ASPECTS OF PROBABILISTIC LITTLEWOOD–PALEY THEORY IN BANACH SPACES

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Abstract. A probabilistic square function estimate, due to Varopoulos in the scalar case, is proved for functions with values in a UMD Banach space. As an application, close to optimal dimension-free norm bounds for certain spectral multipliers of the Laplacian in Bôchner spaces are obtained. The proof uses stochastic integration of vector-valued processes and especially decoupling estimates due to Garling.

Dedicated to Professor Nigel Kalton on the occasion of his (quasi-)60th birthday

1. Introduction

The notion of square-functions originating from J. E. Littlewood and R. E. A. C. Paley, and their use in the estimation of singular integrals, are among the central ideas in Harmonic Analysis. The theory around these topics is also an area where deep connections to Probability manifest themselves, and several authors (in particular, E. M. Stein [19], P. A. Meyer [14], and N. Th. Varopoulos [21]) have exploited stochastic methods to reprove and improve results of the classical Littlewood–Paley theory. Besides offering insight into the original problems, the probabilistic approach has become an indispensable tool in developing Harmonic Analysis of Banach space -valued functions, a theory pioneered by J. Bourgain [1, 2], D. L. Burkholder [3, 4], and T. R. McConnell [12] in the early 80's. This program has further elaborated on the relationships of Harmonic and Stochastic Analysis by showing that the probabilistic UMD property of a Banach space X is equivalent to the validity of several estimates from Harmonic Analysis for X-valued functions.

Quadratic estimates, reformulated in a "randomized" way, have also played a crucial rôle in the vector-valued developments. This theory, however, was essentially restricted to discretely parameterized square functions, until N. J. Kalton and L. Weis's development of a more general framework, which has been a source of great inspiration to, I dare say, everyone who has had the chance to learn about the ideas contained in their still unpublished manuscript [11]. Using equivalents of these new square functions, although formulated in the different language of vector-valued stochastic integrals (cf. [15, 18]), I was recently [10] able to characterize the UMD property in terms of a vector-valued extension of Stein's [19] abstract Littlewood–Paley-type estimates.

The purpose of the present paper is to carry over (parts of) the Littlewood–Paley theory of Varopoulos [21] to the Banach space situation. Varopoulos's approach has a stronger probabilistic flavour than Stein's, and to reach the goal, I will need to employ the stochastic Itô integration of vector-valued *processes*, as opposed to the easier Wiener integration of (deterministic) functions used in [10]. The required integration theory, based on decoupling estimates due to D. J. H. Garling [7], has been developed by McConnell [13], with considerable recent elaboration by J. M. A. M. van Neerven, M. C. Veraar and Weis [16]. I wish to acknowledge that the application of this more advanced stochastic machinery to the vector-valued Harmonic Analysis was first suggested to me by Jan van Neerven, during the time of my finishing the work [10].

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Compared to the Littlewood–Paley–Stein theory [19], which is valid for a general diffusion semigroup T^y on an abstract measure space, the methods of Varopoulos are more restricted (requiring, in particular, some topological structure on the underlying measure space), and I have here chosen to work in an even narrower setting, where T^y is the classical Gaussian semigroup on $\mathbb{R}^n$. One motivation for treating such a specialized case is the considerable simplification of the argument that is obtained here, even to the extent that I feel that the present proofs are the "right" ones for the theorems stated in this paper. There is also a more substantial reason: the new approach is very efficient in the use of the required martingale estimates in that fairly good constants in the various inequalities are obtained. In particular, as a corollary it is found that the norm of the imaginary powers of the Laplace operator, $(-\triangle)^{i\gamma}$, in the limit as $\gamma \to 0$, satisfies an upper bound involving the UMD constant, which differs from the known lower bound due to S. Guerre-Delabrière [9] only by a factor of 2. Thus, not more than this numerical factor is lost in the present estimates. Moreover, it is found that some of the Varopoulos-type square-function estimates only require "one half" of the UMD property (in the sense of the one-sided random martingale transform inequalities introduced by D. J. H. Garling [8]), so that more precise than before relations of the probabilistic and analytic estimates are achieved.

2. Preliminaries

The UMD condition and related notions. Let us recall the fundamental UMD condition of Burkholder [3]. A Banach space X has the property of *unconditional martingale differences*, for short UMD, provided that the best constant $\beta_{p,X}$ is finite for one (and then all; see [3]) $p \in]1,\infty[$ in the following inequality:

$$\Big(\mathsf{E}\Big|\sum_{j=1}^N \epsilon_j d_j\Big|^p\Big)^{1/p} \leq \beta_{p,X}\Big(\mathsf{E}\Big|\sum_{j=1}^N d_j\Big|^p\Big)^{1/p}, \tag{2.1}$$

which should hold for all $N \in \mathbb{N}$, all signs $(\epsilon_j)_{j=1}^N \in \{-1,+1\}^N$, any probability space $(\Omega, \mathscr{F}, \mathsf{P})$ (with expectation $\mathsf{E} := \int_\Omega (\cdot)\,\mathrm{d}\mathsf{P}$) equipped with a filtration $\mathscr{F}_1 \subseteq \ldots \subseteq \mathscr{F}_N \subseteq \mathscr{F}$, and all martingale difference sequences $(d_j)_{j=1}^N \in L^p(\Omega,\mathscr{F},\mathsf{P};X)^N$ adapted to this filtration (i.e., $d_j \in L^p(\Omega,\mathscr{F}_j,\mathsf{P};X)$ and $\mathsf{E}[d_j|\mathscr{F}_{j-1}] = 0$ for all $j = 2,\ldots,N$).

