CONICAL SQUARE FUNCTION ESTIMATES IN UMD BANACH SPACES AND APPLICATIONS TO H^∞-FUNCTIONAL CALCULI

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Abstract. We study conical square function estimates for Banach-valued functions, and introduce a vector-valued analogue of the Coifman–Meyer–Stein tent spaces. Following recent work of Auscher–M$^{\rm c}$Intosh–Russ, the tent spaces in turn are used to construct a scale of vector-valued Hardy spaces associated with a given bisectorial operator A with certain off-diagonal bounds, such that A always has a bounded H^∞-functional calculus on these spaces. This provides a new way of proving functional calculus of A on the Bochner spaces $L^p(\mathbb{R}^n;X)$ by checking appropriate conical square function estimates, and also a conical analogue of Bourgain's extension of the Littlewood-Paley theory to the UMD-valued context. Even when $X=\mathbb{C}$, our approach gives refined p-dependent versions of known results.

1. Introduction

Since the development of the Littlewood-Paley theory, square function estimates of the form

$$\Big\|\Big(\int_0^\infty \big|t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f\big|^2\,\frac{\mathrm{d}t}{t}\Big)^{\frac12}\Big\|_{L^p(\mathbb{R}^n)} \eqsim \|f\|_{L^p(\mathbb{R}^n)},$$

have been widely used in harmonic analysis. When dealing with functions which take values in a UMD Banach space X, such estimates have to be given an appropriate meaning. This is done through a linearisation of the square function using randomisation, which gives (see [14])

$$\Big\|\int_0^\infty t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f\,\frac{\mathrm{d}Wt}{\sqrt{t}}\Big\|_{L^2(\Omega;L^p(\mathbb{R}^n;X))} \eqsim \|f\|_{L^p(\mathbb{R}^n;X)},$$

where the integral is a Banach space-valued stochastic integral with respect to a standard Brownian motion W on a probability space $(\Omega,\mathbb{P})$ (see [25]), or, in a simpler discrete form,

$$\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k 2^k\sqrt{\Delta}e^{-2^k\sqrt{\Delta}}f\Big\|_{L^2(\Omega;L^p(\mathbb{R}^n;X))} \eqsim \|f\|_{L^p(\mathbb{R}^n;X)} \tag{1.1}$$

where (ε_k) is a sequence of independent Rademacher variables on $(\Omega,\mathbb{P})$. The latter was proven by Bourgain in [6], thereby starting the development of harmonic analysis for UMD-valued functions. In recent years, research in this field has accelerated as it appeared that its tools, and in particular square function estimates, are of

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fundamental importance in the study of the H^∞-functional calculus (see [20]) and in stochastic analysis in UMD Banach spaces (see [24]).

To some extent, even the scalar-valued theory (i.e. $X = \mathbb{C}$) has benefited from this probabilistic point of view (see for instance [16, 22]). However this fruitful linearisation has, so far, been limited to the above "vertical" square functions estimates, leaving aside the "conical" estimates of the form

(1.2)
$$\Big(\int_{\mathbb{R}^n}\Big(\iint_{|y-x|<t}\big|t\sqrt{\Delta}e^{-t\sqrt{\Delta}}f(y)\big|^2\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\Big)^{\frac{p}{2}}\mathrm{d}x\Big)^{\frac{1}{p}} \eqsim \|f\|_{L^p(\mathbb{R}^n)}, \quad 1<p\le 2.$$

In the meantime, such estimates have attracted much attention as it was realised that they could be used to extend the real variable theory of Hardy spaces in a way which is suitable to treat operators beyond the Calderón-Zygmund class (see [3, 9, 13]). Indeed, elliptic operators of the form $-\mathrm{div}B\nabla$, where B is a matrix with L^∞ entries, are not, in general, sectorial on L^p for all $1 < p < \infty$. Their study thus requires the L^p-spaces to be replaced by appropriate Hardy spaces, on which they have good functional calculus properties (in the same way as L^1 has to be replaced by H^1 when dealing with the Laplacian). To define such spaces, conical square functions have to be used, since the use of vertical ones would impose severe restrictions on the class of operators under consideration (namely, L^p (R-)sectoriality).

The present paper gives extensions of (1.2) to the UMD-valued context. This starts with the construction of appropriate tent spaces, which is carried out in Section 4 by reinterpreting and extending [11] using the methods of stochastic analysis in Banach spaces from [19, 24, 25]. Relevant notions and results from this theory are recalled in Section 2, while the crucial technical estimate is proven in Section 3. Following ideas developed in [3], we then prove appropriate estimates for operators acting on these tent spaces in Section 5. After collecting some basic results on bisectorial operator in Section 6, this allows us in Section 7 to define Hardy spaces associated with bisectorial operators of the form $A \otimes I_X$, where X is a UMD Banach space, and A acts on $L^2(\mathbb{R}^n, \mathbb{C}^N)$ and satisfies suitable off-diagonal estimates. We prove that $A \otimes I_X$ always has an H^∞-functional calculus on these Hardy spaces. Finally, in Section 8, we specialise to differential operators A, and, in particular, give a conical analogue to Bourgain's square function estimate (1.1).

Specialising to the case $X = \mathbb{C}$, our approach allows to define Hardy spaces (associated with operators) using a class of functions which is wider than in [3]. This is due to the fact that our estimates (see Proposition 7.5) are directly obtained for a given value of p (and actually depend on the type and cotype of L^p), instead of using interpolation.

To conclude this introduction, let us now point out the possible uses of our results. First, one can deduce the boundedness of the functional calculus of an operator $A\otimes I_X$ from conical square function estimates. For instance, with Theorem 8.2, we recover the well-known fact that, if X is UMD and $1 < p < \infty$, $\Delta \otimes I_X$ admits an H^∞-calculus on $L^p(X)$. Note that this characterises the UMD spaces among all Banach spaces and thus indicates that it cannot be expected that the results presented here extend beyond the UMD setting.

Another application is to deduce conical square function estimates for functions with limited decay from such estimates for functions with good decay properties.

In their formulation we use the notations

$$S_\theta^+ = \{z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| < \theta\},$$
$$\Psi_\alpha^\beta(S_\theta^+) = \big\{f \in H^\infty(S_\theta^+) : \ \exists C \ \ |f(z)| \le C \min(|z|^\alpha, |z|^{-\beta}) \ \forall \ z \in S_\theta^+\big\}.$$

Let $\theta, \varepsilon > 0$, and assume that either

$$\psi \in \Psi_1^{n/2+\varepsilon}(S_\theta^+) \quad \text{and} \quad 1 < p < \frac{2n}{n-2}$$

or

$$\psi \in \Psi_{n/2+\varepsilon}^1(S_\theta^+) \quad \text{and} \quad \frac{2n}{n+2} < p < \infty.$$

Theorem 8.2 together with Theorem 7.10 give the following estimates:

$$\int_{\mathbb{R}^n} \Big(\iint_{|y-x|<t} |\psi(t\sqrt{\Delta})u(y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \Big)^{p/2} \mathrm{d}x \eqsim \|u\|_{L^p}^p.$$

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2. Preliminaries

In this section we establish some terminology and collect auxiliary results needed in the main body of the paper.

Let X and Y be Banach spaces and let $\mathscr{L}(X,Y)$ denote the space of all bounded linear operators acting from X into Y. A family of bounded operators $\mathscr{T} \subseteq \mathscr{L}(X,Y)$ is called *γ-bounded* if there is a constant C such that for all integers $k \geqslant 1$ and all $T_1, \dots, T_k \in \mathscr{T}$ and $\xi_1, \dots, \xi_j \in X$ we have

$$\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j T_j \xi_j \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \xi_j \Big\|^2. \tag{2.1}$$

Here, $\gamma_1, \dots, \gamma_k$ are independent standard normal variables defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$. The least admissible constant in (2.1) is denoted by $\gamma(\mathscr{T})$.

By the Kahane-Khintchine inequality, the exponent 2 may be replaced by any exponent $1 \leqslant p < \infty$ at the cost of a possibly different constant.

Upon replacing the standard normal variables by Rademacher variables in (2.1) one arrives at the notion of R-boundedness. Every R-bounded family is γ-bounded, and the converse holds if Y has finite cotype. Since we are primarily interested in UMD spaces Y, which have finite cotype, the distinction between γ-boundedness and R-boundedness is immaterial. We prefer the former since our techniques are Gaussian and therefore the use of Gaussian variables seems more natural.

Let H be a Hilbert space. A linear operator $R : H \to X$ is called *γ-summing* if

$$\|R\|_{\gamma^\infty(H,X)} := \sup \Big(\mathbb{E}\Big\|\sum_{j=1}^k \gamma_j R h_j\Big\|^2\Big)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all integers $k \geqslant 1$ and all finite orthonormal systems $h_1, \dots, h_k$ in H. The space $\gamma^\infty(H,X)$, endowed with the above norm, is a Banach space. The closed subspace of $\gamma^\infty(H,X)$ spanned by the finite rank operators is denoted by $\gamma(H,X)$. A linear operator $R : H \to X$ is said to be *γ-radonifying* if it belongs to $\gamma(H,X)$.

A celebrated result of Hoffman-Jørgensen and Kwapień [12, 21] implies that

$$\gamma^\infty(H,X) = \gamma(H,X)$$

for Banach spaces X not containing an isomorphic copy of c_0.

If H is separable with orthonormal basis $(h_n)_{n\geqslant 1}$, then an operator $R : H \to X$ is γ-radonifying if and only if the sum $\sum_{n\geqslant 1} \gamma_n R h_n$ converges in $L^2(\Omega; X)$, in which case we have

$$\|R\|_{\gamma(H,X)} = \Big(\mathbb{E}\Big\|\sum_{j\geqslant 1} \gamma_j R h_j\Big\|^2\Big)^{\frac{1}{2}}.$$

The following criterium for membership of $\gamma(H,X)$ will be referred to as *covariance domination*.

Proposition 2.1. *Let $S \in \mathscr{L}(H,X)$ and $T \in \gamma(H,X)$ satisfy*

$$\|S^*\xi^*\| \leqslant C\|T^*\xi^*\|, \quad \xi^* \in X^*,$$

with C independent of ξ^*. Then $S \in \gamma(H,X)$ and $\|S\|_{\gamma(H,X)} \leqslant C\|T\|_{\gamma(H,X)}$.

For more details we refer to [19, 24] and the references therein.

Let (A, Σ, μ) be a σ-finite measure space, and X a Banach space. In the formulation of the next result, which is a multiplier result due to Kalton and Weis [19], we identify X-valued functions $f \otimes \xi$, where $f \in L^2(A)$ and $\xi \in X$, with the operator $R_{f\otimes\xi} \in \gamma(L^2(A), X)$ defined by

$$R_{f\otimes\xi} g := \langle f, g\rangle \otimes \xi, \quad g \in L^2(A). \tag{2.2}$$

where $\langle f, h\rangle$ denotes the scalar product on $L^2(A)$.

Lemma 2.2. *Let X be a Banach space, let (A, Σ, μ) be a σ-finite measure space, and let $M : A \to \mathscr{L}(X)$ be a function such that $a \mapsto M(a)\xi$ is strongly μ-measurable for all $\xi \in X$. If the set*

$$\mathscr{M} = \{M(a) : \ a \in A\}$$

is γ-bounded, then the mapping

$$f(\cdot) \otimes \xi \mapsto f(\cdot) \otimes M(\cdot)\xi,$$

extends to a bounded operator M on $\gamma(L^2(A), X)$ of norm $\|M\| \leqslant \gamma(\mathscr{M})$.

Let us also recall that for all $1 \leqslant p < \infty$ the mapping $f \mapsto [h \mapsto f(\cdot)h]$ defines an isomorphism of Banach spaces

$$L^p(A; \gamma(H,X)) \eqsim \gamma(H, L^p(A;X)). \tag{2.3}$$

This follows from a simple application of the Kahane-Khintchine inequality; we refer to [24, Proposition 2.6] for the details. Here, H and X are allowed to be arbitrary Hilbert spaces and Banach spaces, respectively; the norm constants in

the isomorphism are independent of H.

Let $\gamma = (\gamma_n)_{n\geqslant 1}$ be a sequence of independent standard normal variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Recall that a Banach space X is called *K-convex* if the mapping

$$\pi_\gamma : f \mapsto \sum_{n\geqslant 1} \gamma_n \mathbb{E}(\gamma_n f), \quad f \in L^2(\Omega; X),$$

defines a bounded operator on $L^2(\Omega; X)$. This notion is well-defined: if π_γ is bounded for some sequence γ, then it is bounded for all sequences γ. A celebrated result of Pisier [26] states that X is K-convex if and only if X is B-convex if and only if X has nontrivial type.

