Statistical Methods (Autumn 2017): Homework Problems IV

October 13, 2017

1 Problem 1

We wish to calculate the volume of a sphere with radius $R = 1$(a.u.) without using formulas for the volume, explicit integration or anything that requires information about the value of π. This can be done in Cartesian coordinates by a Monte Carlo study using the acceptance-rejection method enclosing the sphere inside a cube.

Solution:

1. Randomly pick points with x, y and z in $[-1, 1]$. This corresponds to picking points inside the smallest possible cube that can enclose a sphere of radius $R = 1$ and this cube has the volume $V_{cube} = 2 * 2 * 2 = 8$.
2. Calculate the distance of the point from the origin (which is also the center of the cube and the sphere): $r = \sqrt{x^2 + y^2 + z^2}$. If $r \leq R$, the point is also inside the sphere and it is accepted. If not, it is rejected.
3. The ratio of accepted points over the total number of points tried q is equal to the ratio of the volume of the sphere and the volume of the cube at the limit of infinite tries. The volume is $V_{sphere} = V_{cube}q$.

Naively, a relative error of 1% means $\sqrt{1/\mathcal{O}(N)} = 1$, so use 10000 tries/run to find the ratio $q = 0.5226$ and $V_{sphere} = 8 * 0.5226 = 4.181 \pm 0.032$ (average over 20 runs of the code, the error quoted is the standard deviation. If you assume binomial errors, the average error calculated is ± 0.040).

ii) Now using the volume we found in 1i), $\pi = \frac{3}{4*1^3} V \approx 3.136 \pm 0.024$, $\sigma_\pi/\pi = \sigma_V/V \approx 0.77\% < 1\%$.

2 Problem 2

The MATLAB code is in the appendix. The cumulative Gaussian distributions are integrated using the built-in quad function, which provides high (enough) precision values for the integrals. The test statistic for muons and electrons is shown in figure 1.

i) The muon selection efficiency ε_μ requiring $w < -1.5$ is given by:

$$\varepsilon_\mu = \int_{-\infty}^{-1.5} g(w|\mu) dt, \tag{1}$$

where $g(w|\mu)$ is the Gaussian centered at $\mu_\mu = -4.5$ (and with $\sigma = 1.5$):

$$g(w|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(w-\mu_\mu)^2/(2\sigma^2)}. \tag{2}$$

Calculations in MATLAB show that this is $\varepsilon_\mu \approxeq 0.98$.

ii) This probability ε_e is calculated similarly ($\mu_e = 0$):

$$\varepsilon_e = \int_{-\infty}^{-1.5} g(w|e) dt, \tag{3}$$

$$g(w|e) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(w-\mu_e)^2/(2\sigma^2)}. \tag{4}$$

This gives the probability $\varepsilon_e \approxeq 0.16$.