We also consider some further estimates closely related to (2.1), where the quantities common with (2.1) have the same range as there. Let $\beta^{\mathbb{C}}_{p,X}$ be the best constant in

$$\Big(\mathsf{E}\Big|\sum_{j=1}^N \zeta_j d_j\Big|^p\Big)^{1/p} \leq \beta^{\mathbb{C}}_{p,X}\Big(\mathsf{E}\Big|\sum_{j=1}^N d_j\Big|^p\Big)^{1/p}, \tag{2.2}$$

where the ζ_j are arbitrary complex numbers of absolute value $|\zeta_j| = 1$. It is clear that $\beta_{p,X} \leq \beta^{\mathbb{C}}_{p,X}$ and a splitting into real and imaginary parts plus a standard convexity argument shows that $\beta^{\mathbb{C}}_{p,X} \leq 2\beta_{p,X}$. By more careful reasoning, using Lemma 4.11.5 of [17], this may be improved to $\beta^{\mathbb{C}}_{p,X} \leq \pi/2 \cdot \beta_{p,X}$.

There is a natural splitting of the inequality (2.1) into two parts. Let us denote by $\beta^{\pm}_{p,X}$ the best constants in the following estimates, where the ε_j designate independent *random* signs on another probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathsf{P}})$ with expectation $\tilde{\mathsf{E}}$, distributed by the symmetric law $\tilde{\mathsf{P}}(\varepsilon_j = -1) = \tilde{\mathsf{P}}(\varepsilon_j = +1) = 1/2$:

$$\frac{1}{\beta^-_{p,X}}\Big(\mathsf{E}\Big|\sum_{j=1}^N d_j\Big|^p\Big)^{1/p} \leq \Big(\mathsf{E}\tilde{\mathsf{E}}\Big|\sum_{j=1}^N \varepsilon_j d_j\Big|^p\Big)^{1/p} \leq \beta^+_{p,X}\Big(\mathsf{E}\Big|\sum_{j=1}^N d_j\Big|^p\Big)^{1/p}. \tag{2.3}$$

It is easy to see that $\max(\beta^-_{p,X}, \beta^+_{p,X}) \leq \beta_{p,X} \leq \beta^-_{p,X}\beta^+_{p,X}$. The two Banach space properties defined by the inequalities (2.3) have been studied by Garling [8] who showed, among other results, their independence from $p \in]1,\infty[$. He called the first and second inequality in (2.3) the *lower and upper estimate for random martingale transforms*, for short LERMT and UERMT, respectively. van Neerven, Veraar and Weis have proposed the alternative names UMD$^-$ and UMD$^+$, which are more suggestive of the close connection between (2.1) and (2.3).

We are going to make use of a well-known theorem of Burkholder [3] which shows that the UMD property (2.1) (or (2.2)) self-improves into an estimate for more general *martingale transforms.* We say that $(v_j)_{j=1}^N \in L^\infty(\Omega, \mathscr{F}, \mathsf{P}; \mathbb{C})^N$ is a *predictable* sequence (adapted to the filtration $(\mathscr{F}_j)_{j=1}^N$) if $v_j \in L^\infty(\Omega, \mathscr{F}_{(j-1)\vee 1}, \mathsf{P}; \mathbb{C})$ for each $j = 1, \ldots, N$. If $(d_j)_{j=1}^N$ is a martingale difference sequence adapted to this same filtration, then so is $(v_j d_j)_{j=1}^N$. Only the case with $v_j \in L^\infty(\Omega, \mathscr{F}_{(j-1)\vee 1}, \mathsf{P}; \mathbb{R})$ is explicitly treated in [3], but the complex-valued result follows in the same way.

2.4. **Theorem** ([3])**.** *Let X be a UMD space, $1 < p < \infty$, and $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space with filtration $(\mathscr{F}_j)_{j=1}^N$. For every martingale difference sequence $(d_j)_{j=1}^N$ and every predictable sequence $(v_j)_{j=1}^N$ adapted to this filtration, there holds*

$$\Big(\mathsf{E}\Big|\sum_{j=1}^N v_j d_j\Big|^p\Big)^{1/p} \leq \beta_{p,X}^{\mathbb{C}} \max_{1\leq j\leq N} \|v_j\|_\infty \Big(\mathsf{E}\Big|\sum_{j=1}^N d_j\Big|^p\Big)^{1/p}.$$

If the v_j are real-valued, $\beta_{p,X}^{\mathbb{C}}$ may be replaced by $\beta_{p,X}$ in the above estimate.

Stochastic integration. We next recall the elements of the theory of stochastic integration, which are relevant for the present purposes. Let $x(t)$ and $y(t)$, $t \geq 0$, be independent standard Brownian motions with continuous paths on $\mathbb{R}^n$ and $\mathbb{R}$, respectively, and $z(t) := (x(t), y(t))$. We write $\mathsf{P}_z \equiv \mathsf{P}_x \otimes \mathsf{P}_y$ for the probability on a measurable space $(\Omega, \mathscr{F})$ governing the motion of $z(t)$ with initial value $z(0) = z = (x, y) \in \mathbb{R}^n \times \mathbb{R}$; this is the Cartesian product of two independent processes as indicated by the notation. Similarly, we write $\mathsf{E}_z = \mathsf{E}_x \otimes \mathsf{E}_y$ for the related expectations. Let us finally introduce the Brownian motions $\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))$, $t \geq 0$, which are independent copies of the above mentioned processes with starting point at 0. For convenience, we may take them to be defined on a different probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathsf{P}})$, with expectation $\tilde{\mathsf{E}}$. Let $(\mathscr{F}_t)_{t\geq 0}$ be any filtration (an increasing family of sub-σ-algebras of $\mathscr{F}$) such that $z(s)$ is $\mathscr{F}_t$-measurable for all $0 \leq s \leq t < \infty$, and let $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ be similarly related to $\tilde{z}(t)$.