If H is a Hilbert space and X is a K-convex Banach space, then the isometry $I_\gamma : \gamma(H, X) \to L^2(\Omega; X)$ defined by

$$I_\gamma R := \sum_{n\geqslant 1} \gamma_n R h_n$$

maps $\gamma(H, X)$ onto a complemented subspace of $L^2(\Omega; X)$. Indeed, for all $R \in \gamma(H, X)$ we have

$$\pi_\gamma I_\gamma R = \sum_{n\geqslant 1} \gamma_n \mathbb{E}\gamma_n \sum_{j\geqslant 1} \gamma_j R h_j = \sum_{n\geqslant 1} \gamma_n R h_n = I_\gamma R.$$

Hence, the range of I_γ is contained in the range of π_γ. Since the range of π_γ is spanned by the functions $\gamma_n \otimes \xi = I_\gamma(h_n \otimes \xi)$, the range is π_γ is contained in the range of I_γ. We conclude that the ranges of π_γ and I_γ coincide and the claim is proved. As an application of this we are able to describe complex interpolation spaces of the spaces $\gamma(H, X)$.

Proposition 2.3. *If H is a Hilbert space and the Banach spaces X_1 and X_2 are K-convex, then for all $0 < \theta < 1$ we have*

$$[\gamma(H, X_1), \gamma(H, X_2)]_\theta = \gamma(H, [X_1, X_2]_\theta) \quad \textit{with equivalent norms.}$$

Proof. In view of the preceding observations this follows from general results on interpolation of complemented subspaces [5, Chapter 5]. □

3. Main estimate

The main estimate of this paper is a γ-boundedness estimate for some averaging operators, which is proven below.

We start by recalling some known results. The first is Bourgain's extension to UMD spaces of Stein's inequality [6] (see [7] for a complete proof).

Lemma 3.1. *Let $1 < p < \infty$ and let X be a UMD space. Let $(\mathscr{F}_m)_{m\in\mathbb{Z}}$ be a filtration on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Then the family of conditional expectations*

$$\mathscr{E} = \{\mathbb{E}(\,\cdot\,|\mathscr{F}_m) : \ m \in \mathbb{Z}\}$$

is γ-bounded on $L^p(\Omega; X)$.

Let us agree that a cube in $\mathbb{R}^n$ is any set Q of the form $x + [0, \ell)^n$ with $x \in \mathbb{R}^n$ and $\ell > 0$. We denote $\ell(Q) := \ell$ and call it the side-length of Q. A *system of dyadic cubes* is a collection $\Delta = \bigcup_{k\in\mathbb{Z}} \Delta_{2^k}$, where Δ_{2^k} is a disjoint cover of $\mathbb{R}^n$ by cubes of side-length 2^k, and each $Q \in \Delta_{2^k}$ is the union of 2^n cubes $R \in \Delta_{2^{k-1}}$. We recall the following geometric lemma of Mei [23]:

Lemma 3.2. *There exist $n+1$ systems of dyadic cubes $\Delta^0, \dots, \Delta^n$ and a constant $C < \infty$ such that for any ball $B \subseteq \mathbb{R}^n$ there is a $Q \in \bigcup_{k=0}^n \Delta^k$ which satisfies $B \subseteq Q$ and $|Q| \leq C\,|B|$.*

The following results can be found in [16]:

Lemma 3.3. *Let X be a UMD space and $1 < p < \infty$. Let $r \in \mathbb{Z}^n \setminus \{0\}$ and $x_Q \in X$ for all $Q \in \Delta$. Then*

$$\mathbb{E}\Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k \sum_{Q\in\Delta_{2^k}} 1_{Q+r\ell(Q)} x_Q \Big\|_p \leq C(1+\log|r|)\mathbb{E}\Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k \sum_{Q\in\Delta_{2^k}} 1_Q x_Q \Big\|_p.$$

Lemma 3.4. *Let X be a UMD space, $1 < p < \infty$, and $m \in \mathbb{Z}_+$. For each $Q \in \Delta$, let $Q', Q'' \in \Delta$ be subcubes of Q of side-length $2^{-m}Q$. Then for all $\ell \in \mathbb{Z}$ and all $x_Q \in X$*

$$\mathbb{E}\Big\| \sum_{k\equiv\ell} \varepsilon_k \sum_{Q\in\Delta_{2^k}} 1_{Q''} x_Q \Big\|_p \leq C\mathbb{E}\Big\| \sum_{k\equiv\ell} \varepsilon_k \sum_{Q\in\Delta_{2^k}} 1_{Q'} x_Q \Big\|_p,$$

where $k \equiv \ell$ is short-hand for $k \equiv \ell \mod (m+1)$.

The previous lemmas will now be used to prove our main estimate.

Proposition 3.5. *Let X be a UMD space, $1 < p < \infty$, and let $L^p(X)$ have type τ. For $\alpha \geqslant 1$, let $\mathscr{A}_\alpha$ be the family of operators*

$$f \mapsto A_B^\alpha f := 1_{\alpha B} \fint_B f \,\mathrm{d}x,$$

where B runs over all balls in $\mathbb{R}^n$. Then $\mathscr{A}_\alpha$ is γ-bounded on $L^p(X)$ with the γ-bound at most $C(1+\log\alpha)\alpha^{n/\tau}$ and C depends only on X, p, τ and n.

Proof. We have to show that

$$\mathbb{E}\Big\| \sum_{j=1}^k \varepsilon_j 1_{\alpha B_j} \fint_{B_j} f_j \,\mathrm{d}x \Big\|_p \leq C\mathbb{E}\Big\| \sum_{j=1}^k \varepsilon_j f_j \Big\|_p.$$

By splitting all the balls B_j into $n+1$ subsets and considering each of them separately, we may assume by Mei's lemma that there is a system of dyadic cubes Δ and $Q_1, \dots, Q_k \in \Delta$ such that $B_j \subseteq Q_j$ and $|Q_j| \leq C\,|B_j|$.

Let m be the integer for which $2^{m-1} \leq \alpha < 2^m$. Let $Q_j^* \in \Delta$ be the unique cube in the dyadic system which has side-length $2^m\ell(Q_j)$ and contains Q_j. Then αB_j is contained in the union of Q_j^* and at most $2^n - 1$ of adjacent cubes $R \in \Delta$ of the same size. Writing $g_j = 1_{B_j} f_j$, we observe that

$$\fint_{B_j} f_j \,\mathrm{d}x = \frac{|Q_j|}{|B_j|} \fint_{Q_j} g_j \,\mathrm{d}x.$$

Since $|Q_j| / |B_j| \leq C$, by the contraction principle it suffices to show that

$$\mathbb{E}\Big\| \sum_{j=1}^k \varepsilon_j 1_{R_j} \fint_{Q_j} g_j \,\mathrm{d}x \Big\|_p \leq C\mathbb{E}\Big\| \sum_{j=1}^k \varepsilon_j g_j \Big\|_p,$$

where $R_j = Q_j^* + r\ell(Q_j^*)$ for some $|r| \leq n$. Thanks to Lemma 3.3, it suffices to consider $r = 0$.

We next write Q_j^* as the union $\bigcup_{i=1}^M Q_{ji}$, where $Q_{ji} \in \Delta$ are the $M := 2^{nm}$ subcubes of Q_j^* of side-length $\ell(Q_j)$. Let us fix the enumeration so that $Q_{j1} = Q_j$. Writing $x_j := \fint_{Q_j} g_j \,\mathrm{d}x$ for short, it follows that

$$\mathbb{E}\Big\|\sum_{j=1}^k \varepsilon_j 1_{Q_j^*} x_j\Big\|_p = \mathbb{E}\Big\|\sum_{i=1}^M\sum_{j=1}^k \varepsilon_j 1_{Q_{ji}} x_j\Big\|_p \le C\mathbb{E}'\mathbb{E}\Big\|\sum_{i=1}^M \varepsilon_i' \sum_{j=1}^k \varepsilon_j 1_{Q_{ji}} x_j\Big\|_p$$
$$\le C\Big(\sum_{i=1}^M \mathbb{E}\Big\|\sum_{j=1}^k \varepsilon_j 1_{Q_{ji}} x_j\Big\|_p^\tau\Big)^{1/\tau}$$

where the first estimate follows from the Khintchine–Kahane inequality and the disjointness of the Q_{ji} for each fixed j, and the second from the assumed type-τ property.

If we assume, for the moment, that all the side-lengths $2^{k(j)} := \ell(Q_j)$ satisfy $k(j) \equiv k(j') \mod (m+1)$, we may apply Lemma 3.4 to continue the estimate with

$$\le C\Big(\sum_{i=1}^M \mathbb{E}\Big\|\sum_{j=1}^k \varepsilon_j 1_{Q_j} x_j\Big\|_p^\tau\Big)^{1/\tau} \le CM^{1/\tau}\mathbb{E}\Big\|\sum_{j=1}^k \varepsilon_j 1_{Q_j} \fint_{Q_j} g_j \,\mathrm{d}x\Big\|_p$$
$$\le CM^{1/\tau}\mathbb{E}\Big\|\sum_{j=1}^k \varepsilon_j g_j\Big\|_p,$$

where the last estimate applied Stein's inequality, observing that the operators $g \mapsto 1_{Q_j} \fint_{Q_j} g \,\mathrm{d}x$ are conditional expectations related to the dyadic filtration induced by Δ. Since $M = 2^{nm} \le 2^n \alpha^n$, we obtain the assertion even without the logarithmic factor in this case.

In general, the above assumption may not be satisfied, but we can always split the indices j into $m+1 \le c(1+\log\alpha)$ subsets which verify the assumption, and this concludes the proof. □

Remark 3.6. The proof simplifies considerably in the important special case $\alpha = 1$.

4. The vector-valued tent spaces $T^{p,2}(X)$

In order to motivate our approach, we begin with a simple characterisation of tent spaces in the scalar-valued case. We put $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times \mathbb{R}_+$ and denote

$$\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : \ |x-y| < t\}.$$

Thus $(y,t) \in \Gamma(x) \Leftrightarrow y \in B(x,t)$, where $B(x,t) = \{y \in \mathbb{R}^n : \ |x-y| < t\}$. We shall write

$$L^p = L^p(\mathbb{R}^n), \qquad L^2(\frac{\mathrm{d}y\ \mathrm{d}t}{t^{n+1}}) = L^2\big(\mathbb{R}_+^{n+1}, \frac{\mathrm{d}y\ \mathrm{d}t}{t^{n+1}}\big),$$

where $\mathrm{d}y$ and $\mathrm{d}t$ denote the Lebesgue measures on $\mathbb{R}^n$ and $\mathbb{R}_+$. Similar conventions will apply to their vector-valued analogues. The dimension $n \geqslant 1$ is considered to be fixed.

For $1 \leqslant p, q < \infty$, the *tent space* $T^{p,q} = T^{p,q}(\mathbb{R}_+^{n+1})$ consists of all (equivalence classes of) measurable functions $f : \mathbb{R}_+^{n+1} \to \mathbb{C}$ with the property that

$$\int_{\mathbb{R}^n} \Big(\int_{\Gamma(x)} |f(y,t)|^q \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\Big)^{\frac{p}{q}} \mathrm{d}x$$

is finite. With respect to the norm

$$\|f\|_{T^{p,q}} := \Big\| \Big(\int_{\Gamma(\cdot)} |f(y,t)|^q \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \Big)^{\frac{1}{q}} \Big\|_{L^p},$$

$T^{p,q}$ is a Banach space. Tent spaces were introduced in the 1980's by Coifman, Meyer, and Stein [8]. Some of the principal results of that paper were simplified by Harboure, Torrea, and Viviani [11], who exploited the fact that

$$J : f \mapsto \big[x \mapsto [(y,t) \mapsto 1_{B(x,t)}(y) f(y,t)]\big]$$

maps $T^{p,q}$ isometrically onto a complemented subspace of $L^p(L^q(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}))$ for $1 < p, q < \infty$.

We now take $q = 2$, and extend the mapping J to functions in $C_c \otimes X$ by $J(g \otimes \xi) := Jg \otimes \xi$ and linearity. Here, C_c denotes the space of continuous functions on $\mathbb{R}^{n+1}_+$ with compact support. Note that by (2.2), $J(g \otimes \xi)$ defines an element of $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$ in a natural way.

Definition 4.1. Let $1 \le p < \infty$. The *tent space* $T^{p,2}(X)$ is defined as the completion of $C_c \otimes X$ with respect to the norm

$$\|f\|_{T^{p,2}(X)} := \|Jf\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))}.$$

It is immediate from this definition that J defines an isometry from $T^{p,2}(X)$ onto a closed subspace of $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$. In what follows we shall always identify $T^{p,2}(X)$ with its image in $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X)$.

Using the identification $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), \mathbb{C}) = L^2(\frac{\mathrm{d}y\,dt}{t^{n+1}})$ we see that our definition extends the definition of tent spaces in the scalar-valued case.

Our first objective is to prove that if X is a UMD space, then $T^{p,2}(X)$ is complemented in $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$.

Proposition 4.2. *Let $1 < p < \infty$, and X a UMD space. The mapping*

$$Nf(x,y,t) := \frac{1_{B(y,t)}(x)}{|B(y,t)|} \int_{B(y,t)} f(z,y,t)\,\mathrm{d}z,$$

initially defined for operators of the form (2.2), extends to a bounded projection in

$$L^p(\gamma(L^2(\tfrac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$$

whose range is $T^{p,2}(X)$.

