The Mathematical theory of wave turbulence
Part II

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Main Theorem

On a periodic box $\mathbb{T}_L^d$ of size L with $d \geqslant 3$, we consider

$$(i\partial_t - \Delta)u(t,x) + \alpha|u|^2 u = 0,$$

- Here α as the characteristic size of the nonlinearity. Recall that

$$u_{\text{in}}(x) = L^{-d/2}\sum_k \widehat{u_{\text{in}}}(k)e^{2\pi i k\cdot x}, \qquad \widehat{u_{\text{in}}}(k) = \sqrt{n_{\text{in}}(k)}\eta_k(\omega).$$

  Here, k ranges over $\mathbb{Z}_L^d = L^{-1}\mathbb{Z}^d$, a lattice with mesh L^{-1} (which tends to continuum as $L \to \infty$), n_{in} is a non-negative Schwartz function on $\mathbb{R}^d$, and $\eta_k(\omega)$ are i.i.d. random variables such that

$$\mathbb{E}\,\eta_k(\omega) = 0, \qquad \mathbb{E}|\eta_k(\omega)|^2 = 1.$$

  We assume that the law of $\eta_k(\omega)$ is rotationally symmetric and has exponential tails (e.g. Gaussian or random phase).

- Recall that we will adopt the scaling law $\alpha = L^{-\gamma}$ with $\gamma \in (0,1]$, and that for $\gamma = 1$ we adopt genericity conditions on the aspect ratios of the large box. No assumptions on the domain are needed for $\gamma < 1$.

Theorem (Deng-H., 2021-2022)

Under the scaling law $\alpha = L^{-\gamma}$ *for any* $\gamma \in (0, 1]$*, there holds:*

1. *Full derivation of the wave kinetic equation: There exists* $\delta < 1$ *fixed, and an absolute constant* $\nu > 0$ *such that for* L *large enough it holds that:*

$$\mathbb{E}|\widehat{u}(T_{\text{kin}} \cdot t, k)|^2 = n(t, k) + O(L^{-\nu})$$

   uniformly in (t, k) *for* $t \in [0, \delta]$*. Here* $n(t, k)$ *solves the wave kinetic equation with data* n_{in}*.*

2. *Propagation of Chaos: Suppose that* $k_1, \ldots, k_r$ *are distinct, then the random variables* $\widehat{u}(t, k_j)$ $(1 \leqslant j \leqslant r)$ *retain their independence in the kinetic limit* $L \to \infty$*.*

3. *Limiting law: The law of* $\widehat{u}(t, k)$ *converges to the density function* $\rho_k(t, v)$ *which evolves according to the linear PDE*

$$\partial_t \rho_k = \frac{\sigma_k(t)}{4} \Delta \rho_k - \frac{\gamma_k(t)}{2} \nabla \cdot (v \rho_k), \qquad v \in \mathbb{R}^2$$

   where $\sigma_k(t) > 0$ *and* $\gamma_k(t)$ *are functions constructed from the solution* $n(t, k)$ *to the wave kinetic equation.*

4. *Propagation of Gaussianity: In particular, if* $\eta_k(\omega)$ *are Gaussian, then* $\rho_k(t, v)$ *is Gaussian with variance* $n(t, k)$ *for any* $t > 0$*.*

Outline

1. Recall yesterday: Tree and couple expansions (Feynman Diagrams)
2. High level overview of the difficulties.
3. Leading diagrams: **regular couples**
4. The strategy of the proof
5. The **rigidity theorem** and the molecules
6. **Cancellations** in the Feynman Series Expansion
7. Regular couples -Part II (Why are they leading? Reduction to prime couples).
8. Higher order statistics (time permitting)

Trees and couples

Recall from yesterday: Trees and couples

- The basic starting point in the proof is to perform a power series expansion of the solution, and write it as

$$u = u^{(0)} + u^{(1)} + \ldots + u^{(N)} + (\text{remainder}).$$

$$\mathcal{F}u^{(n)}(\delta T_{\text{kin}} \cdot t, k) = \sum_{|\mathcal{T}|=n} a_k^{\mathcal{T}}(t), \qquad 0 \leqslant t \leqslant 1.$$

  where the sum is taken over trees $\mathcal{T}$ of order n (n branching nodes).