We then move to the vector-valued situation. Let X be a Banach space. In accordance with the stochastic integration theory of [16], we take the scalar field to be $\mathbb{R}$; in particular, by the dual space we understand $X' := \mathscr{L}(X, \mathbb{R})$. Observe that a complex Banach space may always be viewed also as a real Banach space, where all the original operations are still defined, but we have to think of the mapping $X \in \xi \mapsto \lambda\xi \in X$ not as a scalar multiplication but as a bounded linear operator on X when $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Let us consider $\phi = (\phi_k)_{k=1}^{n+1} : [0, \infty[\times \Omega \to X^{n+1}$, where X is a Banach space. We identify $X^{n+1} \eqsim \mathscr{L}(\mathbb{R}^{n+1}, X)$, in particular when computing the norm in this space. Such a function is called *elementary adaptive* [16] if

$$\phi_k = \sum_{m=0}^{M} \sum_{\ell=1}^{L} 1_{]t_{m-1},t_m]\times A_{\ell m}} \xi_{k\ell m}$$

for some $0 = t_{-1} \leq t_0 < t_1 < \ldots < t_M$, $A_{\ell m} \in \mathscr{F}_{t_{m-1}}$ and $\xi_{k\ell m} \in X$, where it is understood (with slight misuse of notation) that $]t_{-1}, t_0] := [0, t_0]$. A process $\phi : [0, \infty[\times \Omega \to X^{n+1}$ is called *adaptive* if $\omega \mapsto \phi(t, \omega)$ is (strongly) $\mathscr{F}_t$-measurable for every $t \in [0, \infty[$.

The stochastic integral of an elementary adaptive process is defined in the obvious way by

$$\int_0^\infty \phi(t)\,\mathrm{d}z(t) := \sum_{k,\ell,m} [z_k(t_m) - z_k(t_{m-1})] 1_{A_{\ell m}} \xi_{k\ell m}.$$

We may also define the stochastic integral where $z(t)$ is replaced by $\tilde{z}(t)$: both $\phi(t)$ and $\tilde{z}(t)$ may be extended to functions on $\Omega \times \tilde{\Omega}$ in a canonical way, and they satisfy the appropriate assumptions with respect to the product filtration $(\mathscr{F}_t \times \tilde{\mathscr{F}}_t)_{t\geq 0}$.

2.5. **Definition** ([16])**.** A process $\phi : [0, \infty[\times \Omega \to X^{n+1}$ is called L^p*-stochastically integrable* with respect to $(z(t))_{t\geq 0}$ if $\langle \phi_k(\cdot, \cdot), \xi' \rangle \in L^p(\Omega, L^2(0, \infty))$ for $k = 1, \ldots, n+1$, and there are elementary adaptive processes $\phi^{(j)} = (\phi_k^{(j)})_{k=1}^{n+1}$, $j \in \mathbb{N}$, such that

- $\lim_{j\to\infty} \left\langle \phi_k^{(j)}(\cdot, \cdot), \xi' \right\rangle = \langle \phi_k(\cdot, \cdot), \xi' \rangle$ in $L^p(\Omega, L^2(0, \infty))$ for every $\xi' \in X'$, and

- there exists $\Phi \in L^p(\Omega, X)$ such that
$$\Phi = \lim_{j\to\infty} \int_0^\infty \phi^{(j)}(t)\,\mathrm{d}z(t) \quad \text{in } L^p(\Omega, X).$$

In this case, we define $\displaystyle\int_0^\infty \phi(t)\,\mathrm{d}z(t) := \Phi$.

If $X_0 \subseteq X$ is a finite-dimensional subspace, an adaptive process $\phi : [0,\infty[\, \times \Omega \to X_0^{n+1}$ may be analyzed "component-wise", and the classical necessary and sufficient condition for its stochastic integrability is valid:

$$\mathsf{E}_z \Big(\int_0^\infty |\phi(t)|^2 \,\mathrm{d}t \Big)^{p/2} < \infty. \tag{2.6}$$

For most of our purposes, this finite-dimensional situation would be enough as far as the *existence* of stochastic integrals is concerned; however, we want to make *estimates* of the integrals which are independent of the particular subspace X_0. For this purpose, we quote the following result proved by Garling:

2.7. **Theorem** ([7]). *Let X be a Banach space, and let the Brownian motion $z(t)$ be adapted to a filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $v : [0,\infty[\, \times \Omega \to X_0^{n+1}$, where $X_0 \subseteq X$ is a finite-dimensional subspace, be adapted $(\mathscr{F}_t)_{t\geq 0}$ and satisfy* (2.6). *If X is UMD$^-$ resp. UMD$^+$, then there holds*

$$\begin{aligned} \Big(\mathsf{E}_z \Big| \int_0^\infty v(t)\,\mathrm{d}z(t) \Big|^p \Big)^{1/p} &\leq \beta^-_{p,X} \Big(\mathsf{E}_z \tilde{\mathsf{E}} \Big| \int_0^\infty v(t)\,\mathrm{d}\tilde{z}(t) \Big|^p \Big)^{1/p} \qquad \textit{resp.} \\ \Big(\mathsf{E}_z \tilde{\mathsf{E}} \Big| \int_0^\infty v(t)\,\mathrm{d}\tilde{z}(t) \Big|^p \Big)^{1/p} &\leq \beta^+_{p,X} \Big(\mathsf{E}_z \Big| \int_0^\infty v(t)\,\mathrm{d}z(t) \Big|^p \Big)^{1/p}. \end{aligned} \tag{2.8}$$

This concludes our general preliminaries, and we move on to matters more specific to the present paper.