Proof. We follow the proof of Harboure, Torrea, and Viviani [11, Theorem 2.1] for the scalar-valued case, the main difference being that the use of maximal functions is replaced by a γ-boundedness argument using averaging operators.

First we prove that N is a bounded operator. In view of the isomorphism (2.3) it suffices to prove that N acts as a bounded operator on $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), L^p(X))$. This will be achieved by identifying N as a pointwise multiplier on $L^p(X)$ with γ-bounded range, and then applying Lemma 2.2. In fact, putting

$$N(y,t)\,g := \frac{1_{B(y,t)}}{|B(y,t)|} \int_{B(y,t)} g(z)\,\mathrm{d}z, \quad g \in L^p(X),$$

and $f_{y,t}(x) := f(x,y,t) := \widetilde{f}(y,t) \otimes g(x)$, we have

$$Nf(\cdot,y,t) = \widetilde{f}(y,t) \otimes N(y,t)g = \widetilde{f}(y,t) \otimes A_{B(y,t)}g.$$

The γ-boundedness of $\{N(y,t):\ (y,t)\in\mathbb{R}^{n+1}_+\}$ now follows from Proposition 3.5.

Knowing that N is bounded on $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))$, the fact that it is a projection follows from the scalar case, noting that the linear span of the functions of the form $1_{B(x,t)}\otimes(f\otimes\xi)$, with $f\in C_c$, $x\in\mathbb{R}^n$, and $t>0$, is dense in $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))$. □

For $\alpha>0$ the vector-valued tent space $T^{p,2}_\alpha(X)$ may be defined as above in terms of the norm

$$\|f\|_{T^{p,2}_\alpha(X)}:=\|J_\alpha f\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))},$$

where $J_\alpha f:=\big[x\mapsto[(y,t)\mapsto 1_{B(x,\alpha t)}(y)f(y,t)]\big]$.

Theorem 4.3. *Let $1<p<\infty$, and X a UMD space such that $L^p(X)$ has type τ. For all $\alpha>0$, a strongly measurable function $f:\mathbb{R}^{n+1}_+\to X$ belongs to $T^{p,2}(X)$ if and only if it belongs to $T^{p,2}_\alpha(X)$. Moreover, there exists a constant $C=C(p,X)$ such that*

$$\|f\|_{T^{p,2}(X)}\leqslant\|f\|_{T^{p,2}_\alpha(X)}\leqslant C(1+\log\alpha)\alpha^{n/\tau}\|f\|_{T^{p,2}(X)} \tag{4.1}$$

for $f\in T^{p,2}(X)$ and $\alpha>1$.

Proof. It suffices to prove the latter estimate in (4.1). On $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))$, we consider the operator

$$N_\alpha f(x,y,t):=\frac{1_{B(y,\alpha t)}(x)}{|B(y,t)|}\int_{B(y,t)}f(z,y,t)\,\mathrm{d}z.$$

Simple algebra shows that $N_\alpha Jf=J_\alpha f$, and hence

$$\begin{aligned}\|f\|_{T^{p,2}_\alpha(X)}&=\|J_\alpha f\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))}=\|N_\alpha Jf\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))}\\&\leq\|N_\alpha\|_{\mathscr{L}(L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)))}\|Jf\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))}.\end{aligned}$$

By the isomorphism (2.3), we may consider the boundedness of N_α on the space $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),L^p(X))$ instead, and here this operator acts as the pointwise multiplier

$$N_\alpha(\widetilde{f}\otimes g)(\cdot,y,t)=\widetilde{f}(y,t)\otimes A^\alpha_{B(y,t)}g.$$

So, its boundedness with the asserted estimate follows from Proposition 3.5. □

Remark 4.4. If $X=\mathbb{C}$, then one can take $\tau=\min(2,p)$ in Theorem 4.3. Except possibly for the logarithmic factor, (4.1) gives the correct order of growth of $\|f\|_{T^{p,2}_\alpha}$ in terms of the angle $\alpha\geqslant 1$.

To see this, consider functions of the form $f(y,t)=1_{[1,2]}(t)g(y)$. Then

$$\|f\|_{T^{p,2}_\alpha}=\big\|(\eta_\alpha*|g|^2)^{1/2}\big\|_p,$$

where the η_α are functions having pointwise bounds $c1_{B(0,\alpha)}\leq\eta_\alpha\leq C1_{B(0,C\alpha)}$ for some constants $C>1>c>0$ depending only on n.

Let us take $g=|g|^2=1_{B(0,1)}$. Then $(\eta_\alpha*|g|^2)^{1/2}=\tilde{\eta}_\alpha$, where $\tilde{\eta}_\alpha$ is another similar function, and hence

$$\|f\|_{T^{p,2}_\alpha}=\big\|(\tilde{\eta}_\alpha)^{1/2}\big\|_p\eqsim\alpha^{n/p}\eqsim\alpha^{n/p}\|f\|_{T^{p,2}}.$$

This proves the sharpness for $p\leq 2$.

Let us then choose $g=g_\alpha=1_{B(0,\alpha)}$. Then

$$\eta_\alpha*|g_\alpha|^2=\alpha^n\overline{\eta}_\alpha,\qquad\eta_1*|g_\alpha|^2=\underline{\eta}_\alpha,$$

where $\overline{\eta}_\alpha, \underline{\eta}_\alpha$ are yet more similar functions as η_α. Writing $f_\alpha(y,t) = 1_{[1,2]}(t) g_\alpha(y)$, we have

$$\|f_\alpha\|_{T^{p,2}_\alpha} = \big\|(\alpha^n \overline{\eta}_\alpha)^{1/2}\big\|_p = \alpha^{n/2}\big\|(\overline{\eta}_\alpha)^{1/2}\big\|_p \eqsim \alpha^{n/2}\big\|(\underline{\eta}_\alpha)^{1/2}\big\|_p = \alpha^{n/2}\|f_\alpha\|_{T^{p,2}}.$$

This proves the sharpness for $p \geqslant 2$.

In fact, for $p = 2$, a simple application of Fubini's theorem shows that we have the equality $\|f\|_{T^{2,2}_\alpha} = \alpha^{n/2}\|f\|_{T^{2,2}}$ for all $f \in T^{2,2}$ and $\alpha > 0$, so the logarithmic factor is unnecessary in this case.

Sometimes it is useful to use tent space norms defined with a smooth cut-off instead of the sharp cut-off $1_{B(x,t)}(y)$. Given a function $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(w) = 1$ if $|w| \leq \frac{1}{2}$ and $\phi(w) = 0$ if $|w| \geqslant 1$, we are thus led to consider the mapping $J_\phi f := \big[x \mapsto [(y,t) \mapsto \phi(\frac{|y-x|}{t}) f(y,t)]\big]$ and

$$\|f\|_{T^{p,2}_\phi(X)} := \|J_\phi f\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))}.$$

Proposition 4.5. *Let $1 < p < \infty$, and X a UMD space. A strongly measurable function $f : \mathbb{R}^{n+1}_+ \to X$ belongs to $T^{p,2}(X)$ if and only if it belongs to $T^{p,2}_\phi(X)$. Moreover,*

$$\|f\|_{T^{p,2}_\phi(X)} \eqsim \|f\|_{T^{p,2}(X)}$$

for $f \in T^{p,2}(X)$.

Proof. The proof is the same as that of Theorem 4.3. Consider the operators

$$N_\phi f(x,y,t) := \frac{\phi(\frac{|y-x|}{t})}{|B(y,t)|}\int_{B(y,t)} f(z,y,t)\,\mathrm{d}z,$$
$$\widetilde{N}_{\frac{1}{2}} f(x,y,t) := \frac{1_{B(x,\frac{t}{2})}}{\big|B(y,\frac{t}{2})\big|}\int_{B(y,\frac{t}{2})} f(z,y,t)\,\mathrm{d}z.$$

We have $J_\phi = N_\phi J$ and $J_{\frac{1}{2}} = \widetilde{N}_{\frac{1}{2}} J_\phi$. Moreover the operators N_ϕ and $\widetilde{N}_{\frac{1}{2}}$ act as the pointwise multipliers

$$N_\phi(\widetilde{f} \otimes g)(\cdot,y,t) = \widetilde{f}(y,t) \otimes M^\phi_{y,t} A^1_{B(y,t)} g,$$
$$\widetilde{N}_{\frac{1}{2}}(\widetilde{f} \otimes g)(\cdot,y,t) = \widetilde{f}(y,t) \otimes A^1_{B(y,\frac{t}{2})} g.$$

where $M^\phi_{y,t} g(x) := \phi(\frac{|y-x|}{t}) g(x)$. By Lemma 2.2 and Theorem 4.3 the result follows from Proposition 3.5 and Kahane's contraction principle. □

If X is a UMD space, and $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, we have natural isomorphisms

$$(L^p(\gamma(L^2(\tfrac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)))^* \eqsim L^q((\gamma(L^2(\tfrac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))^*) \eqsim L^q(\gamma(L^2(\tfrac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X^*))).$$

The first of these follows from the fact that X, and therefore $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)$, is reflexive, and the second follows from the K-convexity of UMD spaces. Denoting by N the projection of Proposition 4.2, it is easily verified that under the above identification the adjoint N^* is given by the same formula. As a result we obtain the following representation for the dual of $T^{p,2}(X)$:

Theorem 4.6. *If X is a UMD space, and $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, we have a natural isomorphism*

$$(T^{p,2}(X))^* \eqsim T^{q,2}(X^*).$$

As an immediate consequence of Proposition 2.3 we obtain the following result.

Theorem 4.7. *Let $1 < p_0 \leqslant p_1 < \infty$, and let X_0 and X_1 be UMD spaces. Then for all $0 < \theta < 1$ we have*

$$[T^{p_0,2}(X_0), T^{p_1,2}(X_1)]_\theta = T^{p_\theta,2}([X_0, X_1]_\theta), \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. The result follows by combining (2.3) with the following facts: (i) if X is a UMD space, then $L^p(X)$ is a UMD space for all $1 < p < \infty$, (ii) UMD spaces are K-convex, (iii) for $1 \leqslant p_0 \leqslant p_1 < \infty$ we have $[L^{p_0}(X_0), L^{p_1}(X_1)]_\theta = L^{p_\theta}([X_0, X_1]_\theta)$ with p_θ as above. □

We conclude this section with a result showing that certain singular integral operators are bounded from $L^p(X)$ to $T^{p,2}(X)$. This gives a Banach space-valued extension of [11, Section 4].

Theorem 4.8. *Let X be a UMD space. Consider the singular integral operator defined by*

$$Sf(t,y) = \int_{\mathbb{R}^n} k_t(y,z) f(z)\,\mathrm{d}z$$

for $f \in C_c(\mathbb{R}^n)$ and a measurable complex-valued function $(t,y,z) \mapsto k_t(y,z)$. Assume that

(1) $S \in \mathscr{L}(L^2, T^{2,2})$,
(2) *There exists $\alpha > 0$ such that for all $y, z \in \mathbb{R}^n$ and $t > 0$ we have*
$$|k_t(y,z)| \lesssim \frac{t^\alpha}{(|y-z|+t)^{n+\alpha}},$$
(3) *There exists $\beta > 0$ such that for all $t > 0$ and all $y, z, z' \in \mathbb{R}^n$ satisfying $|z-y| + t > 2|z-z'|$ we have*
$$|k_t(y,z) - k_t(y,z')| \lesssim \frac{t^\beta |z-z'|}{(|y-z|+t)^{n+1+\beta}},$$
(4) *For all $t > 0$ and $y \in \mathbb{R}^n$ we have*
$$\int_{\mathbb{R}^n} k_t(y,z)\,\mathrm{d}z = 0.$$

Let $1 < p < \infty$. Then $S \otimes I_X$ extends to a bounded operator from $L^p(X)$ to $T^{p,2}(X)$.