- $$a_k^{\mathcal{T}}(t) = \zeta_{\mathcal{T}} \left(\frac{\delta}{2L^{d-\gamma}}\right)^n \sum_{(k_\mathfrak{n})\in\mathcal{D}} A\left(t, (\Omega_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}}\right) \prod_{\mathfrak{l}\in\mathcal{L}} \underbrace{\sqrt{n_{\text{in}}(k_\mathfrak{l})}\eta_{k_\mathfrak{l}}^{\pm}}_{\widehat{u}_{\text{in}}(k_\mathfrak{l})}$$

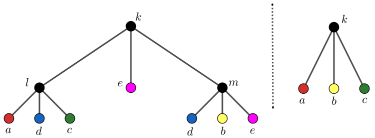
  where

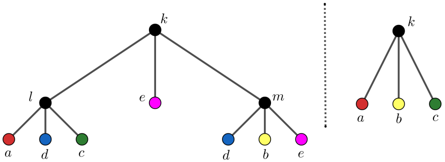
  - $\zeta_{\mathcal{T}}$ is the product of n factors of $\pm i$.
  - The sum over $k_\mathfrak{n} \in \mathbb{Z}_L^d$ is over decorations $\mathcal{D}$ of the tree: these are assignments of $k_\mathfrak{n} \in \mathbb{Z}_L^d$ for each $\mathfrak{n} \in \mathcal{T}$ such that $k_\mathfrak{r} = k$ and $k_\mathfrak{n} = k_{\mathfrak{n}_1} - k_{\mathfrak{n}_2} + k_{\mathfrak{n}_3}$ whenever $\mathfrak{n}$ is a branching node with children $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$.
  - $\Omega_\mathfrak{n} = |k_{\mathfrak{n}_1}|_\beta^2 - |k_{\mathfrak{n}_2}|_\beta^2 + |k_{\mathfrak{n}_3}|_\beta^2 - |k_\mathfrak{n}|_\beta^2$ for every $\mathfrak{n} \in \mathcal{N}$, the set of branching nodes.
  - $\mathcal{L}$ is the set of leaves, and $\eta_{k_\mathfrak{l}}^{\pm} = \eta_{k_\mathfrak{l}}$ if $\mathfrak{l}$ has $+$ sign and $\overline{\eta_{k_\mathfrak{l}}}$ if $\mathfrak{l}$ has $-$ sign.

- In computing, $\mathbb{E}|a_k(t)|^2$ we are thus led to consider $\mathbb{E}\, a_k^{\mathcal{T}_1}(t)\overline{a_k^{\mathcal{T}_2}(t)}$.

$$\mathbb{E}a_k^{\mathcal{T}_1}(t)\overline{a_k^{\mathcal{T}_2}(t)} = \zeta_{\mathcal{T}_1}\zeta_{\mathcal{T}_2}\left(\frac{\delta}{2L^{d-\gamma}}\right)^{n_1+n_2}\sum_{\mathcal{P}}\sum_{(k_\mathfrak{n})\in\mathcal{D}} B\left(t,(\Omega_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_1\cup\mathcal{N}_2}\right) \times \prod_{\mathfrak{l}\in\mathcal{L}_1\cup\mathcal{L}_2}^{+} n_{\mathrm{in}}(k_\mathfrak{l})$$

  where
  - $\mathcal{P}$ runs over all pairings of the leaves in $\mathcal{L}_1\cup\mathcal{L}_2$ so that paired leaves have opposite signs
  - $\mathcal{D}$ is the union of decorations of the two trees $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $k_\mathfrak{l} = k_{\mathfrak{l}'}$ if $(\mathfrak{l},\mathfrak{l}')\in\mathcal{P}$.
  - $B\left(t,(\Omega_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_1\cup\mathcal{N}_2}\right) := A\left(t,(\Omega_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_1}\right) A\left(t,(\Omega_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_2}\right)$.
  - $\prod^+$ runs over all leaves with sign $+$.
- **Couples:** We now define the couple $\mathcal{Q}$ to be the triplet $(\mathcal{T}_1,\mathcal{T}_2,\mathcal{P})$, i.e. it is a couple of trees with their leaves paired. Also, define the order n of a couple to be $n = n_1 + n_2$ where n_j is the order of $\mathcal{T}_j$. A decoration $\mathcal{D}$ as above is now called a decoration of the couple.