Set-up for the theory of Varopoulos. Following Varopoulos [21], we introduce the operators

$$\mathsf{E}^\lambda(F) := \int_{\mathbb{R}^n} \mathsf{E}_{(x,\lambda)}(F)\,\mathrm{d}x, \qquad \mathsf{P}^\lambda(A) := \mathsf{E}^\lambda(1_A),$$

where the action of E^λ is defined for all measurable functions $F \geq 0$ on Ω. These represent the "expectation" and "probability" related to Brownian motion starting in a random point at height λ, except that P^λ is a σ-finite positive measure of infinite total mass, and hence not really a probability. However, as explained in [21], all the results from Probability relevant to the present context remain valid in this extended setting, which may be seen either by inspection of the classical proofs or by splitting into countably many honest probability spaces and summing up.

An important rôle will be played by the stopping time

$$\tau := \inf\{t : y(t) \leq 0\}, \tag{2.9}$$

where, we recall, $y(t)$ is the vertical component of the Brownian motion $z(t) = (x(t), y(t))$.

We write $T^y := e^{y\triangle}$ and $P^y := e^{-y(-\triangle)^{1/2}}$ for the Gaussian and Poisson semigroups, where $\triangle$ is the Laplacian, $\triangle f = \sum_{k=1}^n \partial^2 f/\partial x_k^2$, on $\mathbb{R}^n$. For any reasonable enough, possibly vector-valued, function f on $\mathbb{R}^n$, we denote by u its harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0,\infty[$, i.e., $u(x,y) = P^y f(x)$. Finally, let $\partial u/\partial x := (\partial u/\partial x_k)_{k=1}^n$, so that $\nabla u = (\partial u/\partial x, \partial u/\partial y)$.

3. Square function estimates

The main probabilistic Littlewood–Paley theorem of Varopoulos is the following:

3.1. **Theorem** ([21]). *For all $p \in [2,\infty[$, there exists a constant K_p such that for all $\lambda \geq 0$ and all $f \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$\mathsf{E}^\lambda \Big[\Big(\int_0^\tau \Big| \frac{\partial u}{\partial y}(z(s)) \Big|^2 \mathrm{d}s \Big)^{p/2} \Big] \leq K_p \|f\|_p^p, \qquad u(x,y) := P^y f(x). \tag{3.2}$$

In order to motivate our vector-valued extension, we rewrite the estimate (3.2) in a somewhat different but equivalent form. Recall that $(\tilde{y}(t))_{t\geq 0}$ is a Brownian motion independent of all the earlier-mentioned processes, defined on another probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathsf{P}})$, with expectation $\tilde{\mathsf{E}}$. Then the classical Itô isometry says that

$$\int_0^\tau \Big|\frac{\partial u}{\partial y}(z(s))\Big|^2 \,\mathrm{d}s = \tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial y}(z(s))\,\mathrm{d}\tilde{y}(s)\Big|^2.$$

The stochastic integral $\int_0^\tau \partial u(z(s))/\partial y \,\mathrm{d}\tilde{y}(s)$ is a Gaussian random variable on $\tilde{\Omega}$, and thus all of its L^p norms a comparable, i.e.,

$$\Big(\tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial y}(z(s))\,\mathrm{d}\tilde{y}(s)\Big|^2\Big)^{1/2} \eqsim_p \Big(\tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial y}(z(s))\,\mathrm{d}\tilde{y}(s)\Big|^p\Big)^{1/p}.$$

In conclusion, we find that

$$\mathsf{E}^\lambda\Big[\Big(\int_0^\tau \Big|\frac{\partial u}{\partial y}(z(s))\Big|^2 \,\mathrm{d}s\Big)^{p/2}\Big] \eqsim_p \mathsf{E}^\lambda\tilde{\mathsf{E}}\Big[\Big|\int_0^\tau \frac{\partial u}{\partial y}(z(s))\,\mathrm{d}\tilde{y}(s)\Big|^p\Big],$$

where the left-hand side coincides with that of (3.2). Hence we see that the following result is indeed an extension of Theorem 3.1.

3.3. **Theorem.** *Let X be a UMD$^+$ Banach space and $p \in \,]1,\infty[$. Then for all $\lambda \geq 0$ and all $f \in \mathscr{S}(\mathbb{R}^n) \otimes X$, we have*

$$\Big\{\mathsf{E}^\lambda\tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial y}(z(s))\,\mathrm{d}\tilde{y}(s)\Big|^p\Big\}^{1/p} \leq \beta^+_{p,X}\big(\|f\|_{L^p_X} + \|P^\lambda f\|_{L^p_X}\big) \leq 2\beta^+_{p,X}\|f\|_{L^p_X}. \tag{3.4}$$

Proof. To ensure the existence of the stochastic integrals, we may in the first place replace the stopping time τ by $\tau \wedge T$ and pass to the limit $T \to \infty$ in the end. However, we still write simply τ for convenience.