Proof. We consider the auxiliary operator T taking X-valued functions to ones with values in $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X)$, given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,z) \otimes f(z)\,\mathrm{d}z, \qquad f \in C_c(X),$$

where $K(x,z)$ is the $L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})$-valued kernel defined by

$$K(x,z) : (y,t) \mapsto \phi\big(\frac{|y-x|}{t}\big) k_t(y,z)$$

for some even $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(w) = 1$ if $|w| \leq \frac{1}{2}$, $\phi(w) = 0$ if $|w| \geqslant 1$, and $\int_0^1 \phi(r) r^{n-1}\,\mathrm{d}r = 0$. The claim of the theorem follows if we can show that T extends to a bounded operator from $L^p(X)$ to $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}); X))$. This is proved by applying a version of the $T(1)$ theorem for Hilbert space -valued kernels from [15] (which, in turn, is based on results from [17, 18]). We first remark that the condition

$T(1) = 0$ follows directly from (4), whereas the vanishing integral assumption on ϕ guarantees that $T'(1) = 0$, too. It remains to check the following $L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})$-valued versions of the standard estimates:

$$\sup_{x,z\in\mathbb{R}^n} |x-z|^n \|K(x,z)\|_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} \lesssim 1, \tag{4.2}$$

$$\sup_{\substack{x,x',z\in\mathbb{R}^n\\ |x-z|>2|x-x'|}} \frac{|x-z|^{n+1}}{|x-x'|} \|K(x,z)-K(x',z)\|_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} \lesssim 1, \tag{4.3}$$

$$\sup_{\substack{x,z,z'\in\mathbb{R}^n\\ |x-z|>2|z-z'|}} \frac{|x-z|^{n+1}}{|z-z'|} \|K(x,z)-K(x,z')\|_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} \lesssim 1, \tag{4.4}$$

and the weak boundedness property: for any $\eta, \widetilde{\eta} \in C_c^\infty(B(0,1))$ which satisfy the bounds $\|\eta\|_\infty, \|\widetilde{\eta}\|_\infty, \|\nabla\eta\|_\infty, \|\nabla\widetilde{\eta}\|_\infty \leq 1$, one should have

$$\sup_{(u,r)\in\mathbb{R}^n\times\mathbb{R}_+} \Big\| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,z)\eta\big(\frac{x-u}{r}\big)\widetilde{\eta}\big(\frac{z-u}{r}\big)\frac{\mathrm{d}z\,\mathrm{d}x}{r^n}\Big\|_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} \lesssim 1. \tag{4.5}$$

Proof of (4.2): Using (2) and noting that we have $\phi\big(\frac{|y-x|}{t}\big) = 0$ for $y \notin B(x,t)$,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \Big|\phi\big(\frac{|y-x|}{t}\big)k_t(y,z)\Big|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\\
&\lesssim \int_0^{|x-z|}\int_{B(x,t)} \Big|\frac{t^\alpha}{(|x-z|+t-|y-x|)^{n+\alpha}}\Big|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} + \int_{|x-z|}^\infty \int_{B(x,t)} \frac{\mathrm{d}y\,\mathrm{d}t}{t^{3n+1}}\\
&\lesssim \int_0^{|x-z|} \frac{t^{2\alpha-1}}{|x-z|^{2n+2\alpha}}\,\mathrm{d}t + \int_{|x-z|}^\infty \frac{\mathrm{d}t}{t^{2n+1}} \lesssim |x-z|^{-2n}.
\end{aligned}$$

Proof of (4.3): Using (2) and the mean value theorem and reasoning as above, for x, x', z satisfying $|x-z| > 2|x-x'|$ we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \Big|\Big(\phi\big(\frac{|y-x|}{t}\big) - \phi\big(\frac{|y-x'|}{t}\big)\Big)k_t(y,z)\Big|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\\
&\qquad\lesssim \int_0^\infty \int_{B(x,t)} \Big(\frac{|x-x'|t^\alpha}{t(|y-z|+t)^{n+\alpha}}\Big)^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} + \text{similar}\\
&\qquad\lesssim \int_0^{|x-z|} \int_{B(x,t)} \Big(\frac{|x-x'|t^\alpha}{t(|x-z|+t-|y-x|)^{n+\alpha}}\Big)^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\\
&\qquad\quad + \int_{|x-z|}^\infty |x-x'|^2 \frac{\mathrm{d}t}{t^{2n+3}} + \text{similar}\\
&\qquad\lesssim \int_0^{|x-z|} \frac{t^{2\alpha-3}|x-x'|^2}{|x-z|^{2n+2\alpha}}dt + \frac{|x-x'|^2}{|x-z|^{2n+2}} + \text{similar}\\
&\qquad\lesssim \frac{|x-x'|^2}{|x-z|^{2n+2}},
\end{aligned}$$

where the words "similar" above refer to a copy of the other terms appearing in the same step, with all the occurrences of x and x' interchanged.

Proof of (4.4): Using (3), for x, z, z' satisfying $|x-z| > 2|z-z'|$ we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \Big|\phi\big(\frac{|y-x|}{t}\big)\big(k_t(y,z)-k_t(y,z')\big)\Big|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \\
&\lesssim \int_0^\infty \int_{B(x,t)} \Big(\frac{t^\beta|z-z'|}{(|z-y|+t)^{n+1+\beta}}\Big)^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \\
&\lesssim \int_0^{|x-z|} \int_{B(x,t)} \Big(\frac{t^\beta|z-z'|}{(|z-x|+t-|y-x|)^{n+1+\beta}}\Big)^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} + \int_{|x-z|}^\infty \frac{|z-z'|^2}{t^{2n+3}}\,\mathrm{d}t \\
&\lesssim \int_0^{|x-z|} \frac{t^{2\beta-1}|z-z'|^2}{|z-x|^{2n+2+2\beta}}\,\mathrm{d}t + \int_{|x-z|}^\infty \frac{|z-z'|^2}{t^{2n+3}}\,\mathrm{d}t \lesssim \frac{|z-z'|^2}{|x-z|^{2n+2}}.
\end{aligned}$$

Proof of (4.5): Using the Cauchy-Schwarz inequality and (1) we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi\big(\frac{|y-x|}{t}\big)k_t(y,z)\eta\big(\frac{x-u}{r}\big)\widetilde{\eta}\big(\frac{z-u}{r}\big)\frac{\mathrm{d}z\,\mathrm{d}x}{r^n}\Big|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \\
&\qquad \lesssim \frac{1}{r^n}\int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big|\phi\big(\frac{|y-x|}{t}\big)\int_{\mathbb{R}^n} k_t(y,z)\widetilde{\eta}\big(\frac{z-u}{r}\big)\,\mathrm{d}z\Big|^2 \frac{\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}x}{t^{n+1}} \\
&\qquad \lesssim \frac{1}{r^n}\big\|S\big(\widetilde{\eta}\big(\frac{\cdot-u}{r}\big)\big)\big\|_{T^{2,2}}^2 \lesssim \|\widetilde{\eta}\|_{L^2}^2 \lesssim 1.
\end{aligned}$$

This concludes the proof. □

5. Off-diagonal estimates and their consequences

We start by recalling some terminology.

Definition 5.1. Let $M, t > 0$. An operator $T \in \mathscr{L}(L^2)$ is said to have *off-diagonal estimates of order M at the scale of t* if there is a constant C such that

$$\|Tf\|_{L^2(E)} \le C\langle d(E,F)/t\rangle^{-M}\|f\|_{L^2(F)}$$

for all Borel sets $E, F \subseteq \mathbb{R}^n$ and all $f \in L^2(\mathbb{R}^n)$ with support in F. Here, $\langle a\rangle = 1+|a|$ and $d(E,F) = \inf\{|x-y| : \ x \in E, \ y \in F\}$. The set of such operators is denoted by $OD_t(M)$.

Note that a single operator belongs to $OD_t(M)$ if and only if it belongs to $OD_s(M)$ whenever $s, t > 0$. However, the related constant C will typically not be the same. The scale of the off-diagonal estimates becomes very relevant when we want uniformity in the constants for a family of bounded operators. Thus we say that $(T_z)_{z\in\Sigma} \subseteq L^2$, where $\Sigma \subseteq \mathbb{C}$, satisfies *off-diagonal estimates of order M* if $T_z \in OD_{|z|}(M)$ for all $z \in \Sigma$ with the same constant C.

Theorem 5.2. *Let $1 < p < \infty$, X be a UMD Banach space, and $L^p(X)$ have type τ. Let $(T_t)_{t>0}$ be a uniformly bounded family of operators on L^2 satisfying off-diagonal estimates of order M for some $M > n/\tau$. Then the operator T, defined on $C_c \otimes X$ by*

$$T(g\otimes\xi)(y,t) := T_t(g(\cdot,t))(y)\otimes\xi,$$

extends uniquely to a bounded linear operator on $T^{p,2}(X)$.

Proof. Let us consider a function $f = \sum_i g_i \otimes \xi_i \in C_c \otimes X$. We define the sets

$$\begin{aligned}
C_0(x,t) &:= B(x,2t), \\
C_m(x,t) &:= B(x,2^{m+1}t)\setminus B(x,2^m,t), \quad m = 1,2,\dots,
\end{aligned}$$

so that there is a disjoint union $\bigcup_{m=0}^{\infty} C_m(x,t) = \mathbb{R}^n$. Let $(u_m)_{m=0}^{\infty}$ be the functions

$$u_m : x \mapsto \big[(y,t) \mapsto 1_{B(x,t)}(y) T_t\big(1_{C_m(x,t)} f(\cdot,t)\big)(y)\big],$$

where

$$T_t\big(1_{C_m(x,t)} f(\cdot,t)\big)(y) := \sum_i T_t(1_{C_m(x,t)} g_i(\cdot,t))(y) \otimes \xi_i.$$

We then have the formal expansion $J(Tf) = \sum_{m=0}^{\infty} u_m$, and for a fixed $x \in \mathbb{R}^n$, we separately estimate the $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X)$-norms of each $u_m(x)$.

Fix $\xi^* \in X^*$. Let us also write $\langle f(y,t), \xi^* \rangle := \sum_i g_i(y,t) \langle \xi_i, \xi^* \rangle$. For $m = 0$ we estimate, using the uniform boundedness of the operators T_t on L^2,

$$\begin{aligned} \|u_0(x)^* \xi^*\|^2_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} &= \int_{\mathbb{R}^{n+1}_+} 1_{B(x,t)}(y) \big| T_t\big(1_{B(x,2t)} \langle f(\cdot,t), \xi^* \rangle\big)(y) \big|^2 \, \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \\ &\lesssim \int_{\mathbb{R}^{n+1}_+} 1_{B(x,2t)}(y) |\langle f(y,t), \xi^* \rangle|^2 \, \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}. \end{aligned}$$

Hence, by covariance domination (Proposition 2.1),

$$\|u_0(x)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)} \lesssim \|(y,t) \mapsto 1_{B(x,2t)}(y) f(y,t)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)},$$

and we conclude that

$$\|u_0\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))} \lesssim \|f\|_{T^{p,2}_2(X)} \lesssim \|f\|_{T^{p,2}(X)}.$$

For $m \geqslant 1$, the off-diagonal estimates of order M imply

$$\begin{aligned} \|u_m(x)^* \xi^*\|^2_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} &= \int_{\mathbb{R}^{n+1}_+} 1_{B(x,t)}(y) \big| T_t\big(1_{C_m(x,t)} \langle f(\cdot,t), \xi^* \rangle\big)(y) \big|^2 \, \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}} \\ &\leq 2^{-2mM} \int_{\mathbb{R}^{n+1}_+} 1_{B(x,2^{m+1}t)}(y) \big|\langle f(y,t), \xi^* \rangle\big|^2 \, \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}. \end{aligned}$$

Hence, by covariance domination,

$$\|u_m(x)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)} \lesssim 2^{-mM} \|(y,t) \mapsto 1_{B(x,2^{m+1}t)}(y) f(y,t)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)},$$

and from Theorem 4.3 we conclude that

$$\|u_m\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))} \lesssim 2^{-mM} \|f\|_{T^{p,2}_{2^{m+1}}(X)} \lesssim 2^{-mM} \cdot m \cdot 2^{mn/\tau} \|f\|_{T^{p,2}(X)}.$$

Keeping in mind that $M > n/\tau$, we may sum over m to see that the formal expansion $J(Tf) = \sum_{m=0}^{\infty} u_m$ converges absolutely in $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$, and we obtain the desired result. □

Remark 5.3. The $T^{p,2}(X)$-boundedness of the operator T as considered above can be seen as a (p and X dependent) property of the (parametrised) operator family $(T_t)_{t>0} \subseteq \mathscr{L}(L^2)$. Let us call this property *tent-boundedness*. A simple example of a tent-bounded family consists of the translations $T_t f(x) = f(x + ty)$, where y is some unit vector. Indeed, these are obviously uniformly bounded in L^2 (and in L^p as well) and satisfy off-diagonal estimates of any order. In contrast to this, even when $X = \mathbb{C}$, it is well known that this family is not γ-bounded in L^p unless $p = 2$.

We next consider operators of the form

$$(Tf)_t := \int_0^\infty T_{t,s} f_s \frac{\mathrm{d}s}{s}, \qquad f \in C_c \otimes X,$$

where $T_{t,s} \in \mathscr{L}(L^2)$. This is first done separately for upper and lower diagonal "kernels" $T_{t,s}$.

Proposition 5.4. *Let $1 < p < \infty$, X be a UMD space, and let $L^p(X)$ have type τ. Let $(U_{t,s})_{0<t\leq s<\infty}$ be a uniformly bounded family of operators on L^2 such that $(U_{t,s})_{s\geqslant t} \in OD_s(M)$ uniformly in t for some $M > n/\tau$. Let further $\alpha > n/2$. Then*

$$(UF)_t = \int_t^\infty \big(\frac{t}{s}\big)^\alpha U_{t,s} F_s \frac{\mathrm{d}s}{s}$$

extends to a bounded operator on $T^{p,2}(X)$.