- With this in hand, we can summarize

$$\mathbb{E}|a_k(t)|^2 = \sum_{\mathcal{Q}} \left(\frac{\delta}{2L^{d-\gamma}}\right)^n \zeta_{\mathcal{Q}} \sum_{(k_\mathfrak{n})\in\mathcal{D}} B\left(t, (\Omega_\mathfrak{n})_{\mathfrak{n}\in\mathcal{N}_\mathcal{Q}}\right) \prod_{\mathfrak{l}\in\mathcal{L}_\mathcal{Q}}^{+} n_{\text{in}}(k_\mathfrak{l}) + \text{remainder}$$
$$= \sum_{\mathcal{Q}} \mathcal{K}_\mathcal{Q}(t,k) + \text{remainder terms}$$

  - $\sum_\mathcal{Q}$ is over all couples $\mathcal{Q}$ of two trees of total order $\leqslant N$.
  - $\zeta_\mathcal{Q} = \zeta_{\mathcal{T}_1}\zeta_{\mathcal{T}_2}$, $\mathcal{N}_\mathcal{Q} = \mathcal{N}_1 \cup \mathcal{N}_2$ and $\mathcal{L}_\mathcal{Q} := \mathcal{L}_1 \cup \mathcal{L}_2$.
- Similar formulas hold for $\mathbb{E}a_k^{\mathcal{T}_1}(t)\overline{a_k^{\mathcal{T}_2}(s)}$ for $t \neq s$ lead to expressions $\mathcal{K}_\mathcal{Q}(t,s,k)$.
- **Key fact:** there are C^n trees of order n, but there are $C^n n!$ couples of order $2n$.

High level overview of difficulties

Expected Difficulties

- Enemy 1: There are $O(n!)$ couples at order n, which presents a major obstruction to the convergence of the power series expansion. This means that blanket or uniform estimates on couples of order n (even sharpest ones) would not be sufficient, and one has to go deeper into the couple analysis.

- Enemy 2: With high probability, the iterate $u^{(n+1)}$ is only better than $u^{(n)}$ by a factor of $\sqrt{\delta}$ (rather than a factor of $L^{-\varepsilon}$). This is called *probabilistic criticality*.

Criticality

- T_{kin} can be understood as the longest timescale for which the power series expansion of the solution into Feynman diagrams is valid.
- The problem is *subcritical* for timescales $T \ll T_{\text{kin}}$. In [Deng, H. 2019], we treated the full subcritical regime where $|t| \leqslant L^{-\varepsilon} T_{\text{kin}}$. There, each $u^{(n+1)}$ is better than $u^{(n)}$ by a factor of $L^{-\varepsilon}$. This gives that $u^{(n)} = O(L^{-\varepsilon n})$, and as such, we only needed to expand the solution up to $N = O(\varepsilon^{-1})$ to prove the approximation. The estimation of this long, but finite, expansion is highly nontrivial, but it can be done by exploiting the subtle combinatorial structure of the trees.
- In our critical setting here, the best thing we can hope for $u^{(N)}$ is an estimate of the form $O(\delta^{N/2})$, and for this to be an error term for the approximation (i.e. $\lesssim L^{-\nu}$), we need to have $N \geqslant \frac{\log L}{\log(\delta^{-1})}$, which diverges with L. This creates all sorts of new difficulties.

Factorial divergence

- Recall that $\mathbb{E}|u^{(n)}|^2$ is the sum of $O(n!)$ couples. If all these couples have size $\sim \delta^n$ (as many of them do!), we are doomed to failure.

- **Strategy:** *Classify the couples into groups, such that a) those saturating or almost saturating the worst-case-scenario estimates are relatively few (say $O(C^n)$ instead of $n!$), while b) the remaining (factorially many) couples satisfy much better estimates than the worst-case scenario, i.e. feature a gain of powers of L which is sufficient to offset the factorial loss?*

- In other words, this strategy requires two things:
  A) Identifying those couples with saturated or almost-saturated bounds; one would first hope that these are exactly the couples that converge to the iterates of the (WKE). It is crucial that there are only $O(C^n)$ of them.
  B) A rigidity theorem that says that: once we adequately remove the couples from Step A), we are left with couples that feature a gain that is large enough to offset the factorial number of such couples.
  C) The above handles the estimates on the iterates $u^{(n)}$ and the convergence of the partial sums of the series. Then, we have to prove that the remainder of the expansion is indeed small.