Let $(\tilde{x}(t))_{t\geq 0}$ be our independent copy of $(x(t))_{t\geq 0}$, also independent of the other processes. From the contraction principle and the decoupling inequality (2.8) we get

$$\begin{aligned}\Big\{\mathsf{E}^\lambda\tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial y}\,\mathrm{d}\tilde{y}(s)\Big|^p\Big\}^{1/p} &\leq \Big\{\mathsf{E}^\lambda\tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial x}\,\mathrm{d}\tilde{x}(s) + \int_0^\tau \frac{\partial u}{\partial y}\,\mathrm{d}\tilde{y}(s)\Big|^p\Big\}^{1/p} \\ &\leq \beta^+_{p,X}\Big\{\mathsf{E}^\lambda\Big|\int_0^\tau \frac{\partial u}{\partial x}\,\mathrm{d}x(s) + \int_0^\tau \frac{\partial u}{\partial y}\,\mathrm{d}y(s)\Big|^p\Big\}^{1/p}\end{aligned} \tag{3.5}$$

By Itô's formula we have

$$u[z(\tau)] - u[z(0)] = \int_0^\tau \frac{\partial u}{\partial x}(z(s))\,\mathrm{d}x(s) + \int_0^\tau \frac{\partial u}{\partial y}(z(s))\,\mathrm{d}y(s),$$

and thus we have proved that

$$\Big\{\mathsf{E}^\lambda\tilde{\mathsf{E}}\Big|\int_0^\tau \frac{\partial u}{\partial y}\,\mathrm{d}\tilde{y}(s)\Big|^p\Big\}^{1/p} \leq \beta^+_{p,X}\big(\mathsf{E}^\lambda\,|u[(z(\tau))] - u[z(0)]|^p\big)^{1/p}.$$

The right-hand side may now be estimated by the triangle inequality to get the final upper bound

$$\beta^+_{p,X}\big\{\big(\mathsf{E}^\lambda\,|u[(z(\tau))]|^p\big)^{1/p} + \big(\mathsf{E}^\lambda\,|u[(z(0))]|^p\big)^{1/p}\big\} = \beta^+_{p,X}\big\{\|f\|_p + \|P^\lambda f\|_p\big\};$$

we refer to [21], Proposition 3.1, for the last equality, which is a result about nonnegative functions (observe the norm signs) and does not involve anything new in the present setting. □

4. Multiplier theorems

We now consider applications of the square function estimates to the boundedness in $L^p(\mathbb{R}^n, X)$ of the Laplacian spectral multipliers

$$m(-\triangle)f = \int_0^\infty m(\lambda)\,\mathrm{d}E(\lambda)f, \tag{4.1}$$

where E is the spectral measure of $-\triangle$. The multipliers which we are going to treat are of the modified Laplace transform type,

$$m(\lambda) = 2\lambda \int_0^\infty tM(t)e^{-2\lambda^{1/2}t}\,\mathrm{d}t, \qquad M \in L^\infty(0,\infty;\mathbb{C}). \tag{4.2}$$

For such m, the operator $m(-\triangle)$ may also be written in terms of the Poisson semigroup P^t as

$$m(-\triangle)f = \int_0^\infty yM(y)\frac{1}{2}\frac{\partial^2}{\partial y^2}P^{2y}f\,\mathrm{d}y. \tag{4.3}$$

There is, of course, nothing new in the boundedness itself for these $m(-\triangle)$. It follows, for instance, from Bourgain's [2] or McConnell's [12] vector-valued Fourier multiplier theorems, and is also a special case of the Littlewood–Paley–Stein theory in [10]. However, the question we wish to address here concerns the actual size of the operator norms $\|m(-\triangle)\|_{\mathscr{L}(L^p(\mathbb{R}^n,X))}$.

Let us start with some preparations. Following Varopoulos [21], we introduce the stochastic multiplier transform, an obvious variant of (4.3),

$$U := \int_0^\tau M[z(s)]\frac{\partial u}{\partial y}[z(s)]\,\mathrm{d}y(s), \qquad u(x,y) = P^y f(x),$$

where $f \in \mathscr{S}(\mathbb{R}^n)\otimes X$, say. With

$$g_\lambda(x) := \mathsf{E}[U|z(\tau) = x],$$

it is shown in [21], in the scalar-valued case, that $g_\lambda \to m(-\triangle)f$ weakly in $L^2(\mathbb{R}^n)$, and hence

$$\|m(-\triangle)f\|_p \le \limsup_{\lambda\to\infty}\|g_\lambda\|_p \le \limsup_{\lambda\to\infty}\big(\mathsf{E}^\lambda|U|^p\big)^{1/p}. \tag{4.4}$$

It is clear by linearity that (4.4) remains true for the vector-valued test functions $f \in \mathscr{S}(\mathbb{R}^n)\otimes X$.

The common way of exploiting square functions in the estimation of an operator T such as our $m(-\triangle)$ consists of three steps: the norm of Tf is controlled by a square function norm of the same quantity, this by a square function norm of f alone (which is often easy, if the square function and the operator are suitably related), and this by the norm of f. In the present setting, this approach yields the following easy consequence of Theorem 3.3.