Proof. Let $F \in C_c \otimes X$ be arbitrary and fixed. It suffices to estimate the norm of the functions $u_k \in L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$ defined by

$$u_k : x \mapsto \big[(y,t) \mapsto 1_{B(x,t)} \int_t^\infty \big(\frac{t}{s}\big)^\alpha U_{t,s}(1_{C_k(x,s)} F_s)(y) \frac{\mathrm{d}s}{s}\big], \quad k = 0, 1, \dots,$$

where $C_0(x,s) := B(x, 2s)$, and $C_k(x,s) := B(x, 2^{k+1}s) \setminus B(x, 2^k s)$ for $k \geqslant 1$.

Let $x \in \mathbb{R}^n$ be fixed for the moment. To estimate the relevant $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X)$-norm at this point, we wish to use covariance domination. Hence let $\xi^* \in X^*$, write $f_s := \langle F_s(\cdot), \xi^* \rangle \in L^2$ for short, and consider the quantity

$$\langle (u_k(x))(y,t), \xi^* \rangle = 1_{B(x,t)} \int_t^\infty \big(\frac{t}{s}\big)^\alpha U_{t,s}(1_{C_k(x,s)} f_s)(y) \frac{\mathrm{d}s}{s}.$$

Its norm in $L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})$ is dominated by

$$\begin{aligned}
&\Big(\int_0^\infty \Big[\int_t^\infty \big(\frac{t}{s}\big)^\alpha \|1_{B(x,t)} U_{t,s}(1_{C_k(x,s)} f_s)\|_{L^2} \frac{\mathrm{d}s}{s} \Big]^2 \frac{\mathrm{d}t}{t^{n+1}} \Big)^{1/2} \\
&\leq \Big(\int_0^\infty \Big[\int_t^\infty \big(\frac{t}{s}\big)^{2\epsilon} \frac{\mathrm{d}s}{s} \Big] \Big[\int_t^\infty \big(\frac{t}{s}\big)^{2(\alpha-\epsilon)} \|1_{B(x,t)} U_{t,s}(1_{C_k(x,s)} f_s)\|_{L^2}^2 \frac{\mathrm{d}s}{s} \Big] \frac{\mathrm{d}t}{t^{n+1}} \Big)^{1/2} \\
&\lesssim \Big(\int_0^\infty \int_t^\infty \big(\frac{t}{s}\big)^{2(\alpha-\epsilon)} \big(2^{-kM} \|1_{B(x,2^{k+1}s)} f_s\|_{L^2}\big)^2 \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t^{n+1}} \Big)^{1/2} \\
&\eqsim 2^{-kM} \Big(\int_0^\infty \|1_{B(x,2^{k+1}s)} f_s\|_{L^2}^2 \frac{\mathrm{d}s}{s^{n+1}} \Big)^{1/2},
\end{aligned}$$

where in the last step we exchanged the order of integration and integrated out the t variable; the convergence required that $2(\alpha - \epsilon) > n$, which holds for sufficiently small $\epsilon > 0$, since $\alpha > n/2$.

The right-hand side of our computation is 2^{-kM} times the $L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})$-norm of $1_{B(x,2^{k+1}s)} \langle F_s(y), \xi^* \rangle$, so that covariance domination gives us

$$\|u_k(x)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)} \lesssim 2^{-kN} \|(J_{2^{k+1}} F)(x)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)}.$$

Taking L^p-norms and using Theorem 4.3 yields

$$\|u_k\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))} \lesssim 2^{-kM} \|F\|_{T^{p,2}_{2^{k+1}}(X)} \lesssim 2^{-kM}(1+k) 2^{kn/\tau} \|F\|_{T^{p,2}(X)}.$$

Recalling that $M > n/\tau$, we find that the formal expansion $J(UF) = \sum_{k=0}^{\infty} u_k$ converges absolutely in $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X))$, and we obtain the desired estimate $\|UF\|_{T^{p,2}(X)} \lesssim \|F\|_{T^{p,2}(X)}$. □

Proposition 5.5. *Let $1 < p < \infty$, X be a UMD space, and let $L^p(X)$ have type τ. Let $(L_{t,s})_{0<s\leq t<\infty}$ be a uniformly bounded family of operators on L^2 such that $(L_{t,s})_{t\geqslant s} \in OD_t(N)$ uniformly in s for some $N > n/\tau$. Let further $\beta > n(1/\tau - 1/2)$. Then*

$$(LF)_t = \int_0^t \big(\frac{s}{t}\big)^\beta L_{t,s} F_s \frac{\mathrm{d}s}{s}$$

extends to a bounded operator on $T^{p,2}(X)$.

Proof. The proof follows a similar approach as the previous one. This time, we expand $J(LF)$ in a double series $\sum_{k,m=0}^{\infty} v_{k,m}$, where

$$v_{k,m} : x \mapsto \big[(y,t) \mapsto \int_{2^{-(m+1)}t}^{2^{-m}t} \big(\frac{s}{t}\big)^\beta 1_{B(x,t)}(y) L_{t,s}(1_{C_k(x,t)} F_s)(y) \frac{\mathrm{d}s}{s}\big].$$

Again, we wish to estimate the $\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}), X)$-norm of $v_{k,m}(x)$ by covariance domination, for which purpose we take $\xi^* \in X^*$, write $f_s := \langle F_s(\cdot), \xi^*\rangle$, and compute

$$\begin{aligned}
&\|\langle v_{k,m}(x), \xi^*\rangle\|_{L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})} \\
&\leq \Big(\int_0^\infty \Big[\int_{2^{-(m+1)}t}^{2^{-m}t} 2^{-m\beta} \|1_{B(x,t)} L_{t,s}(1_{C_k(x,t)} F_s)\|_{L^2} \frac{\mathrm{d}s}{s}\Big]^2 \frac{\mathrm{d}t}{t^{n+1}}\Big)^{1/2} \\
&\lesssim 2^{-m\beta}\Big(\int_0^\infty \int_{2^{-(m+1)}t}^{2^{-m}t} \big(2^{-kN}\|1_{B(x,2^{k+1}t)} F_s\|_{L^2}\big)^2 \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t^{n+1}}\Big)^{1/2} \\
&\lesssim 2^{-m(\beta+n/2)} 2^{-kN}\Big(\int_0^\infty \|1_{B(x,2^{k+m+2}s)} F_s\|_{L^2}^2 \frac{\mathrm{d}s}{s^{n+1}}\Big)^{1/2}.
\end{aligned}$$

This is $2^{-m(\beta+n/2)}2^{-kN}$ times the $L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}})$-norm of $1_{B(x,2^{k+m+2}s)}(y)\langle F_s(y), \xi^*\rangle$; hence by covariance domination

$$\|v_{k,m}(x)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)} \lesssim 2^{-m(\beta+n/2)}2^{-kN}\|(J_{2^{k+m+2}}F)(x)\|_{\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X)}.$$

Taking L^p-norms and using Theorem 4.3 we get

$$\begin{aligned}
\|v_{k,m}\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}),X))} &\lesssim 2^{-m(\beta+n/2)}2^{-kN}\|F\|_{T^{p,2}_{2^{k+m+2}}(X)} \\
&\lesssim 2^{-m(\beta+n/2)}2^{-kN}(1+k+m)2^{(k+m)n/\tau}\|F\|_{T^{p,2}(X)},
\end{aligned}$$

and we can sum up the series over k and m since $\beta + n/2 > n/\tau$ and $N > n/\tau$. □

Combining the previous two propositions with a duality argument, we finally obtain:

Theorem 5.6. *Let $1 < p < \infty$, X be a UMD space, and let $L^p(X)$ have type τ and cotype γ. Let $(T_{t,s})_{0<t,s<\infty}$ be a uniformly bounded family of operators on L^2 such that:*

(i) $(T_{t,s})_{s>t} \in OD_s(M)$ uniformly in t,
(ii) $(T_{t,s})_{t>s} \in OD_t(N)$ uniformly in s.

Then

$$(TF)_t = \int_0^\infty \min\{(\tfrac{t}{s})^\alpha, (\tfrac{s}{t})^\beta\} T_{t,s} F_s \frac{\mathrm{d}s}{s}$$

extends to a bounded operator on $T^{p,2}(X)$ *if at least one of the following four conditions is satisfied:*

(*a*) $M > n/\tau$, $\alpha > n/2$, $N > n/\tau$, *and* $\beta > n(1/\tau - 1/2)$,
(*b*) $M > n/\tau$, $\alpha > n/2$, $N > n(1-1/\gamma)$, *and* $\beta > n/2$,
(*c*) $M > n(1-1/\gamma)$, $\alpha > n(1/2-1/\gamma)$, $N > n/\tau$, *and* $\beta > n(1/\tau - 1/2)$,
(*d*) $M > n(1-1/\gamma)$, $\alpha > n(1/2-1/\gamma)$, $N > n(1-1/\gamma)$, *and* $\beta > n/2$.

Proof. We split T into a sum $U+L$ of upper and lower triangular parts as considered in the previous two propositions. Part (*a*) is an immediate consequence, since the conditions on M and α guarantee the boundedness of U and those on N and β that of L.

For part (*b*), the boundedness of U follows as before. As for L, we observe that its (formal) adjoint on $T^{p',2}(X^*)$ is the upper triangular operator

$$(L^*G)_t = \int_t^\infty (\tfrac{t}{s})^\beta T^*_{s,t} G_s \frac{\mathrm{d}s}{s},$$

where $T^*_{s,t} \in OD_s(N)$ and $L^{p'}(X^*) = (L^p(X))^*$ has type $\gamma' = \gamma/(\gamma-1)$. We know that this operator is bounded on $T^{p',2}(X^*)$ under the conditions that $N > n/\gamma' = n(1-1/\gamma)$ and $\beta > n/2$.

Parts (*c*) and (*d*) are proved similarly by considering U^* and L, and U^* and L^*, respectively. □

The most important case for us is when $N = M$, and we record this as a corollary for later reference. In this situation, the condition (*b*) of Theorem 5.6 becomes redundant, since it is always contained in condition (*a*).

Corollary 5.7. *Let* $1 < p < \infty$, X *be a UMD space, and let* $L^p(X)$ *have type* τ *and cotype* γ. *Let* $(T_{t,s})_{0<t,s<\infty}$ *be a uniformly bounded family of operators on* L^2 *such that* $T_{t,s} \in OD_{\max\{t,s\}}(M)$ *uniformly in* t *and* s. *Then*

$$(TF)_t = \int_0^\infty \min\{(\tfrac{t}{s})^\alpha, (\tfrac{s}{t})^\beta\} T_{t,s} F_s \frac{\mathrm{d}s}{s} \tag{5.1}$$

extends to a bounded operator on $T^{p,2}(X)$ *if at least one of the following three conditions is satisfied:*

(*a*) $M > n/\tau$, $\alpha > n/2$, *and* $\beta > n(1/\tau - 1/2)$,
(*c*) $M > n \cdot \max\{1/\tau, 1-1/\gamma\}$, $\alpha > n(1/2-1/\gamma)$, *and* $\beta > n(1/\tau - 1/2)$,
(*d*) $M > n(1-1/\gamma)$, $\alpha > n(1/2-1/\gamma)$, *and* $\beta > n/2$.

Remark 5.8. If $X = \mathbb{C}$ (or more generally a Hilbert space), then one can take $\tau = \min(2,p)$ and $\gamma = \max(2,p)$ in Corollary 5.7. For $p \in [2,\infty)$ (so that $\tau = 2$), part (*a*) provides the following sufficient condition for the $T^{p,2}$-boundedness of (5.1): $M, \alpha > n/2$, and $\beta > 0$. For $p \in (1,2]$ (so that $\gamma = 2$), part (*d*) in turn gives $M, \beta > n/2$, and $\alpha > 0$. This recovers the corresponding result in [3] in the Euclidean case for $p \in (1,\infty)$. Note that in [3] the end-points $p \in \{1,\infty\}$ are also considered; in fact, the proof for $p \in (1,2)$ goes via interpolating between estimates available in the atomic space $T^{1,2}$ and the Hilbert space $T^{2,2}$. See also [1], where a weak type $(1,1)$ estimate is obtained.

6. Bisectorial operators and functional calculus

In this section we collect some generalities concerning bisectorial operators and their H^∞-calculus. We denote by S_θ the (open) bisector of angle θ, i.e. $S_\theta = S_\theta^+ \cup S_\theta^-$ with $S_\theta^+ = \{z \in \mathbb{C}\setminus\{0\} : \ |\arg(z)| < \theta\}$ and $S_\theta^- = -S_\theta^+$. We denote by Γ_θ the boundary of S_θ, which is parametrised by arc-length and oriented anticlockwise around S_θ.

