Computational methods for acoustic inverse problems

Lauri Oksanen
University of Helsinki

Based on a joint work with
Maarten de Hoop and Paul Kepley

Inverse boundary value problem

Consider the wave equation

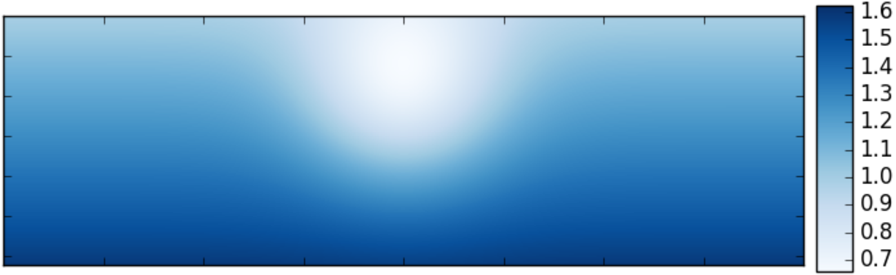
$$\begin{aligned}
&\partial_t^2 u - c^2(x)\Delta u = 0, \quad \text{in } (0,\infty)\times\Omega,\\
&\partial_\nu u|_{x\in\partial\Omega} = f,\\
&u|_{t=0} = \partial_t u|_{t=0} = 0,
\end{aligned}$$

and define the Neumann-to-Dirichlet map by $\Lambda f = u|_{x\in\partial\Omega}$.
This corresponds to Green's function on the surface $\partial\Omega$ of the region Ω.

Inverse problem. Determine c given Λ.

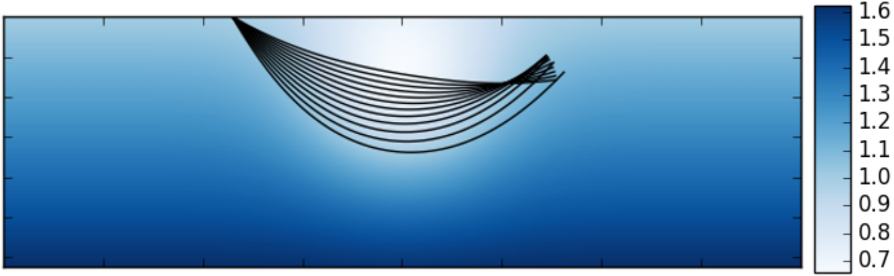
Global uniqueness holds, that is, Λ determines c without any further information (such as a good initial guess for c).

Example



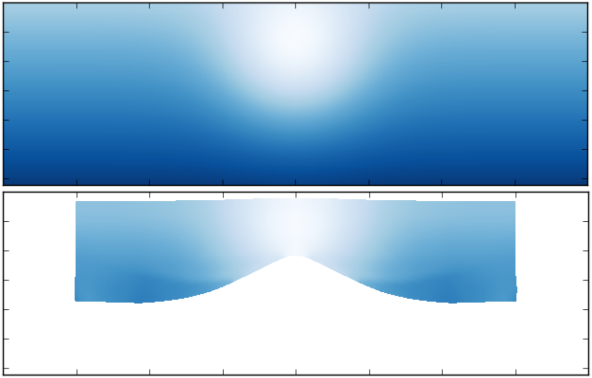
Speed of sound $c(x)$. Surface is on top.
The speed increases in the depth and there is a slow region in the middle.

Example



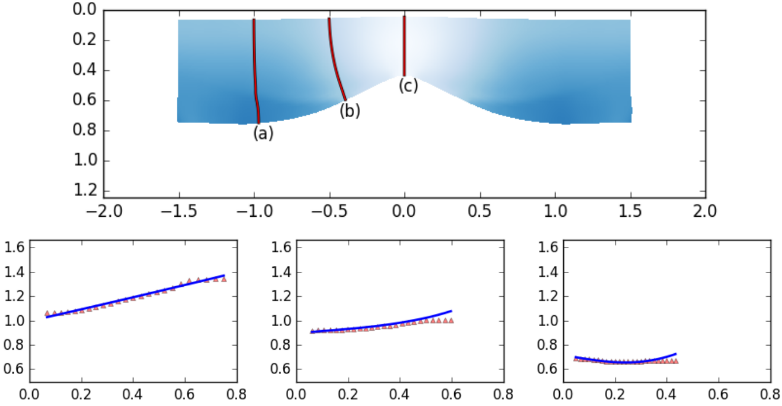
Shortest paths with respect to travel time focus behind the lens.
These are geodesics with respect to the Riemannian metric $c^2(x)dx^2$.