4.5. **Theorem.** *Let X be a UMD space and $p \in]1,\infty[$. If $M \in L^\infty(0,\infty;\mathbb{C})$, then*

$$\|m(-\triangle)f\|_p \le \frac{\pi}{2}\beta^-_{p,X}\beta^+_{p,X}\|M\|_\infty\|f\|_p, \qquad f \in L^p(\mathbb{R}^n,X).$$

If $M \in L^\infty(0,\infty;\mathbb{R})$, one may drop the factor $\pi/2$ above.

Proof. According to (4.4), it suffices to estimate $\limsup_{\lambda\to\infty}\big(\mathsf{E}^\lambda|U|^p\big)^{1/p}$.

By the decoupling inequality (2.8) and the contraction principle (cf. Lemma 4.11.5 of [17] for the complex-valued case), we find that

$$\begin{aligned}\big(\mathsf{E}^\lambda|U|^p\big)^{1/p} &\le \beta^-_{p,X}\Big\{\mathsf{E}^\lambda\tilde{\mathsf{E}}\Big|\int_0^\tau M[z(s)]\frac{\partial u}{\partial y}[z(s)]\,\mathrm{d}\tilde{y}(s)\Big|^p\Big\}^{1/p}\\ &\le \beta^-_{p,X}\cdot\frac{\pi}{2}\|M\|_\infty\Big\{\mathsf{E}^\lambda\tilde{\mathsf{E}}\Big|\int_0^\tau\frac{\partial u}{\partial y}[z(s)]\,\mathrm{d}\tilde{y}(s)\Big|^p\Big\}^{1/p},\end{aligned}$$

where the factor $\pi/2$ may be omitted if M is real-valued. Theorem 3.3 tells that the L^p norm on the right is bounded by

$$\beta^+_{p,X}\big(\|f\|_p + \|P^\lambda f\|_p\big) \to \beta^+_{p,X}\|f\|_p \qquad \text{as } \lambda\to\infty,$$

and this completes the proof. □

When hunting good constants, it turns out that the above three-step approach loses information. It is more desirable to combine all steps together, exploiting the full strength of the UMD property in one decisive strike, instead of dividing its power between the two inequalities (2.3) or their implications in (2.8). Our tool for implementing this strategy is Burkholder's Theorem 2.4, which involves the best constant. (This is, indeed, the beauty and strength of Burkholder's result. The analogous estimates with $\beta^-_{p,X}\beta^+_{p,X}$ and $\pi/2\cdot\beta^-_{p,X}\beta^+_{p,X}$ in place of $\beta_{p,X}$ and $\beta^{\mathbb{C}}_{p,X}$, respectively, are rather direct consequences of the definitions and convexity.) More precisely, we shall use Theorem 2.4 via the following corollary for stochastic integrals, whose proof we postpone to the appendix.

4.6. **Proposition.** *Let X be a UMD space and $p \in]1,\infty[$. Let the process*

$$\Phi : \mathbb{R}_+ \times \Omega \to \mathscr{L}(\mathbb{R}^{n+1}, X)$$

be L^p-stochastically integrable with respect to an $(n+1)$-dimensional Brownian motion $(z(t))_{t\geq 0}$. Let $M_k \in L^\infty(\mathbb{R}_+ \times \Omega; \mathbb{C})$ be adaptive processes with $|M_k(t,\omega)| \leq 1$. Then

$$\Phi M(t,\omega) := (\Phi_k(t,\omega)M_k(t,\omega))_{k=1}^{n+1} \in \mathscr{L}(\mathbb{R}^{n+1}, X)$$

is also L^p-stochastically integrable, and there holds

$$\mathsf{E}\Big\| \int_0^\infty \Phi(t)M(t)\,\mathrm{d}z(t)\Big\|_X^p \leq (\beta_{p,X}^{\mathbb{C}})^p \mathsf{E}\Big\| \int_0^\infty \Phi(t)\,\mathrm{d}z(t)\Big\|_X^p.$$

If $M_k \in L^\infty(\mathbb{R}_+ \times \Omega; \mathbb{R})$, then $\beta_{p,X}^{\mathbb{C}}$ may be replaced by $\beta_{p,X}$.

Let us point out that the conclusion of the Proposition with *some* constant $c_{p,X}$ in place of $\beta_{p,X}^{\mathbb{C}}$ and $\beta_{p,X}$ is an immediate consequence of results from [16], but this is not what we want here. With the help of Proposition 4.6, we reach the following improvement of Theorem 4.5 (recall that $2/\pi \cdot \beta_{p,X}^{\mathbb{C}} \leq \beta_{p,X} \leq \beta_{p,X}^- \beta_{p,X}^+$):

4.7. **Theorem.** *Let X be a UMD space and $p \in]1,\infty[$. If $M \in L^\infty(0,\infty;\mathbb{C})$, then*

$$\|m(-\triangle)f\|_p \leq \beta_{p,X}^{\mathbb{C}} \|M\|_\infty \|f\|_p, \qquad f \in L^p(\mathbb{R}^n, X).$$

If $M \in L^\infty(0,\infty;\mathbb{R})$, then one may replace $\beta_{p,X}^{\mathbb{C}}$ by $\beta_{p,X}$.