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Regular Couples - Part I

Remark on sums converging to integrals and revisiting scaling laws

- In the following computations, we will need to approximate a sum over the lattice $\mathbb{Z}_L^d$ by an integral over $\mathbb{R}^d$. This takes the caricature form

$$L^{-2d} \sum_{(k_1,k_2)\in\mathbb{Z}_L^{2d}} W(k_1,k_2)\chi(T\Omega) \sim \int_{\mathbb{R}^{2d}} W(k_1,k_2)\chi(T\Omega)\, dk_1 dk_2,$$

  where Ω is a quadratic form like $\Omega(k_1,k_2) = \langle Ak_1, k_2\rangle$, and χ is some cutoff function (assume to be C_0^∞). Also, $T \sim T_{\text{kin}} \sim L^{2\gamma}$.
- For this inequality to hold, we need the equidistribution of the lattice points $\mathbb{Z}_L^{2d}$ in the region

$$\{(k_1,k_2) \in \mathbb{R}^{2d} : |\Omega(k_1,k_2)| \lesssim T^{-1}\}.$$

- If $T \leqslant L$, then this is fairly robust and extends to much more general Ω functions (i.e. other dispersion relations).
- If $T \gg L$, this starts to be a deep question in analytic number theory, and it depends on the diophantine nature of A. For example, if $A = Id$ (square torus), then we don't expect this to be true if $T \geqslant L^2$ (since $\Omega \in L^{-2}\mathbb{Z}$).
- This leads to the condition $T_{\text{kin}} \ll L^2$ on the square torus, which is the range $0 \leqslant \gamma < 1$ on the scaling law. If the torus is generically irrational, then T can be as large as L^{d-}, which gives the bigger range $0 \leqslant \gamma < d/2^-$.

The first iterate of (WKE)/ Physics Derivation

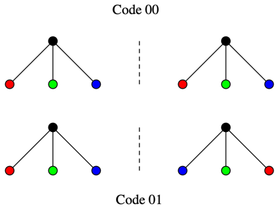
- An explicit computation shows that $\mathbb{E}|a_k^{(1)}(t)|^2$ is given by

$$2t^2\left(\frac{\delta}{2L^{d-\gamma}}\right)^2\sum_{k_1-k_2+k_3=k} n_{\text{in}}(k_1)n_{\text{in}}(k_2)n_{\text{in}}(k_3)\left(\frac{\sin(\delta\pi L^{2\gamma}t\Omega)}{\delta\pi L^{2\gamma}t\Omega}\right)^2$$

$$\sim \frac{\delta t}{2}\,(\delta L^{2\gamma}t)\int_{k_1-k_2+k_3=k} n_{\text{in}}(k_1)n_{\text{in}}(k_2)n_{\text{in}}(k_3)\underbrace{\left(\frac{\sin(\delta\pi L^{2\gamma}t\Omega)}{\delta\pi L^{2\gamma}t\Omega}\right)^2}_{\widetilde{A}(\delta L^{2\gamma}t\Omega);\ \ \widetilde{A}:=(\sin(\pi x)/\pi x)^2\in L^1} dk_1dk$$

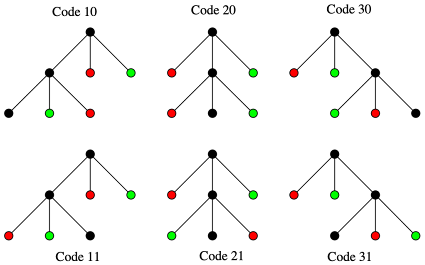
$$\sim \delta t\int_{k_1-k_2+k_3=k} n_{\text{in}}(k_1)n_{\text{in}}(k_2)n_{\text{in}}(k_3)\delta_{\mathbb{R}}(\Omega)dk_1dk_2dk_3.$$

which is part of the first iterate of the wave kinetic equation. This comes from



- We call those couples (1,1) minicouples.

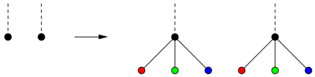
- The rest of the first iterate comes from $\mathbb{E}\, a_k^{(0)} \overline{a_k^{(2)}}$ and $\mathbb{E}\, a_k^{(2)} \overline{a_k^{(0)}}$, which are represented by the pairing of the following trees with a trivial tree with one node.



