucht-Fraisse games of length $n_i < \omega$. Coherence allows us to "knit together" these strategies into one winning strategy.

Necessity of $\Box^{fin}_{\lambda,D}$

Suppose $\lambda \geq \omega$ and D is a regular ultrafilter on λ. Then:
$(\forall M(||L(M)|| \leq \lambda \rightarrow M^{\lambda}/D \text{ is } \lambda^{++} - \text{universal})) \Rightarrow \Box^{fin}_{\lambda,D}$.

Necessity of $\square^{fin}_{\lambda,D}$

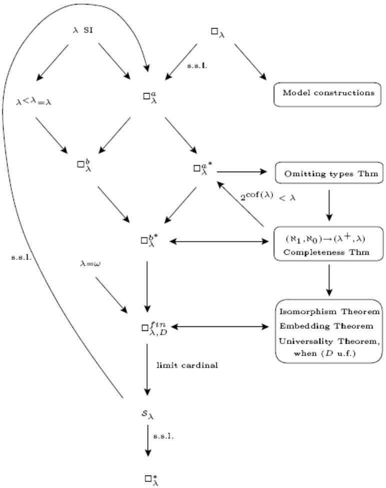
Suppose $\lambda \geq \omega$ and D is a regular filter on λ. Then:
$(\forall M, N(||L(M)||, ||L(N)|| \leq \lambda \ \& \ M \equiv N \rightarrow$
Second player has a winning strategy in $EF_{\lambda^+}(M^\lambda/D, N^\lambda/D) \Rightarrow$
$\square^{fin}_{\lambda,D}$.

$$\text{GCH} + \lambda \text{ regular}$$
$$\Downarrow$$
$$\lambda = \lambda^{<\lambda}$$
$$\Downarrow$$
$$(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$$
$$\Updownarrow$$
$$\square_\lambda^{b^*}$$
$$\Downarrow \qquad (\Updownarrow \text{ for s.s.l. } \lambda \text{ and } D \text{ gen. by } \leq \lambda \text{ sets})$$
$$\square_{\lambda,D}^{fin}$$
$$\Updownarrow$$

Isomorphism Theorems for λ

Embedding Theorems for λ

Universality Theorems for λ (and D u.f.)


λ SI
□λ
s.s.l.
λ<λ=λ
□aλ
Model constructions
□bλ
□a*λ
Omitting types Thm
2cof(λ) < λ
□b*λ
(ℵ1,ℵ0)→(λ+,λ)
Completeness Thm
s.s.l.
λ=ω
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Isomorphism Theorem
Embedding Theorem
Universality Theorem,
when (D u.f.)
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s.s.l.
□*λ

Independence results

The following equivalent conditions are **true** for all regular λ and all regular filters D on λ, if $\lambda = \lambda^{<\lambda}$ (Chang), and for singular λ if $V = L$ holds (Jensen). They are **false** consistently with GCH for $\lambda = \aleph_\omega$ and some regular filter D on λ (Litman-Shelah, assuming the consistency of supercompact cardinals).

1. $\square^{fin}_{\lambda,D}$.
2. Embedding Theorem for D and λ
3. Isomorphism Theorem for D and λ

The Missing Case: D an ultrafilter

The following equivalent conditions are **false** consistently for some regular ultrafilter D on λ if λ singular strong limit of cofinality κ and there is a strongly compact cardinal between κ and λ:

1. $\square^{fin}_{\lambda,D}$.
2. Isomorphism Theorem for D and λ
3. Universality Theorem for D and λ

The Proof

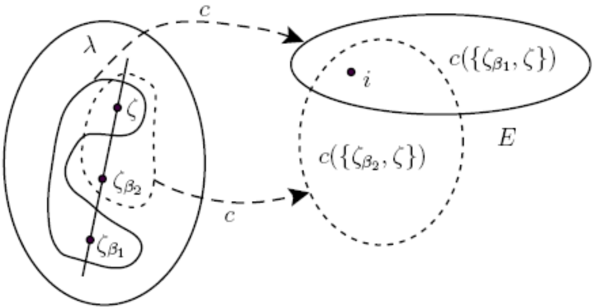
How to find an u.f. D so that $\square^{fin}_{\lambda,D}$ fails? Note: for maximal D, $\square^{fin}_{\lambda,D}$ is the weakest. Use strong compactness to get a particular partition property, via a κ^+-complete ultrafilter on λ^+. This makes sense, as $\square^{fin}_{\lambda,D}$ is a particularly strong form of regularity – the opposite of completeness.

Our partition property

Definition: Let $Pr_2(\lambda, \kappa)$ denote the following property of λ and κ with $\kappa < \lambda$:

> *Suppose* $c : [\lambda]^2 \to E$*, where* E *is a filter on* κ*. Then there is an* $i < \kappa$ *such that for all* $\chi < \lambda$ *there is an increasing sequence* ζ_β*,* $\beta < \chi$*, of ordinals* $< \lambda$ *such that for all* $\beta_1 < \beta_2 < \chi$ *there is* $\zeta > \zeta_{\beta_2}$ *such that* $i \in c(\{\zeta_{\beta_1}, \zeta\}) \cap c(\{\zeta_{\beta_2}, \zeta\})$

(From the appendix of Shelah's *Cardinal Arithmetic*.)


c
λ
ζ
ζ_{β_2}
ζ_{β_1}
c
i
$c(\{\zeta_{\beta_1}, \zeta\})$
E
$c(\{\zeta_{\beta_2}, \zeta\})$

$Pr_2(\lambda, \kappa)$ holds for λ weakly compact – because then there is always a big homogeneous set.