A closed, densely defined, linear operator A acting in a Banach space Y is called *bisectorial* (of *angle* ω, where $0 < \omega < \frac{1}{2}\pi$) if the spectrum of A is contained in $\overline{S_\omega}$ and for all $\omega < \theta < \frac{1}{2}\pi$ there exists a constant C_θ such that for all nonzero $z \in \mathbb{C} \setminus S_\theta$

$$\|(I + zA)^{-1}\| \leqslant C_\theta \frac{|z|}{d(z, S_\theta)}.$$

For $\alpha, \beta > 0$ we set

$$\begin{aligned}
\Psi_\alpha(S_\theta) &= \big\{f \in H^\infty(S_\theta) : \ \exists C \ \ |f(z)| \leq C \min(|z|^\alpha, 1) \text{ for all } z \in S_\theta\big\},\\
\Psi^\beta(S_\theta) &= \big\{f \in H^\infty(S_\theta) : \ \exists C \ \ |f(z)| \leq C \min(1, |z|^{-\beta}) \text{ for all } z \in S_\theta\big\},\\
\Psi^\beta_\alpha(S_\theta) &= \big\{f \in H^\infty(S_\theta) : \ \exists C \ \ |f(z)| \leq C \min(|z|^\alpha, |z|^{-\beta}) \text{ for all } z \in S_\theta\big\}
\end{aligned}$$

and $\Psi(S_\theta) = \bigcup_{\alpha,\beta>0} \Psi^\beta_\alpha(S_\theta)$.

Let $\omega < \theta < \frac{1}{2}\pi$ be fixed. For $\psi \in \Psi(S_\theta)$, we define

$$\psi(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} \psi(z)(z - A)^{-1} \, \mathrm{d}z.$$

The resolvent bounds for A imply that this integral converges absolutely in $\mathscr{L}(Y)$. If one has, in addition, the quantitative estimate

$$\|\psi(A)\|_{\mathscr{L}(Y)} \lesssim \|\psi\|_\infty,$$

then A is said to have $H^\infty(S_\theta)$-calculus on Y.

Lemma 6.1. *Let A be bisectorial of angle ω and let $\theta > \omega$.*

1. *For $\phi_1, \phi_2 \in \Psi(S_\theta)$ we have $\phi_1(A)\phi_2(A) = (\phi_1 \cdot \phi_2)(A)$; this is also true if $\phi_2 \in H^\infty(S_\theta)$ is a rational function, in which case $\phi_2(A)$ is defined in the usual way by using the resolvents of A.*
2. *For all $\psi_1 \in \Psi(S_\theta)$, $\psi_2 \in H^\infty(S_\theta)$, $\psi_3 \in \Psi(S_\theta)$ we have*

$$\psi_1(A)(\psi_2\psi_3)(A) = (\psi_1\psi_2)(A)\psi_3(A).$$

Proof. The first claim is the well-known homomorphism property, which in both cases can be proved by writing out the definition of $\phi_1(A)\phi_2(A)$, performing a partial fraction expansion, and using Cauchy's theorem. The second claim follows from the homomorphism property for $\psi_2 \in \Psi(S_\theta)$, and the general case can be obtained from this by approximation (cf. [20, Theorem 9.2(i)]). □

Lemma 6.2. *Let A be bisectorial of angle ω and let $\theta > \omega$. Let $\mathsf{D}(A)$ and $\mathsf{R}(A)$ denote the domain and range of A, respectively. Then,*

$$\overline{\mathsf{R}(A)} = \overline{\mathsf{R}(A) \cap \mathsf{D}(A)} = \overline{\mathsf{R}(A(I + A)^{-2})} = \overline{\bigcup_{\psi \in \Psi(S_\theta)} \mathsf{R}(\psi(A))}.$$

Proof. If $f = \psi(A)g \in \mathsf{R}(\psi(A))$, let $f_\varepsilon := A(\varepsilon + A)^{-1} f \in \mathsf{R}(A)$. Then

$$f - f_\varepsilon = \varepsilon(\varepsilon + A)^{-1}\psi(A)g = \frac{1}{2\pi i}\int_\Gamma \frac{\varepsilon}{\varepsilon + z}\psi(z)(z - A)^{-1} g \,\mathrm{d}z.$$

The integrand is bounded by $\left|\psi(z)z^{-1}\right| \in L^1(\Gamma, |\,\mathrm{d}z|)$ and tends pointwise to zero as $\varepsilon \to 0$. Hence $f_\varepsilon \to f$ by dominated convergence.

Next we observe that $f^\varepsilon = (I + \varepsilon A)^{-1} f \to f$ as $\varepsilon \to 0$. Indeed, if $f \in \mathsf{D}(A)$, then $f - f^\varepsilon = \varepsilon \cdot (I + \varepsilon A)^{-1} A f$ has norm at most $C\varepsilon$, since the second factor stays uniformly bounded. Since the operators $(I + \varepsilon A)^{-1}$ are uniformly bounded and $\mathsf{D}(A)$ is dense, the convergence remains true for all f. If now $f \in \mathsf{R}(A)$, then $f^\varepsilon \in \mathsf{R}(A) \cap \mathsf{D}(A)$.

To complete the chain, let $f \in \mathsf{R}(A) \cap \mathsf{D}(A)$. Then for some $g \in \mathsf{D}(A^2)$ we have $f = Ag = A(I + A)^{-2}(I + A)^2 g = \psi(A)h$, where $\psi(z) = z/(1 + z)^2 \in \Psi$ and $h = (I + A)^2 g$. This completes the proof. □

We say that $\psi \in \Psi_\alpha^\beta(S_\theta)$ is degenerate if (at least) one of the restrictions $\psi|_{S_\theta^\pm}$ vanishes identically; otherwise it is called non-degenerate. The following two lemmas go back to Calderón, cf. [27, Section IV.6.19]. For the convenience of the reader we include simple proofs.

Lemma 6.3 (Calderón's reproducing formula, I)**.** *Let $\psi \in \Psi_\alpha^\beta(S_\theta)$ be non-degenerate. If $\alpha' \geqslant \alpha$ and $\beta' \geqslant \beta$, there exists $\widetilde{\psi} \in \Psi_{\alpha'}^{\beta'}(S_\theta)$ such that*

$$\int_0^\infty \psi(tz)\widetilde{\psi}(tz)\frac{\mathrm{d}t}{t} = 1, \quad z \in S_\theta. \tag{6.1}$$

Proof. Let $\overline{\psi}(z) := \overline{\psi(\overline{z})}$. Let $m \geqslant \max(\alpha' - \alpha, \beta' - \beta)$ and denote

$$c_\pm := \int_0^\infty \frac{(\pm t)^m}{(1 + t^2)^m}\psi(\pm t)\overline{\psi}(\pm t)\frac{\mathrm{d}t}{t}.$$

By non-degeneracy, $c_\pm > 0$. Hence the function $\widetilde{\psi}(z) = c_\pm^{-1} z^m (1 + z^2)^{-m}\overline{\psi(\overline{z})}$ for $z \in S_\theta^\pm$ has the desired properties. □

Lemma 6.4 (Calderón's reproducing formula, II)**.** *Let $\psi, \widetilde{\psi} \in \Psi(S_\theta)$ satisfy (6.1). Then*

$$\int_0^\infty \psi(tA)\widetilde{\psi}(tA)f\frac{\mathrm{d}t}{t} = f, \qquad f \in \overline{\mathsf{R}(A)},$$

where the left side is defined as an indefinite Riemann integral in L^2.

Proof. Let first $f = \phi(A)g$ for some $\phi \in \Psi(S_\theta)$. Then

$$\begin{aligned}
\int_0^\infty \psi(tA)\widetilde{\psi}(tA)f\frac{\mathrm{d}t}{t} &= \int_0^\infty (\psi(t\cdot)\widetilde{\psi}(t\cdot)\phi(\cdot))(A)g\frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \frac{1}{2\pi i}\int_{\Gamma_{\theta'}} \psi(tz)\widetilde{\psi}(tz)\phi(z)(z - A)^{-1} g \,\mathrm{d}z\frac{\mathrm{d}t}{t} \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\theta'}}\int_0^\infty \psi(tz)\widetilde{\psi}(tz)\frac{\mathrm{d}t}{t}\phi(z)(z - A)^{-1} g \,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\theta'}} \phi(z)(z - A)^{-1} g \,\mathrm{d}z = \phi(A)g = f
\end{aligned}$$

by Lemma 6.1, absolute convergence and Fubini's theorem. To conclude, we recall from Lemma 6.2 that functions as above are dense in $\overline{\mathsf{R}(A)}$, and notice that

$\int_a^b \psi(sz)\widetilde{\psi}(sz)\,\mathrm{d}s/s$ are uniformly in $H^\infty(S_\theta)$ so that the corresponding operators obtained by the formal substitution $z := A$ are uniformly bounded by the functional calculus. From this the convergence of the indefinite Riemann integral to the asserted limit follows easily. □

7. Hardy spaces associated with bisectorial operators

We now move on to more specific spaces and operators. We are concerned with systems of N^2 operators on L^2, which we shall view as single operators on $(L^2)^N$. Off-diagonal estimates for families of such operators can be defined by the same formal requirement as in the case $N = 1$ above. It is easy to see that off-diagonal estimates for a family of operators in $(L^2)^N$ are equivalent to off-diagonal estimates for the N^2 families of operators in L^2 corresponding to the entries in the matrix representation of the original operators.

In the vector-valued context we shall frequently use the natural identifications

$$(L^2)^N \otimes X = (L^2 \otimes \mathbb{C}^N) \otimes X = L^2 \otimes (\mathbb{C}^N \otimes X) \subset L^2(X^N).$$

Under these identifications, the elementary tensor $(0, \dots, 0, f, 0, \dots, 0) \otimes x$ (with f at the n-th entry) is identified with elementary tensor $f \otimes (0, \dots, 0, x, 0, \dots, 0)$ (with x at the n-th entry).

Throughout this section we fix a UMD Banach space X and an exponent $1 < p < \infty$, and suppose that the following assumptions are satisfied:

Assumption 7.1. *The numbers $1 \leqslant \tau \leqslant 2$ and $2 \leqslant \gamma \leqslant \infty$ are fixed in such a way that $L^p(X)$ has type τ and cotype γ.*

Assumption 7.2. *The operator A in $(L^2)^N$ is bisectorial of angle $0 < \omega < \pi/2$. For $\omega < \theta' < \theta < \pi/2$, it also has an $H^\infty(S_\theta)$-calculus on $(L^2)^N$. Moreover, the family $((I + \zeta A)^{-1})_{\zeta \in \mathbb{C} \setminus S_\theta}$ satisfies off-diagonal estimates of order M, where $M > n \cdot \min\{1/\tau, 1 - 1/\gamma\}$.*

With only the above assumptions at hand, A may fail to be bisectorial even for $N = 1$, and in particular to have an H^∞-calculus, in L^p for some values of $p \neq 2$. The tensor extension $A \otimes I_X$ may already fail these properties in $L^2(X)$. To study problems involving operators $f(A)$ in such spaces, we are thus led to define an appropriate scale of Hardy spaces associated with A. When A is the Hodge–Dirac operator or the Hodge–de Rham Laplacian on a complete Riemannian manifold, this has been done in [3]. We build on the ideas of this paper.

Lemma 7.3. *For $\omega < \theta < \pi/2$ and $\varepsilon > 0$, let $g \in H^\infty(S_\theta)$, and let $\psi \in \Psi^\varepsilon_{M+\varepsilon}(S_\theta)$. Then $\{(g \cdot \psi(t\cdot))(A)\}_{t>0}$ satisfies off-diagonal estimates of order M, and the off-diagonal constant has an upper bound which depends linearly on $\|g\|_\infty$.*

Proof. Let us denote by $\delta := d(E, F)$ the 'distance' of two Borel sets E and F as defined previously. Then, using the fact that $(I - z^{-1} A)^{-1} \in OD_{1/|z|}(M)$ uniformly

in $z \in S_\theta$,

$$\begin{aligned}
&\|1_E(g\cdot\psi(t\cdot))(A)1_F f\| \\
&\quad= \Big\|\frac{1}{2\pi i}\int_{\Gamma_{\theta'}} g(z)\psi(tz)1_E\big(I-\frac{1}{z}A\big)^{-1}1_F f\frac{\mathrm{d}z}{z}\Big\| \\
&\quad\lesssim \int_{\Gamma_{\theta'}} \min\big\{(t|z|)^{M+\varepsilon},(t|z|)^{-\varepsilon}\big\}(\delta|z|)^{-M}\|f\|\frac{|\mathrm{d}z|}{|z|} \\
&\quad\lesssim \int_0^{1/t} t^{M+\varepsilon}r^{M+\varepsilon}\cdot\delta^{-M}r^{-M}\|f\|\frac{\mathrm{d}r}{r}+\int_{1/t}^{\infty} t^{-\varepsilon}r^{-\varepsilon}\cdot\delta^{-M}r^{-M}\|f\|\frac{\mathrm{d}r}{r} \\
&\quad\eqsim t^M\delta^{-M}\|f\|,
\end{aligned}$$

and this proves the claim. □

Lemma 7.4. *Let $\alpha,\beta,\varepsilon>0$, and*

$$\psi\in\Psi^{\beta+\varepsilon}_{\max\{M-\beta,\alpha\}+\varepsilon}(S_\theta),\quad \widetilde{\psi}\in\Psi^{\alpha+\varepsilon}_{\max\{M-\alpha,\beta\}+\varepsilon}(S_\theta),\quad \phi\in\mathbb{C}1\oplus\Psi(S_\theta).$$

Then

$$\psi(tA)\phi(A)\widetilde{\psi}(sA)=\min\Big\{\big(\frac{t}{s}\big)^\alpha,\big(\frac{s}{t}\big)^\beta\Big\}S_{t,s},$$

where $(S_{t,s})_{t,s>0}$ is a uniformly bounded family of operators acting on $(L^2)^N$ such that $S_{t,s}\in OD_{\max\{t,s\}}(M)$, uniformly in t and s.