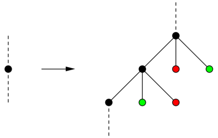
- We call such trees minitrees and the resulting couples (2,0) minicouples. Those minicouples ((1,1) and (2,0)) converge to the first iterate of the kinetic equation.
- *As such, all the remaining iterates of the (WKE) should be obtained* **only** *from couples constructed using the minicouples as building blocks.*
- **Regular Couples** are exactly such couples. They are built using the minicouples above according to the following inductive recipe:

Regular Couples

- Inductive definition of regular couples: The trivial couple formed of two paired nodes is regular. Given a regular couple $\mathcal{Q}$ of order n, then the regular couples of order $n+2$ are obtained by applying one of the following two operations
  - Operation $\mathbb{A}$: Replace two paired leaves in $\mathcal{Q}$ by a $(1,1)$ minicouple.



  - Operation $\mathbb{B}$: Replace any node in $\mathcal{Q}$ by a minitree.



- As such, all regular couples have an even order. They satisfy the estimate $\mathcal{K}_{\mathcal{Q}}(t) \sim \delta^{n/2}$ and their number is C^n. Despite their complexity, they can be computed somewhat "explicitly".
- Remark: Not all regular couples converge to iterates of the (WKE). Only a subfamily thereof does, namely the dominant couples. The non-dominant regular couples either cancel with each other or vanish in the limit.

The enemies and the strategy

The Enemies

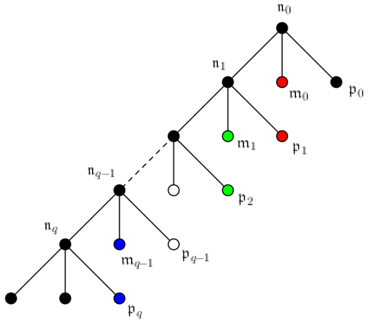
- We just saw that $\mathbb{E}|a^{(1)}(t)|^2 \sim \delta t$. By Gaussian hypercontractivity, we obtain that $|a^{(1)}(t)| \sim \sqrt{\delta}$ with overwhelming probability.

◇ E1) **Criticality.** To derive (WKE) at the kinetic timescale, we need to reach $t = O(1)$, so this means that $a^{(1)}(t) \sim \sqrt{\delta} a^{(0)}(t)$, and more generally, we only have $a^{(n)}(t) < \sqrt{\delta} a^{(n-1)}(t)$ at best. So at best $a^{(n)}(t) \lesssim (\sqrt{\delta})^n$.

- In fact, this estimate is sharp since we saw that $\mathbb{E}|a^{(n)}(t)|^2 = \sum_{\mathcal{Q}} \mathcal{K}_{\mathcal{Q}}$ is a sum over couples of order $2n$, and $\mathcal{K}_{\mathcal{Q}} \sim \delta^n$ when Q is a regular couple of order $2n$.
- This means that we can only stop the expansion after $N \sim \frac{\log L}{\log \delta}$ which diverges with L.

◇ E2) **Factorial Divergence.** The number of couples of order $2n$ is $C^n n!$, so if the best we can show is that their size is at most δ^n (the size of their largest elements), we are doomed.

◇ E3) **Divergent Non-regular couples** Added to the above two expected difficulties is the existence of non-regular couples of order n whose estimates are saturated ($\sim \delta^{n/2}$) or even worse (when $\gamma < 1$). We already encountered one such family, the irregular chains, in our earlier work [DH 2019].



The strategy

1. For most couples of order $2n$, we have that $\mathcal{K}_{\mathcal{Q}} \ll \delta^n$, namely $\delta^n L^{-m}$ with m large enough to offset the factorial losses somehow.
2. In the worst possible scenario, we have that $\mathcal{K}_{\mathcal{Q}} \sim \delta^n$ but this happens only for C^n couples rather than $C^n n!$ of them.
3. Any non-regular couples with worse bounds (e.g. irregular chains) have very specific structures that allow to uncover hidden **cancellations** between them.

• Points 1) and 2) are given by the Rigidity theorem, which identifies precisely the families of couples with (almost-)saturated bounds, and the gain in the remaining ones. Point 3) is based on a case by case study of each of those families of large couple identified in the Rigidity Theorem.