Proposition Suppose $\kappa < \lambda$ and E is a κ^+-complete uniform ultrafilter on λ^+. Then $Pr_2(\lambda^+, \kappa)$.

Corollary Suppose $\kappa < \theta \leq \lambda$ where θ is strongly compact. Then $Pr_2(\lambda^+, \kappa)$ holds.

(**Proof** of corollary: Let F be the λ^+-complete filter $\{A \subseteq \lambda^+ : |\lambda^+ \setminus A| < \lambda^+\}$. By strong compactness of θ, there is a θ-complete uniform ultrafilter E on λ^+ extending F. Now use the **proposition**.)

(θ *strongly compact* means: for any set S, every θ-complete filter on S can be extended to a θ-complete u.f. on S.)

The Ultrafilter

Definition Suppose $\lambda = sup_{\xi<\kappa}\lambda_\xi$, D_ξ is a filter on λ_ξ for $\xi < \kappa$, and E is a filter on κ. We then define

$$\Sigma_E D_\xi = \{A \subseteq \lambda : \{\xi : A \cap \lambda_\xi \in D_\xi\} \in E\}.$$

$\Sigma_E D_\xi$ is always a filter on λ, and moreover an ultrafilter, if E and each D_ξ are.
This is a general construction.

Theorem

Let us assume

(a) $Pr_2(\lambda^+, \kappa)$.

(b) $\lambda = \sup\{\lambda_\xi : \xi < \kappa\}$.

(c) D_ξ is a regular ultrafilter on λ_ξ such that $\lambda_\xi \setminus \bigcup_{\zeta<\xi} \lambda_\zeta \in D_\xi$.

(e) E is a regular ultrafilter on κ.

Then $D = \Sigma_E D_\xi$ is a regular ultrafilter on λ with $\neg\Box^{fin}_{\lambda,D}$.

We do not know about the failure of $\square^{fin}_{\lambda,D}$ for regular λ, e.g. $\lambda = \omega_2$, but note:

Remark: Let *Fr* be the canonical regular filter on ω_1. If $\square^{fin}_{\omega_1,Fr}$ fails, then ω_2 is inaccessible in L. (ω_1 can be replaced by any regular cardinal.) (because then $\square_{\omega_1}$ fails.)

Theorem (M. Viale)

Assume λ is a singular cardinal of countable cofinality and PID holds. Then there is a regular filter D on λ generated by λ many sets such that $\square^{fin}_{\lambda,D}$ fails.
PID is Todorcevic's P-Ideal Dichotomy.
Corollary $\square^{fin}_{\lambda,D}$ is not equivalent to $\square^*_\lambda$. This is because $\square^*_\lambda$ is consistent with PFA, which implies PID, and therefore the failure of $\square^{fin}_{\lambda,D}$; whereas on the other hand $\square^{fin}_{\lambda,D}$ implies $\square^*_\lambda$ for singular strong limit λ.

Doubly regular filters

Definition

A filter D on λ is called *doubly$^+$ regular*, if there are pairwise disjoint sets $u_i, i < \lambda$, each of cardinality λ, and regular filters D_i on u_i such that for all $A \subseteq \lambda$:

$$[\text{for a club of } i < \lambda(A \cap u_i \in D_i)] \Rightarrow A \in D.$$

There always are doubly$^+$ regular (ultra)filters on a regular cardinal. Doubly$^+$ regular filters are always regular.

Theorem

If D is a doubly$^+$ regular filter on a regular cardinal, then $\square^{fin}_{\lambda,D}$ holds.

Open questions

1. For regular λ we know that $\lambda^{<\lambda} = \lambda$ implies $\Box^{fin}_{\lambda,D}$ if D any regular D (generated by at most λ sets). What if $\lambda^{<\lambda} = \lambda$ does **not** hold, e.g. if $2^\omega = \lambda^+$?
2. For singular λ we know that if λ is s.s.l, then "$\Box^{fin}_{\lambda,D}$for any regular D (generated by at most λ sets)" is between $\Box_\lambda$ and $\Box^*_\lambda$. But what if λ is **not** a strong limit, e.g. if $\lambda = \aleph_\omega < 2^\omega$.
3. We know $\neg\Box_\lambda$ for singular λ implies $0^\#$ exists. Does the failure of $\Box^{fin}_{\lambda,D}$ imply $0^\#$? λ singular, regular?
4. We know $ZFC \vdash \Box^{fin}_{\omega,\mathcal{F}}$. $ZFC \vdash \Box^{fin}_{\omega_1,D}$ for every regular ultrafilter D on ω_1? Follows from CH. True if D is doubly$^+$ regular.

Thank You!