Reconstruction



Reconstruction from simulated measurements with 241 point like sources on the top edge [de Hoop-Kepley-L.O.'18]

Reconstruction



(a) (b) (c)

True c (blue) and its reconstuction (red) along a shortest path

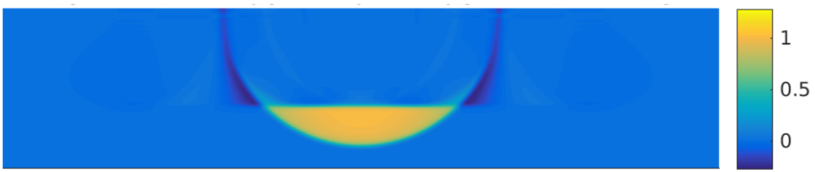
Boundary Control method

Inverse problem. Determine c given Λ.

- Solved using the **Boundary Control** method [Belishev'87]
- The method is based on hyperbolic unique continuation [Tataru'95] which is unstable
- Regularization can be used to trade resolution for stability
- Prospect: find a good initial guess for data fitting, for example, full waveform inversion

Rough idea of the Boundary Control method

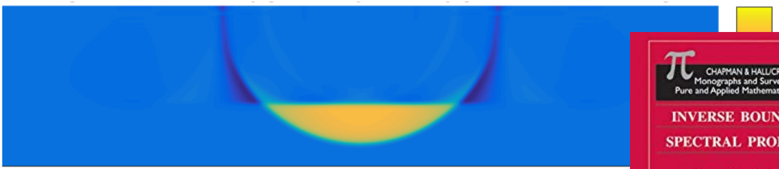
1. Construct localized waves by solving blind control problems
2. Probe the unknown medium by using localized waves



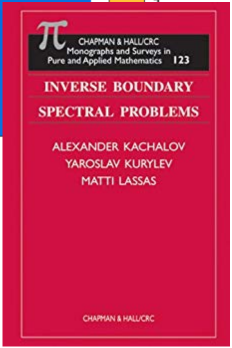
Localized pressure field at a fixed time, computed in the case $c = 1$

Rough idea of the Boundary Control method

1. Construct localized waves by solving blind control problems
2. Probe the unknown medium by using localized waves



Localized pressure field at a fixed time, computed in



Control problem from the boundary

Given a function ϕ on Ω and a regularization parameter $\alpha > 0$, **minimize**

$$\left\| u^f(T) - \phi \right\|_{L^2(\Omega)}^2 + \alpha \left\| f \right\|_{L^2((0,T)\times\partial\Omega)}^2$$

subject to u^f satisfying the wave equation

$$\begin{aligned}
&\partial_t^2 u - c^2(x)\Delta u = 0, \quad \text{in } (0,\infty)\times\Omega,\\
&\partial_\nu u|_{x\in\partial\Omega} = f,\\
&u|_{t=0} = \partial_t u|_{t=0} = 0.
\end{aligned}$$

When $T > 0$ is large enough, the unique minimizer f_α satisfies

$$u^{f_\alpha}(T) \to \phi, \quad \alpha \to 0.$$

Blind control problem

Some control problems can be solved without knowing the speed of sound:

$$\operatorname*{argmin}_{f} \left\| u^f(T) - 1 \right\|^2_{L^2(\Omega; c^{-2}dx)} + \alpha \|f\|^2_{L^2((0,T)\times\partial\Omega)} \tag{1}$$

subject to u^f satisfying the wave equation.

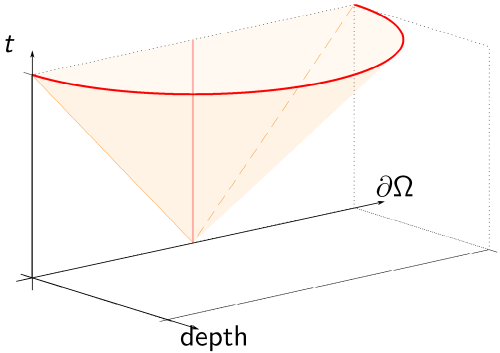
Minimization (1) is equivalent with solving $(K+\alpha)f = b$, where

$$K = J\Lambda - R\Lambda RJ, \quad b(t) = T - t.$$

Here Λ is the Neumann-to-Dirichlet map, and

$$Jf(t) = \frac{1}{2}\int_t^{2T-t} f(s)ds, \quad Rf(t) = f(T-t).$$

Finite speed of propagation and supports

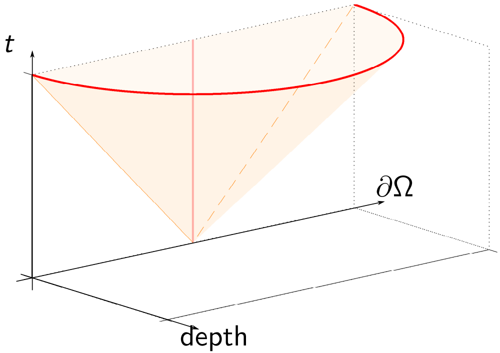


If the source f vanishes outside the pink line,

then u vanishes outside the cone,

and $u(T)$, for fixed T, vanishes outside the half disk inside the red arc.