The Rigidity Theorem

Classification of Couples I: Couples with saturated bounds

- The rigidity theorem identifies the of couples that have almost saturated estimates:
- One such family are the regular couples, which are the leading ones that can be matched (order by order) to the iterates of the wave kinetic equation (up to some couples whose contribution cancel out in the limit).
- Another family, which are the irregular chains, also lead to saturated estimates when $\gamma = 1$ and even worse estimates when $\gamma < 1$. This is the one we saw above, and was identified in [DH'19]. Fortunately, they exhibit an **elaborate cancellation of couples**.
- These are the only divergent families that appear for $\gamma > 2/3$. For $\gamma < 2/3$ several new families of divergent couples arise. We will discuss them later.
- The good news is that there only $O(C^n)$ such couples, and each such family has a precise structure that allows to analyze it separately.

Classification of couples II

- If one performs a type of surgery on an arbitrary couple to remove all its regular sub-couples, all its irregular chains, and all other structures that lead to bad estimates on the previous slide, we are left with a reduced structure of size r. This r measures how far the original couple is from saturating the estimates.
- **Key points of the rigidity theorem:**
  1. The estimate on this reduced couple features a gain L^{-cr}!
  2. The number of possibilities of such structures is $C^N r!$, so this gain of L^{-cr} is enough to offset the factorial divergence since

$$L^{-cr}C^N r! \ll L^{-\nu} \quad \text{if} \quad r \leqslant N \sim \log L.$$

- The proof of this rigidity theorem involves recasting the problem of counting couples into counting another family of combinatorial structures that we call molecules. This is followed by running something akin to a “computer program” to reduce and count these molecules effectively.

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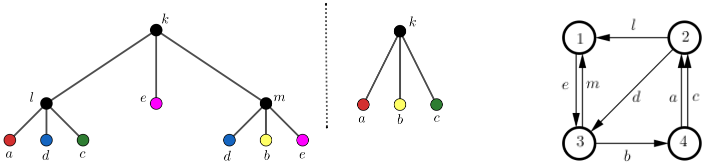


- We shall transform the couple into graphs we call molecules, which are more flexible than couples for the purpose of estimates. Basically, we choose any parent node and its three children, and call this 4−node subset an atom; then draw bonds between atoms with common vectors to form a molecule.
- We can then apply a carefully designed scanning algorithm to the molecule (which will remain a molecule throughout the process) and prove the counting estimate with the desired gain.

Cancellations in the Feynman Series

Cancellation I

- There are couples, such as the *irregular chains* we saw before, that have intrinsic divergence; however, there are *cancellation* structures between different couples:

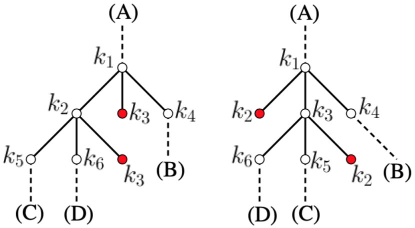


Figure: Irregular chains and cancellation. Here k_j are wave numbers, and the remaining parts of the couple, denoted by (A)–(D), are the same. We call those two couples **twists**.

Cancellation II

- When $\gamma \approx 1$ the irregular chains are the only divergent structures.
- When $\gamma < 2/3$, *much more complicated* divergence structures arise.
- Miraculously, they *still come in pairs of cancellation*! However, this cancellation procedure becomes considerably more elaborate.

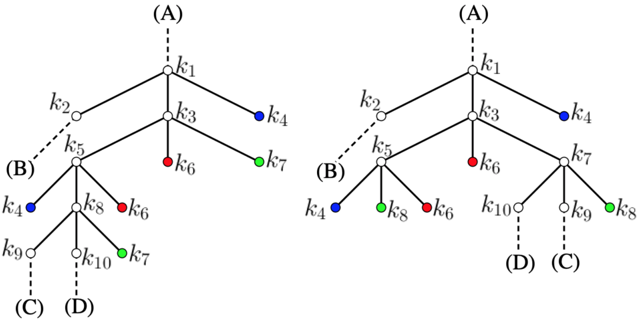


Figure: New divergent structures and their **twists**. Miracle: Those couples cancel each other!

Molecules

- We do not yet have a *physical* interpretation of these.
- However the notion of *molecules* plays a big role in both identifying those divergent structures and finding the cancellation.
- **Key:** Canceling couples correspond to *exactly the same* molecule.