Proof. We start by observing that

$$U = \sum_{k=1}^n \int_0^\tau \frac{\partial u}{\partial x_k}[z(t)] \cdot 0 \cdot \mathrm{d}x_k(t) + \int_0^\tau \frac{\partial u}{\partial y}[z(t)] \cdot M[z(t)] \cdot \mathrm{d}y(t)$$

is an integral of the kind considered in Proposition 4.6 with $\Phi(t) = \nabla u[z(t)]$ and $(0,\ldots,0,M[z(t)])$ in place of $M(t)$. Thus, an application of the Proposition yields

$$\big(\mathsf{E}^\lambda |U|^p\big)^{1/p} \leq \beta_{p,X}^{\mathbb{C}} \|M\|_\infty \Big\{\mathsf{E}^\lambda \Big| \int_0^\tau \frac{\partial u}{\partial x}[z(s)]\,\mathrm{d}x(s) + \int_0^\tau \frac{\partial u}{\partial y}[z(s)]\,\mathrm{d}y(s)\Big|^p\Big\}^{1/p},$$

or the similar estimate with $\beta_{p,X}$ under the reality assumption. The right-hand side is of the form already handled in the proof of Theorem 3.3. □

4.8. **Corollary.** *Let X be a UMD space and $p \in]1,\infty[$. Then*

$$\big\|(-\triangle)^{i\gamma} f\big\|_p \leq \frac{2\beta_{p,X}^{\mathbb{C}}}{|\Gamma[2(1-i\gamma)]|} \|f\|_p, \qquad f \in L^p(\mathbb{R}^n, X).$$

Proof. This is immediate from the identity

$$\lambda^{i\gamma} = 2\lambda \int_0^\infty tM(t)e^{-2\lambda^{1/2}t}\,\mathrm{d}t, \qquad M(t) = \frac{2\cdot(2t)^{-i2\gamma}}{\Gamma[2(1-i\gamma)]}$$

and Theorem 4.7. □

The Corollary implies in particular that

$$\limsup_{\gamma\to 0} \big\|(-\Delta)^{i\gamma}\big\|_{\mathscr{L}(L_X^p(\mathbb{R}))} \leq 2\beta_{p,X}^{\mathbb{C}}. \tag{4.9}$$

A comparison of this with the following result of S. Guerre-Delabriere shows that this estimate, and therefore the ones from which it was derived, are close to optimal:

4.10. **Theorem** ([9])**.** *Let X be a Banach space with $(-\triangle)^{i\gamma} \in \mathscr{L}(L^p(\mathbb{R},X))$ for $\gamma \neq 0$. Then X is a UMD space, and more precisely*

$$\beta_{p,X}^{\mathbb{C}} \leq \liminf_{\gamma\to 0} \big\|(-\Delta)^{i\gamma}\big\|_{\mathscr{L}(L^p(\mathbb{R},X))}. \tag{4.11}$$

The precise estimate above is not explicitly formulated in [9], but an investigation of the proof reveals that a version of (4.11) with $\beta_{p,X}$ in place of $\beta^{\mathbb{C}}_{p,X}$ is actually proved there, and Theorem 4.10 as stated follows *mutatis mutandis*.

5. Final comments

The results (4.9) and (4.11) show that the two different numerical indicators of the UMD property—$\beta_{p,X}$ and $\liminf_{\gamma\to 0}\left\|(-\triangle)^{i\gamma}\right\|_{\mathscr{L}(L^p(\mathbb{R},X))}$—are linearly bounded in terms of each other. This is in contrast to the best known relations between $\beta_{p,X}$ and $\alpha_{p,X} := \|H\|_{\mathscr{L}(L^p(\mathbb{R},X))}$, the norm of the Hilbert transform, which are

$$\beta_{p,X}\leq \alpha_{p,X}^2 \quad [1], \qquad \alpha_{p,X}\leq \beta^-_{p,X}\beta^+_{p,X}\leq \beta^2_{p,X}\quad [4,7].$$

It is an interesting open problem whether one or both of these could be improved to linear bounds. The Hilbert transform, which is not a spectral multiplier of the Laplacian, is not in the scope of the present approach, although some of the proofs of its boundedness are not very far from the ones in this paper; still, two decoupling estimates seem always unavoidable to control it.

An obvious consequence of Theorem 4.7 (or Theorem 4.5) is the fact that the spectral multipliers of $-\triangle$ of the treated type—in particular, the imaginary powers $(-\triangle)^{i\gamma}$—satisfy norm bounds in $L^p(\mathbb{R}^n,X)$ which are independent of n. Such dimension-free estimates have been of interest to many authors, and the mentioned result is very classical for $X=\mathbb{C}$; cf. [20]. Since the exact value of the UMD constants of every Hilbert space H have been obtained by Burkholder (see [5]), the result being

$$\beta_{p,H}=\beta^{\mathbb{C}}_{p,H}=p^*-1:=\max\{p,p'\}-1=\max\{p-1,\frac{1}{p-1}\},$$

Theorem 4.7 implies the following corollary. I do not know if these (in view of Theorem 4.10 close to optimal) explicit bounds have been proved before.

5.1. **Corollary.** *Let $p\in\,]1,\infty[$ and m be as in (4.2). Then $\|m(-\triangle)\|_{\mathscr{L}(L^p(\mathbb{R}^n))}\leq\|M\|_\infty(p^*-1)$, and in particular $\left\|(-\triangle)^{i\gamma}\right\|_{\mathscr{L}(L^p(\mathbb{R}^n))}\leq 2(p^*-1)/\left|\Gamma[2(1-i\gamma)]\right|$.*

Let me finally mention that the reader who followed the text this far may also find the recent paper of O. Dragičević and A. Volberg [6] quite interesting. While they work in finite-dimensional Hilbert spaces, proving a different square function estimate for controlling another family of operators, there is considerable ideological similarity between their methods and the present ones.