Finite speed of propagation and supports



If the source f vanishes outside the pink line,

then u vanishes outside the cone,

and $u(T)$, for fixed T, vanishes outside the half disk inside the red arc.

$\mathrm{supp}(f)$	$\mathrm{supp}(u(T))$
	
	

Blind control problem with a support constraint

Let $\Gamma \subset \partial\Omega$ and $r \in (0, T)$. Consider the minimization

$$\operatorname*{argmin}_f \left\| u^f(T) - 1 \right\|^2_{L^2(\Omega; c^{-2}dx)} + \alpha \|f\|^2_{L^2((0,T)\times\partial\Omega)}$$

subject to u^f satisfying the wave equation and

$$\operatorname{supp}(f) \subset [T - r, T] \times \Gamma.$$

The unique minimizer f_α satisfies

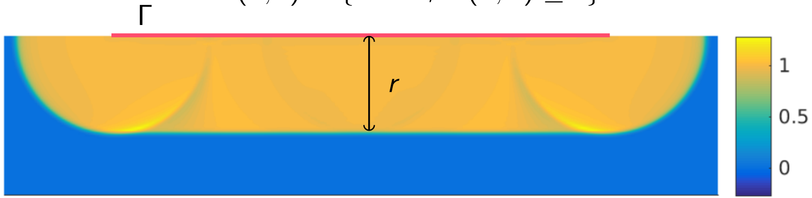
$$\lim_{\alpha\to 0} u^{f_\alpha}(T, x) = 1_{M(\Gamma,r)}(x) := \begin{cases} 1, & x \in M(\Gamma, r) \\ 0, & \text{otherwise} \end{cases},$$

where $M(\Gamma, r) \subset \Omega$ is the maximal set that the waves from Γ can reach in r time units. It is often called a domain of influence.

Different support constraints

Let $d(x, y)$ be the travel time between a pair of points in Ω (i.e. the distance function on the Riemannian manifold $(\Omega, c^{-2}dx^2)$). Then

$$M(\Gamma, r) = \{x \in \Omega;\ d(x, \Gamma) \leq r\}.$$



Blind control problem with wide Γ

Blind control problem with narrow Γ and larger r

Localized waves in theory

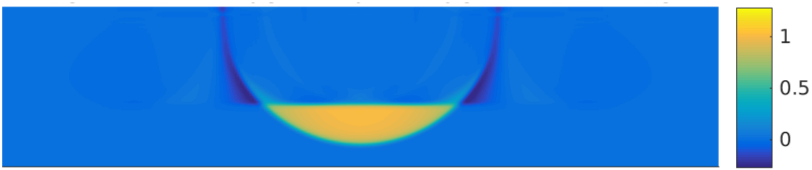
1. Support of f constrained in the union of the wide and narrow Γ

2. Support of f constrained in the wide Γ alone

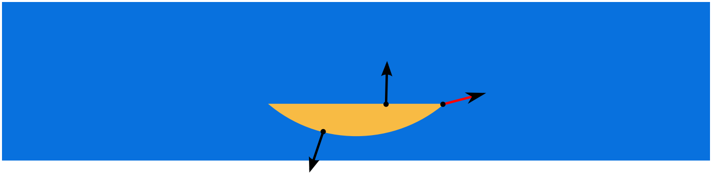
3. Difference

Localized waves in practice

Localized waves have artifacts due to the instability of the control problem



The sharp corners can not be produced in a stable way by using wave fronts coming from the surface on top[1]



[1]This can be turned to a theorem using microlocal analysis

Ongoing work: regularization of blind control problems

Regularization on the discrete level after a finite element discretization

- Gives optimal convergence for **stable** control problems [Burman-Feizmohammadi-Münch-L.O.]
- Gives optimal convergence for unstable unique continuation problems in the frequency domain [Burman-Nechita-L.O.'19]

Open questions

- How to treat **unstable** control problems?
- How to treat **blind** control problems?