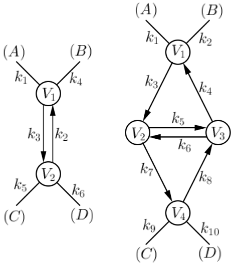


Figure: Molecules coming from canceling couples. Left: irregular chains. Right: new divergent structures.

Couples, molecules, and algorithm

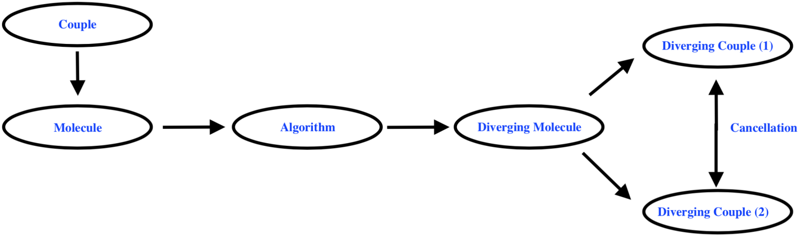


Figure: An illustration of the main ideas and steps leading to the proof and discovery of diverging couples and cancellation. The estimation/counting algorithm plays the key role here.

Regular couples - Part II

Why are they leading?

Reduction to prime couples.

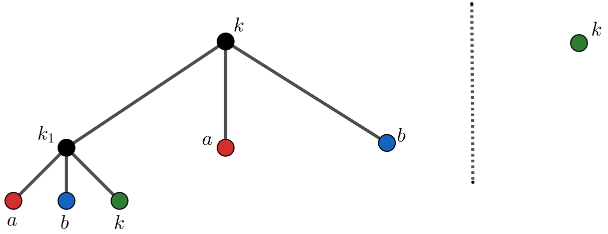
Basic couple estimates

- In caricature, the expression $\mathcal{K}_\mathcal{Q}$ is of the form

$$\sum_{(k_\mathfrak{n})\in\mathcal{D}} \mathcal{B}(t,\Omega_\mathfrak{n})\mathbf{1}_{|k_\mathfrak{n}|\leqslant 1} \sim \sum_{m_\mathfrak{n},\mathfrak{n}\in\mathcal{N}_\mathcal{Q}} B(t,m_\mathfrak{n}) \sum_{\substack{(k_\mathfrak{n})\in\mathcal{D}\\ \Omega_\mathfrak{n}\sim m_\mathfrak{n}}} \mathbf{1}_{|k_\mathfrak{n}|\leqslant 1}$$

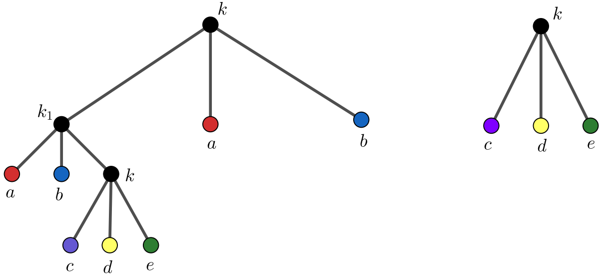
  So, if $\sum_{m_\mathfrak{n}} |B(t,m_\mathfrak{n})|$ is bounded, we are reduced to obtaining a uniform bound on the number of decorations of the couple such that $\Omega_\mathfrak{n} = m_\mathfrak{n} + O(L^{-2})$. Here assume that $\gamma = 1$ for definiteness.

- For the (2,0) minicouple below, this is the number of $(k_1, a, b) \in \left(\mathbb{Z}_L^d \cap B(0,1)\right)^3$ such that: $k_1 - a + b = k$, $\quad |k_1|^2 - |a|^2 + |b|^2 - |k|^2 = m + O(L^{-2})$



This number is $\sim L^{2d-2}$ (3-vector counting).