Appendix A. Proof of Proposition 4.6

Proof. Let first Φ be elementary adaptive,

$$\Phi_k(t,\omega)=\sum_{i,j}1_{]t_{i-1},t_i]}(t)1_{A_{ij}}(\omega)\xi_{ijk},\qquad A_{ij}\in\mathscr{F}_{t_{i-1}},\qquad \xi_{ijk}\in X.$$

Then the existence of the asserted integral poses no problem, and

$$\mathsf{E}\Big|\int_0^\infty 1_{]t_{i-1},t_i]}1_{A_{ij}}M_k(t)\,\mathrm{d}z_k(t)\Big|^p\eqsim_p\mathsf{E}\Big(\int_{t_{i-1}}^{t_i}\left|M_k(t)1_{A_{ij}}\right|^2\,\mathrm{d}t\Big)^{p/2}\leq\|M_k\|^p_{L^p(\Omega,L^2(0,T))},$$

where $T:=\max\operatorname{supp}[t\mapsto\Phi(t)]<\infty$. Since $\int_0^\infty\Phi(t)M(t)\,\mathrm{d}z(t)$ consists of a finite number of terms which can be estimated as above, we may by density in $L^p(\Omega,L^2(0,T))$ assume that the M_k are elementary adaptive processes, too. By making a common refinement if necessary, we may take their expansions to involve the same intervals $]t_{i-1},t_i]$ as that of Φ. Thus

$$\int_0^\infty\Phi(t)M(t)\,\mathrm{d}z(t)=\sum_{i,k}\Big(\sum_\ell 1_{B_{i\ell}}\lambda_{i\ell k}\Big)\Big(\sum_j 1_{A_{ij}}\xi_{ijk}[z_k(t_i)-z_k(t_{i-1})]\Big)$$

The latter factors above, ordered primarily with increasing i and secondarily with increasing k, constitute a martingale difference sequence with respect to the filtration

$$\cdots\subset\mathscr{F}_{t_{i-1}}\subset\sigma(\mathscr{F}_{t_{i-1}},\mathscr{Z}_{i-1,1})\subset\cdots\subset\sigma(\mathscr{F}_{t_{i-1}},\mathscr{Z}_{i-1,1},\ldots\mathscr{Z}_{i-1,n})\subset\mathscr{F}_{t_i}\subset\cdots,$$

where $\mathscr{F}_t := \sigma[z(s) : 0 \leq s \leq t]$ and $\mathscr{Z}_{i-1,k} := \sigma[z_k(t_i) - z_k(t_{i-1})]$. The sum of this difference sequence is $\int_0^\infty \Phi(t)\,\mathrm{d}z(t)$. The first factors, on the other hand, constitute a bounded predictable transforming sequence. Hence the asserted estimate follows from the definition of the UMD constant $\beta_{p,X}$, and the proof is complete for elementary adaptive processes Φ.

Let then Φ be arbitrary as in the assumptions. Thus (cf. Definition 2.5) there is a sequence $(\Phi^{(\nu)})_{\nu=1}^\infty$ of elementary adaptive processes such that

$$\left\langle \Phi_k^{(\nu)}(\cdot,\cdot),\xi'\right\rangle \to \langle \Phi_k(\cdot,\cdot),\xi'\rangle \quad \text{in} \quad L^p(\Omega, L^2(0,\infty))$$

for all $k = 1,\ldots,n+1$ and $\xi' \in X'$, and

$$\int_0^\infty \Phi^{(\nu)}(t)\,\mathrm{d}z(t) \to \int_0^\infty \Phi(t)\,\mathrm{d}z(t) \quad \text{in} \quad L^p(\Omega, X)$$

as $\nu \to \infty$. Then also

$$\left\langle \Phi_k^{(\nu)} M_k(\cdot,\cdot),\xi'\right\rangle \to \langle \Phi_k M_k(\cdot,\cdot),\xi'\rangle \quad \text{in} \quad L^p(\Omega, L^2(0,\infty)),$$

and the fact that $\int_0^\infty \Phi^{(\nu)}(t)\,\mathrm{d}z(t)$ is Cauchy, together with the first part of the proof, implies that $\int_0^\infty \Phi^{(\nu)}(t)M(t)\,\mathrm{d}z(t)$ is Cauchy, hence convergent, in $L^p(\Omega,X)$. The functions $\Phi^{(\nu)}M$ are typically not elementary progressive but, as observed in the first part of the proof, we may, for each ν, find an elementary progressive $M^{(\nu)}$ so that

$$\mathsf{E}\left|\int_0^\infty \Phi^{(\nu)}(t)M^{(\nu)}(t)\,\mathrm{d}y(t) - \int_0^\infty \Phi^{(\nu)}(t)M(t)\,\mathrm{d}y(t)\right|^p < \epsilon_\nu^p,$$

and then also

$$\left\|\left\langle \Phi_k^{(\nu)} M_k^{(\nu)}(\cdot,\cdot),\xi'\right\rangle - \left\langle \Phi_k^{(\nu)} M_k(\cdot,\cdot),\xi'\right\rangle\right\|_{L^p(\Omega,L^2(0,\infty))} < K_p\,\epsilon_\nu\,|\xi'|$$

for $k = 1,\ldots,n+1$ and $\xi' \in X'$. In this way, we can realize ΦM as the limit of the elementary adaptive functions $\Phi^{(\nu)}M^{(\nu)}$ in the appropriate norms so as to extend the required estimate to all L^p-stochastically integrable Φ. □

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