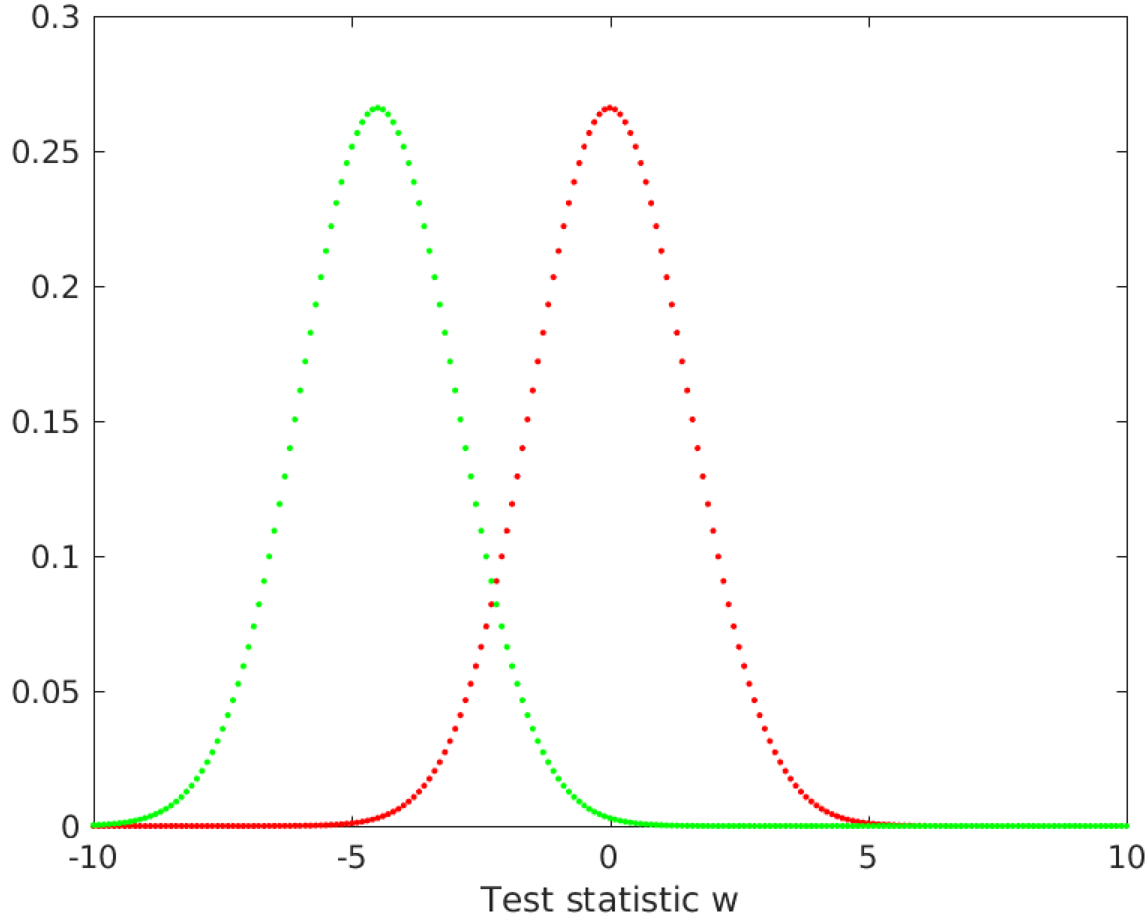

Figure 1: Plot of the test statistic w for muons (green) and electrons (red).

iii) The purity is given by the formula below (where $a_\mu = 0.10$ and $a_e = 0.90$ are the a priori probabilities of the muon and electron hypotheses). This formula is Bayes' theorem for a continuous w (Bayes' theorem was used in problem set 1 for a discrete variable, now the sum is replaced by an integral).

$$p_\mu = \frac{\int_{-\infty}^{-1.5} a_\mu g(w|\mu)dw}{\int_{-\infty}^{-1.5} a_e g(w|e) + a_\mu g(w|\mu)dw} = \frac{a_\mu \int_{-\infty}^{-1.5} g(w|\mu)dw}{a_e \int_{-\infty}^{-1.5} g(w|e)dw + a_\mu \int_{-\infty}^{-1.5} g(w|\mu)dw} \tag{5}$$

Computations in MATLAB show that this is $p_\mu \approxeq 0.41$ for the cut $w < 1.5$.

iv) Lowering the cut parameter reduces electron contamination but also reduces the number of signal events. The optimal cut for a purity of 2/3 can be found by gradually lowering the cut and checking the purity after each change. This can be quickly done using a while loop. The results calculated in MATLAB show that a cut $w < -2.46$ gives a purity $p_\mu \approxeq 0.67$.

A MATLAB code for ex. 1 Ex1.m

```
%iterate 1E4 times
iter = 10000;
q=zeros(20,1);
qerr=zeros(20,1);

for j = 1:20
```

```
    %num in sphere
    ns = 0;
    for i = 1:iter
       %random coordinates uniformly between -1 & +1
       x = -1+2*rand();
       y = -1+2*rand();
       z = -1+2*rand();
       %distance from center of cube & sphere
       r = sqrt(x^2+y^2+z^2);
       if r <= 1
          ns = ns+1;
       end
     end
     q(j)=ns/iter;
     qerr(j)=sqrt(ns*(1-ns/iter))/iter; %naive binomial error
end
vol=q*8
volerr=qerr*8
mean(vol)
mean(volerr) %naive error
std(vol) %actual standard deviation

%1ii: pi=0.75*V

pival=mean(vol)*0.75
pierr=std(vol)*0.75
```

B MATLAB code for ex. 2 Ex2.m

```
format compact;
w = -10:0.1:10;
mue = 0.0; %Mean for electrons
mumu = -4.5; %Mean for muons
sigma = 1.5; %Standard deviation

amu = 0.10; %Muon probability in sample
ae = 0.90; %Electron probability in sample

thecut = -1.5; %Cut given

ge = normpdf(w,mue,sigma);
gmu = normpdf(w,mumu,sigma);

plot(w,ge,'.r')
hold on
plot(w,gmu,'.g')
xlabel 'Test statistic w'
print -dpng 'TestStat.png'

display 'Muon efficiency: '
muoneff = quad(@(x)normpdf(x,mumu,sigma),-1000000,thecut)

display 'Accepted electron probability: '
electronprob = quad(@(x)normpdf(x,mue,sigma),-1000000,thecut)

display 'Purity given wcut = -1.5'
```

```
purity = amu*muoneff/(amu*quad(@(x)normpdf(x,mumu,sigma),-1000000,thecut)...
    +ae*quad(@(x)normpdf(x,mue,sigma),-1000000,thecut))
wcut = thecut;
newpurity = purity;

while(newpurity<2/3)
    wcut = wcut - 0.001;
    newpurity = amu*quad(@(x)normpdf(x,mumu,sigma),-1000000,wcut)/...
        (amu*quad(@(x)normpdf(x,mumu,sigma),-1000000,wcut)...
        +ae*quad(@(x)normpdf(x,mue,sigma),-1000000,wcut));
end
display 'w-cut value: '
wcut
display 'Purity: '
newpurity
```