Order-4 regular couple

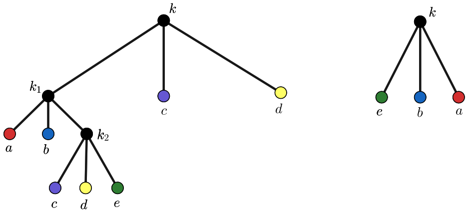


Here the size of $\mathcal{K}_{\mathcal{Q}}$ is bounded by the number of (k_1, a, b, c, d, e) such that

$$k_1 - a + b = k, \qquad |k_1|^2 - |a|^2 + |b|^2 - |k|^2 = m_1 + O(L^{-2})$$
$$c - d + e = k, \qquad |c|^2 - |d|^2 + |e|^2 - |k|^2 = m_2 + O(L^{-2})$$

So the four sets of quadratic relations imposed at each branching node collapse to just two decoupled independent relations. Each is a 3-vector counting problem, so the number of choices is $L^{2d-2} \times L^{2d-2}$ by applying the 3-vector counting estimate twice.

Order-4 non-regular couple



Here the counting problem for $(a, b, c, d, e, k_1, k_2)$ is given by

$$k_1 - c + d = k, \qquad |k_1|^2 - |c|^2 + |d|^2 - |k|^2 = m_1 + O(L^{-2})$$
$$a - b + k_2 = k_1, \qquad |a|^2 - |b|^2 + |k_2|^2 - |k_1|^2 = m_2 + O(L^{-2})$$
$$e - b + a = k, \qquad |e|^2 - |b|^2 + |a|^2 - |k|^2 = m_3 + O(L^{-2})$$

Counting (e, b, a) first gives L^{2d-2} choices, and leaves us with counting (k_1, k_2, c, d):

$$k_1 - c + d = k, \qquad |k_1|^2 - |c|^2 + |d|^2 - |k|^2 = m_1 + O(L^{-2})$$
$$k_2 - k_1 = b - a, \qquad |a|^2 - |b|^2 + |k_2|^2 - |k_1|^2 = m_2 + O(L^{-2})$$

which features an improved bound of $L^{2d-2-\frac{1}{4}}$ over the "trivial" bound L^{2d-2}.

Conclusion for regular couples

- The very first step of the proof is to analyze the expressions $\mathcal{K}_{\mathcal{Q}}(t,s,k)$ when $\mathcal{Q}$ is a regular couple. With some effort, this can be computed "almost explicitly" after some effort. It satisfies the estimate

$$\left\|\widehat{\mathcal{K}_{\mathcal{Q}}}(\lambda_1,\lambda_2,k)\right\|_{L^1_{\lambda_1,\lambda_2}} \leqslant (C\delta)^{\frac{n}{2}}\langle k\rangle^{-40d}.$$

- This means that effectively, one can think of $\mathcal{K}_{\mathcal{Q}}(t,s,k)$ as a linear combination of $e^{i\lambda_1 t}e^{i\lambda_2 t}\langle k\rangle^{-40d}$. So they basically behave like $n_{\text{in}}(k)$.
- This allows to collapse an arbitrary couple $\mathcal{Q}$ into a smaller one $\mathcal{Q}_{sk}$ (skeleton couple) which contains no regular subcouples inside it. We call such couples *prime*.
- As such, the contribution of all non-regular couples can be written as a sum over prime couples $\mathcal{Q}_{sk}$. The aim would be to show that this contribution is an error term. Yu Deng will continue from here tomorrow.

A remark on higher order statistics

- The key point here is to derive the asymptotics of the higher moments like

$$\mathbb{E}\left(a_{k_1}^{\pm} \dots a_{k_r}^{\pm}\right).$$

- This leads to structures more general than couples, namely gardens, which are r trees whose leaves are paired to each other in sets of size two. In the non-Gaussian setting, one also has to introduce overgardens which are r trees whose leaves are paired to each other in sets of size possibly larger than two (over-pairing).
- The analysis of gardens and overgardens follows a similar methodology as for couples, but features many interesting and new features. The asymptotics of the special moments $\mathbb{E}\left(|a_{k_1}|^2 \dots |a_{k_r}|^2\right)$ are shown to be given by solutions of the so-called wave kinetic heierarchy with factorized initial data.
- The distinction between Gaussian and non-Gaussian initial distribution starts to exhibit itself when higher powers of $|a_{k_j}|^2$ are present, at which point overgardens start to have leading order contributions. We obtain the limiting dynamics of such moments, like $\mathbb{E}|a_k|^{2p}$, which seem to be new in the literature.
- Once the limit dynamics of all the higher moments is derived, one can deduce the equation for the PDF thanks to the uniqueness of the moment problem in our setting.

Thanks for your attention!