Proof. We have

$$\psi(tA)\phi(A)\widetilde{\psi}(sA)=(t/s)^\alpha\psi_0(tA)\phi(A)\widetilde{\psi}_0(sA)=(s/t)^\beta\psi_1(tA)\phi(A)\widetilde{\psi}_1(sA),$$

where

$$\begin{aligned}
\psi_0(z)&:=z^{-\alpha}\psi(z)\in\Psi^{\alpha+\beta+\varepsilon}_\varepsilon, & \widetilde{\psi}_0(z)&:=z^\alpha\widetilde{\psi}(z)\in\Psi^\varepsilon_{M+\varepsilon},\\
\psi_1(z)&:=z^\beta\psi(z)\in\Psi^\varepsilon_{M+\varepsilon}, & \widetilde{\psi}_1(z)&:=z^{-\alpha}\widetilde{\psi}(z)\in\Psi^{\alpha+\beta+\varepsilon}_\varepsilon.
\end{aligned}$$

The case $s\geqslant t$ of the claim follows from Lemma 7.3 (with s in playing the role of t in that Lemma) with $g(z)=\psi_0(tz)\phi(z)$ and $\widetilde{\psi}_0$ in place of ψ, while for the other case we take $g(z)=\phi(z)\widetilde{\psi}_1(sz)$ and ψ_1 in place of ψ. □

It is immediate to check that $T^{p,2}(X^N)\eqsim(T^{p,2}(X))^N$. Just as we extended the action of some operators on L^2 to $T^{p,2}(X)$, we may use this isomorphism to extend operators on $(L^2)^N$ to $T^{p,2}(X^N)$ by using their matrix representation and the extension procedure already discussed when $N=1$.

Proposition 7.5. *Let $\psi,\tilde{\psi}\in\Psi(S_\theta)$ and $\phi\in\mathbb{C}1\oplus\Psi(S_\theta)$. Then*

$$(TF)_t=\int_0^\infty\psi(tA)\phi(A)\psi(sA)F_s\frac{\mathrm{d}s}{s}$$

extends to a bounded operator on $T^{p,2}(X^N)$ if at least one of the following conditions is satisfied:

(a) *$M>n/\tau$, $\psi\in\Psi^{n(1/\tau-1/2)+\varepsilon}_{n/2+\varepsilon}$, and $\tilde{\psi}\in\Psi^{n/2+\varepsilon}_{n(1/\tau-1/2)+\varepsilon}$,*

(c) *$M>\max\{n/\tau,n(1-1/\gamma)\}$, $\psi\in\Psi^{n(1/\tau-1/2)+\varepsilon}_{n/2+n\max\{1/\gamma'-1/\tau,0\}+\varepsilon}$, and $\tilde{\psi}\in\Psi^{n(1/2-1/\gamma)+\varepsilon}_{n/2+n\max\{1/\tau-1/\gamma',0\}+\varepsilon}$,*

(d) *$M>n(1-1/\gamma)$, $\psi\in\Psi^{n/2+\varepsilon}_{n(1/2-1/\gamma)+\varepsilon}$, and $\tilde{\psi}\in\Psi^{n(1/2-1/\gamma)+\varepsilon}_{n/2+\varepsilon}$,*

where $\varepsilon>0$ is arbitrary.

Proof. This is directly, if slightly tediously, verified as a corollary of Lemma 7.4 and Corollary 5.7, so that the different conditions of Proposition 7.5 correspond to those of Corollary 5.7. As for the application of Corollary 5.7, we simply apply this to the N^2 matrix elements of the full operator, each of which acts on $T^{p,2}(X)$. □

Definition 7.6. We say that a pair of functions $(\psi, \tilde{\psi}) \in \Psi(S_\theta) \times \Psi(S_\theta)$ has *sufficient decay* if they verify at least one of the conditions (a), (c), or (d) of Proposition 7.5.

Remark 7.7. (i) Note that the notion of sufficient decay as defined above assumes that the parameters appearing in Assumptions 7.1 and 7.2 have been fixed. Also observe that if the parameters are such that for instance $n(1-1/\gamma) < M \leq n/\tau$, then only the condition (d) above is applicable.

(ii) If $(\psi, 0) \in \Psi(S_\theta) \times \Psi(S_\theta)$ has sufficient decay, by Calderón's reproducing formula there exists a $\tilde{\psi} \in \Psi(S_\theta)$ which satisfies (6.1) and decays as rapidly as desired; in particular, we may arrange so that the pair $(\psi, \tilde{\psi})$ also has sufficient decay. A similar remark applies if we start from a $\tilde{\psi} \in \Psi(S_\theta)$ such that $(0, \tilde{\psi})$ has sufficient decay.

For $f = \sum_i g_i \otimes \xi_i \in (L^2)^N \otimes X$ and $\psi \in \Psi(S_\theta)$ we shall write

$$(Q_\psi f)(y,t) := \sum_i \psi(tA) g_i(y) \otimes \xi_i := \psi(tA) f(y).$$

Definition 7.8. For $1 \leq p < \infty$ and a non-degenerate $\psi \in \Psi(S_\theta)$, the *Hardy space* $H^p_{A,\psi}(X^N)$ *associated with* A *and* ψ is the completion of the space

$$\{f \in \overline{\mathsf{R}(A)} \otimes X \subseteq (L^2)^N \otimes X : \ Q_\psi f \in T^{p,2}(X^N)\}$$

with respect to the norm

$$\|f\|_{H^p_{A,\psi}(X^N)} := \|Q_\psi f\|_{T^{p,2}(X^N)}.$$

It is clear that $\|\cdot\|_{H^p_{A,\psi}(X^N)}$ is a seminorm on $\overline{\mathsf{R}(A)} \otimes X$; that it is actually a norm will be seen shortly.

By definition, the operator

$$(Q_\psi f)(\cdot, t) := \psi(tA) f$$

embeds the Hardy space $H^p_{A,\psi}(X^N)$ isometrically into the tent space $T^{p,2}(X^N)$. Of importance will also be another operator acting to the opposite direction. For $\widetilde{\psi} \in \Psi(S_\theta)$, we define $S_{\widetilde{\psi}} f \in (L^2)^N \otimes X$ by

$$S_{\widetilde{\psi}} F := \int_0^\infty \widetilde{\psi}(sA) F(s, \cdot) \frac{\mathrm{d}s}{s} \tag{7.1}$$

for those functions $F \in L^1_{\mathrm{loc}}(\mathbb{R}_+; (L^2)^N) \otimes X$ for which the integral exists as a limit in $L^2(\mathbb{C}^N)$ of the finite integrals $\int_a^b$, where $a \to 0$ and $b \to \infty$.

By Calderón's reproducing formula, for a given $\psi \in \Psi(S_\theta)$, there exists a $\tilde{\psi} \in \Psi(S_\theta)$ such that the defining formula (7.1) makes sense for all $F \in Q_\psi(\overline{\mathsf{R}(A)} \otimes X)$, and we have

$$S_{\tilde{\psi}} Q_\psi f = f, \qquad f \in \overline{\mathsf{R}(A)} \otimes X. \tag{7.2}$$

Hence, if $\|f\|_{H^p_{A,\psi}(X^N)} = 0$ for some $f \in \overline{\mathsf{R}(A)} \otimes X$, this means by definition that $Q_\psi f = 0$, and the identity (7.2) yields immediately $f = 0$. Thus $\|\cdot\|_{H^p_{A,\psi}(X^N)}$ is indeed a norm.

Proposition 7.9. *Let $(\psi, \widetilde{\psi}) \in \Psi(S_\theta) \times \Psi(S_\theta)$ be a pair with sufficient decay. If $f \in T^{p,2}(X^N)$ is such that the defining formula (7.1) is valid, then $S_{\widetilde{\psi}} f \in H^p_{A,\psi}(X^N)$, and the mapping $f \mapsto S_{\widetilde{\psi}} f$ extends uniquely to a bounded operator from $T^{p,2}(X^N)$ to $H^p_{A,\psi}(X^N)$.*

Proof. Write $g := S_{\widetilde{\psi}} f$. First we check that $g \in \overline{\mathsf{R}(A)} \otimes X$: this is clear from the defining formula, since $\psi(sA)f(\cdot, s) \in \overline{\mathsf{R}(A)}$ for each $s > 0$ by Lemma 6.2, and Bochner integration in the Banach space $(L^2)^N$ preserves the closed subspace $\overline{\mathsf{R}(A)}$.

By Proposition 7.5,

$$(y,t) \mapsto \psi(tA)g(y) = \int_0^\infty \psi(tA)\widetilde{\psi}(sA)f(y,s)\frac{\mathrm{d}s}{s}$$

defines an element $\psi(\cdot A)g$ of $T^{p,2}(X^N)$ and we have

$$\|S_{\widetilde{\psi}} f\|_{H^p_{A,\psi}(X^N)} = \|\psi(\cdot A)g\|_{T^{p,2}(X^N)} \lesssim \|f\|_{T^{p,2}(X^N)}.$$

The subspace of $T^{p,2}(X^N)$ where the defining formula (7.1) is valid contains e.g. $(C_c)^N \otimes X$ and is therefore dense in $T^{p,2}(X^N)$. Hence the mapping $S_{\widetilde{\psi}}$ has a unique extension to a bounded operator from $T^{p,2}(X^N)$ to $H^p_{A,\psi}(X^N)$. □

Next we show that $H^p_{A,\psi}(X^N)$ is independent of $\psi \in \Psi(S_\theta)$, provided $(\psi, 0)$ has sufficient decay. A typical function with this property is

$$\psi(z) = (\sqrt{z^2})^{n(\frac{1}{2}-\frac{1}{\gamma})+1} e^{-\sqrt{z^2}},$$

where γ denotes the cotype of $L^p(X)$. This gives the classical definition by the Poisson kernel when $X = \mathbb{C}$ and $1 < p \leq 2$, taking $\gamma = 2$.

Theorem 7.10. *Let $\psi, \underline{\psi} \in \Psi(S_\theta)$ be two functions such that $(\psi, 0)$ and $(\underline{\psi}, 0)$ have sufficient decay. Then:*

(i) $H^p_{A,\psi}(X^N) = H^p_{A,\underline{\psi}}(X^N) =: H^p_A(X^N)$.

(ii) *A has an H^∞-functional calculus on $H^p_A(X^N)$.*

Proof. Let $\phi \in \mathbb{C}1 \oplus \Psi(S_\theta)$ be arbitrary and fixed. Let $f \in \overline{\mathsf{R}(A)} \otimes X$. By Calderón's reproducing formula, there exists $\widetilde{\psi} \in \Psi(S_\theta)$ (with any prescribed decay) such that

$$\psi(tA)\phi(A)f = \int_0^\infty \psi(tA)\phi(A)\widetilde{\psi}(sA)\underline{\psi}(sA)f\frac{\mathrm{d}s}{s}.$$

Thus

$$\|\phi(A)f\|_{H^p_{A,\psi}(X^N)} = \|TQ_{\underline{\psi}}f\|_{T^{p,2}(X^N)},$$

where T is the operator on $T^{p,2}(X^N)$ given by

$$TF(y,t) = \int_0^\infty \psi(tA)\phi(A)\widetilde{\psi}(sA)F(y,s)\frac{\mathrm{d}s}{s}.$$

From Proposition 7.5 we deduce that

$$\|\phi(A)f\|_{H^p_{A,\psi}(X^N)} \lesssim \|Q_{\underline{\psi}}f\|_{T^{p,2}(X^N)} = \|f\|_{H^p_{A,\underline{\psi}}(X^N)}.$$

Taking $\phi = 1$, this gives (i). Taking $\phi \in \Psi(S_\theta)$, we obtain (ii). □

The following, by now quite simple result has some useful consequences:

Proposition 7.11. *If $(0,\widetilde{\psi})$ has sufficient decay, then the bounded mapping $S_{\widetilde{\psi}}: T^{p,2}(X^N) \to H^p_A(X^N)$ is surjective.*

Proof. By Remark 7.7, we find a $\psi \in \Psi(S_\theta)$ such that (7.2) is satisfied and $(\psi,\widetilde{\psi})$ has sufficient decay. Now let $f \in H^p_A(X^N) = H^p_{A,\psi}(X^N)$ be arbitrary and let $\lim_{n\to\infty} f_n = f$ in $H^p_{A,\psi}(X^N)$ with $f_n \in \overline{\mathsf{R}(A)} \otimes X$. The functions $g_n := Q_\psi f_n$ belong to $T^{p,2}(X^N)$ and $\|g_n - g_m\|_{T^{p,2}(X^N)} = \|f_n - f_m\|_{H^p_{A,\psi}(X^N)}$ for all m,n. It follows that the sequence (f_n) is Cauchy in $T^{p,2}(X^N)$ and therefore converges to some $f \in T^{p,2}(X^N)$. From $f_n = S_{\widetilde{\psi}} g_n$ and the continuity of $S_{\widetilde{\psi}}$ it follows that $f = S_{\widetilde{\psi}} g$. □

Corollary 7.12. *Let $(0,\widetilde{\psi})$ have sufficient decay. An equivalent description of the Hardy space is*

$$H^p_A(X^N) = \widetilde{H}^p_{A,\widetilde{\psi}}(X^N) := \{S_{\widetilde{\psi}} F : \ F \in T^{p,2}(X^N)\},$$

and an equivalent norm is given by

$$\|f\|_{\widetilde{H}^p_{A,\widetilde{\psi}}(X^N)} := \inf\{\|F\|_{T^{p,2}(X^N)} : \ f = S_{\widetilde{\psi}} F\}.$$

As a further consequence we deduce an interpolation result for Hardy spaces from the following general principle (see Theorem 1.2.4 in [28]): Let X_0, X_1 and Y_0, Y_1 be two interpolation couples such that there exist operators $S \in \mathscr{L}(Y_i, X_i)$ and $Q \in \mathscr{L}(X_i, Y_i)$ with $SQx = x$ for all $x \in X_i$ and $i = 0,1$. Then $[X_0, X_1]_\theta = S[Y_0, Y_1]_\theta$. Here we take $(\psi,\widetilde{\psi})$ as in the Calderón reproducing formula with sufficient decay, $S = S_{\widetilde{\psi}}$ and $Q = Q_\psi$.

Corollary 7.13. *Let X be a UMD space. For all $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$, and $N \geqslant 1$ we have*

$$[H^{p_0}_A(X^N), H^{p_1}_A(X^N)]_\theta = H^{p_\theta}_A(X^N)$$

with equivalent norms, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

8. Hardy spaces associated with differential operators

The construction described in Section 7 is particularly relevant when dealing with differential operators $A = D_B$ in $L^2 \oplus (L^2)^n$, where

$$D_B = \begin{pmatrix} 0 & -\mathrm{div} B \\ \nabla & 0 \end{pmatrix}$$

with B a multiplication operator on $(L^2)^n$ given by an $(n \times n)$-matrix with L^∞ entries. Such operators have been considered in connection with the celebrated square root problem of Kato, which was originally solved in [2]. A new proof based on first order methods was devised in [4], where it was shown that D_B is bisectorial on $L^2(\mathbb{C} \oplus \mathbb{C}^n)$ and satisfies off-diagonal estimates of any order.

In [16], the H^∞-functional calculus of $D_B \otimes I_X$ in $L^p(X) \oplus (L^p(X))^n = L^p(X \oplus X^n)$ is described in terms of R-boundedness of the resolvents. Although these resolvent conditions, and hence the functional calculus, may fail on $L^p(X \oplus X^n)$ in general, it follows from Section 7 that these operators do have an H^∞-functional

calculus on $H^p_{D_B}(X \oplus X^n)$, which in particular implies Kato type estimates in this space.

To express these estimates, observe first that $\overline{\mathsf{R}(D_B)} = \overline{\mathsf{R}(\mathrm{div}B)} \oplus \overline{\mathsf{R}(\nabla)}$. Let us hence write a function $f \in \overline{\mathsf{R}(D_B)} \otimes X$ as (f_0, f_1), where

$$f_0 \in \overline{\mathsf{R}(\mathrm{div}B)} \otimes X \subseteq L^2 \otimes X, \qquad f_1 \in \overline{\mathsf{R}(\nabla)} \otimes X \subseteq (L^2)^n \otimes X = L^2 \otimes X^n$$

denote the X-valued and X^n-valued parts of f, respectively. Defining

$$H^p_{D_B}(X \oplus X^n) := H^p_{D_B,\psi}(X \oplus X^n)$$

by means of the (even!) function $\psi(z) = (\sqrt{z^2})^N e^{-\sqrt{z^2}}$ with N large enough, we note that $\psi(tD_B) = \phi(t^2 D_B^2)$, where $\phi(z) = \sqrt{z}^N e^{-\sqrt{z}}$ and the operator

$$D_B^2 = \begin{pmatrix} -\mathrm{div}B\nabla & 0 \\ 0 & -\nabla \mathrm{div}B \end{pmatrix},$$

and hence $\phi(t^2 D_B^2)$, is diagonal with respect to the splitting $f = (f_0, f_1)$. In particular this shows that

$$\|(f_0, f_1)\|_{H^p_{D_B}(X \oplus X^n)} \eqsim \|(f_0, 0)\|_{H^p_{D_B}(X \oplus X^n)} + \|(0, f_1)\|_{H^p_{D_B}(X \oplus X^n)}.$$

Hence also the full space $H^p_{D_B}(X \oplus X^n)$ (constructed as the completion of $\overline{\mathsf{R}(D_B)} \otimes X$ with respect to the above-given norm) has the natural direct sum splitting into "X-valued" and "X^n-valued" components. Let us denote these components by $H^p_{D_B}(X)$ and $H^p_{D_B}(X^n)$, so that

$$\|f_0\|_{H^p_{D_B}(X)} := \|(f_0, 0)\|_{H^p_{D_B}(X \oplus X^n)},$$
$$\|f_1\|_{H^p_{D_B}(X^n)} := \|(0, f_1)\|_{H^p_{D_B}(X \oplus X^n)}.$$

Then we are ready to state:

Theorem 8.1. *Let X be a UMD space, $1 < p < \infty$, and D_B be as above. Then*

$$\|\sqrt{-\mathrm{div}B\nabla}u\|_{H^p_{D_B}(X)} \eqsim \|\nabla u\|_{H^p_{D_B}(X^n)}$$

for all $u \in \mathsf{D}(\nabla) \otimes X \subseteq L^2 \otimes X$.

Proof. We know from [4] that $(I + zD_B)^{-1}$ satisfies off-diagonal estimates of arbitrary order and that D_B has an $H^\infty(S_\theta)$-calculus on $L^2 \oplus (L^2)^n$.

Consider the function $\phi(z) = z/\sqrt{z^2} \in H^\infty(S_\theta)$. By the boundedness of the H^∞-calculus and the identity $1/\phi(z) = \phi(z)$,

$$\|\phi(D_B)f\|_{H^p_{D_B}(X \oplus X^n)} \eqsim \|f\|_{H^p_{D_B}(X \oplus X^n)}, \qquad f \in \overline{\mathsf{R}(D_B)} \otimes X. \tag{8.1}$$

Observing that

$$\phi(D_B) = \begin{pmatrix} 0 & -\mathrm{div}B(-\nabla B\mathrm{div})^{-1/2} \\ \nabla(-\mathrm{div}B\nabla)^{-1/2} & 0 \end{pmatrix}$$

and using (8.1) for $f = (f_0, 0)$ gives

$$\|\nabla(-\mathrm{div}B\nabla)^{-1/2} f_0\|_{H^p_{D_B}(X^n)} \eqsim \|f_0\|_{H^p_{D_B}(X)}, \qquad f_0 \in \overline{\mathsf{R}(\mathrm{div}B)} \otimes X. \tag{8.2}$$

Let then $u \in \mathsf{D}(\nabla) \otimes X$. By the solution of Kato's problem we have $\mathsf{D}(\nabla) = \mathsf{D}(\sqrt{-\mathrm{div}B\nabla})$. Substituting

$$f_0 = \sqrt{-\mathrm{div}B\nabla}u \in \mathsf{R}(\sqrt{-\mathrm{div}B\nabla}) \otimes X \subseteq \overline{\mathsf{R}(-\mathrm{div}B\nabla)} \otimes X \subseteq \overline{\mathsf{R}(\mathrm{div}B)} \otimes X$$

in (8.2), we obtain the assertion. We used above the inclusion $\mathsf{R}(\sqrt{A}) \subseteq \overline{\mathsf{R}(A)}$, which is true for all sectorial operators (see [10], Corollary 3.1.11). □

Let $D = D_I$ be the unperturbed operator. Observe that $D^2(f,0) = (\Delta f, 0)$ and then, whenever ψ is even, $\psi(tD)(f,0) = (\psi(t\sqrt{\Delta})f, 0)$. The space $H^p_D(X)$ is then the classical Hardy space.

Theorem 8.2. *Let X be UMD. Then $H^p_D(X) = L^p(X)$ for all $1 < p < \infty$.*

Proof. Let us denote by N the smallest integer greater than $\frac{n}{2}$ and, for functions $f \in C_c$, define

$$Sf(y,t) = \int_{\mathbb{R}^n} k_t(y,z) f(z)\,\mathrm{d}z,$$

where

$$k_t(y,z) = t^N \frac{\partial^N}{\partial t^N}\Big(t^{-n} p\big(\frac{(y-z)}{t}\big)\Big),$$

and $p(w) = 1/(1+w^2)^{\frac{n+1}{2}}$. For a fixed $t > 0$, $f \mapsto Sf(\cdot,t)$ is thus a Fourier multiplier with symbol $m_t(\xi) = (t|\xi|)^N e^{-t|\xi|}$. This implies assumptions (1) and (4) in Theorem 4.8. Assumptions (2) and (3), with $\alpha = \beta = 1$, follow from direct computations of the N-th derivative of $t \mapsto t^{-n} p(\frac{|x|}{t})$ and the mean value theorem. Now, for $f \in L^p(X)$, letting

$$Pf(y,t) := \psi(tD)(f,0)(y) = ((t\sqrt{\Delta})^N e^{-t\sqrt{\Delta}} f(y), 0)$$

and applying Theorem 4.8, we thus obtain that

$$\|f\|_{H^p_D(X)} \lesssim \|f\|_{L^p(X)}$$

for all $f \in L^p(X)$. Now let $f \in L^p(X)$ and $g \in L^{p'}(X^*)$, and denote by $\langle f, g\rangle$ their duality product. By Calderón's reproducing formula there exists $\widetilde{\psi}$ (with arbitrary decay) such that

$$\langle f, g\rangle = \int_0^\infty \langle \psi(t\Delta) f, \widetilde{\psi}(t\Delta)^* g\rangle \frac{\mathrm{d}t}{t}.$$

Therefore

$$|\langle f, g\rangle| \lesssim \|f\|_{H^p_D(X)} \|g\|_{H^{p'}_D(X^*)} \lesssim \|f\|_{H^p_D(X)} \|g\|_{L^{p'}(X)},$$

and hence $\|f\|_{L^p(X)} \lesssim \|f\|_{H^p_D(X)}$. □

9. Extensions

It is possible to incorporate a Hilbert space -parameter in the definition of the tent spaces in such a way that the results of the previous sections concerning Hardy spaces may be generalised to deal with operators in $L^2(H)$, with H possibly infinite-dimensional. In fact, the mapping $J : C_c \otimes X \to L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}); X))$ can also be defined on $C_c(H) \otimes X$ with the same formal expression, and then we let $T^{p,2}(H;X)$ be the completion of this space with respect to the norm

$$\|f\|_{T^{p,2}(H;X)} := \|Jf\|_{L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}};H),X))}.$$

The so-far considered space $T^{p,2}(X)$ is seen to be the special case with $H = \mathbb{C}$. It is also easy to check that for finite dimensional $H = \mathbb{C}^N$ we simply get

$$T^{p,2}(\mathbb{C}^N; X) \eqsim T^{p,2}(X^N). \tag{9.1}$$

The proofs in the special case treated above carry over to show that also the general $T^{p,2}(H;X)$ becomes a complemented subspace of $L^p(\gamma(L^2(\frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}};H),X))$; it is equivalent to the spaces $T^{p,2}_\alpha(H;X)$ with different apertures $\alpha \in (0,\infty)$; and so forth.

Given a bisectorial operator A acting in $L^2(H)$, satisfying off-diagonal resolvent bounds and with a bounded H^∞-calculus on this space, we define the associated Hardy space $H^p_{A,\psi}(H;X)$ as the completion of $\overline{\mathsf{R}(A)} \otimes X$ with respect to the norm $\|Q_\psi f\|_{T^{p,2}(H;X)}$. Due to (9.1), this is equivalent to what we did before. The various results concerning these spaces, such as the independence of ψ under sufficient decay, carry over to this generality with the same proofs. In fact, the proofs are even slightly more streamlined in the general framework, since with H built inside the tent space, we can treat the operators on $L^2(H)$ as a whole, instead of breaking them down to the matrix entries as we did in the finite-dimensional case.

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