Singular convolution integrals with operator-valued kernel

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Abstract

We study operators $f \mapsto Kf$ of the form $(Kf)(t) = \int_{\mathbf{R}^n} k(t-s)f(s)\,\mathrm{d}s$, where f is a vector-valued function and k an operator-valued singular kernel. Sufficient conditions for boundedness on L^p-spaces of UMD-valued functions are given in terms of a Hörmander-type condition involving R-boundedness. The results cover large classes of kernels and also provide new proofs of recent operator-valued Fourier multiplier theorems. Moreover, they give new information about families of singular integral operators.

1 Introduction

Singular integrals have been the object of extensive study since the pioneering work of A. P. Calderón and A. Zygmund [4] in the 50's. Their results showed that large classes of singular integral operators are automatically bounded on the whole scale of the reflexive $L^p(\mathbf{R}^n)$ spaces (i.e., $p \in]1,\infty[$) as soon as they are bounded on $L^2(\mathbf{R}^n)$ and the kernels satisfy certain conditions which hold and can be verified for many operators appearing in applications. Moreover, the required L^2-boundedness is obtained for free (and therefore goes often almost without being mentioned) with the help of the Fourier transform and Plancherel's theorem.

The first results of Calderón and Zygmund concerning convolutions by homogeneous kernels $k(t) = \Omega(t^0)/|t|^n$, $t^0 := t/|t|$, have been generalized in several directions by the same authors and many others, and useful sufficient conditions for L^p-boundedness are now known both in terms of the kernel k (as in the original results) and in terms of the multiplier or the symbol $m = \hat{k}$ (Fourier transform in the sense of distributions). A classical theorem giving sufficient conditions in terms of the multiplier is due to S. G. Mihlin, and a generalization was later proved by L. Hörmander as a corollary of his results on singular integrals [14]. In this connection Hörmander gave the world the condition bearing his name, today usually formulated as

$$\int_{|t|>2|s|} |k(t-s) - k(t)|\,\mathrm{d}t \le A < \infty, \tag{1.1}$$

and being a sufficient condition on k to boundedly extend the operator $f \mapsto k * f$, bounded on $L^{\tilde{p}}(\mathbf{R}^n)$ for one $\tilde{p} \in]1,\infty[$, to the whole scale of the spaces $L^p(\mathbf{R}^n)$, $p \in]1,\infty[$.

The question of whether these results could be extended to the Lebesgue–Bôchner spaces $L^p(\mathbf{R}^n; X)$ of vector-valued functions was taken up by several authors already in the 60's. It was observed by A. Benedek, A. P. Calderón and R. Panzone [2] that the boundedness on $L^{\tilde{p}}(\mathbf{R}^n; X)$ for one $\tilde{p} \in]1,\infty[$ of a convolution operator, together with Hörmander's condition, implies the boundedness on $L^p(\mathbf{R}^n; X)$ for all $p \in]1,\infty[$ also in the general situation of vector-valued functions and an operator-valued kernel. However, to actually get the boundedness, without *a priori* assumptions, even for the single $\tilde{p}$ (something that was immeadiate for $\tilde{p} = 2$ in the scalar-valued, or more generally, a Hilbert space setting) turned out to be a significantly more difficult task.

By the 80's it was understood that the boundedness of vector-valued singular integrals, in particular, the prototype example given by the Hilbert transform, is intimately connected with the geometry of Banach spaces. Indeed, it was shown by D. L. Burkholder and J. Bourgain that the boundedness of the Hilbert transform on $L^p(\mathbf{T}; X)$, $p \in]1,\infty[$, is equivalent to the so called

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UMD-property of the underlying Banach space X. Moreover, the boundedness of this one operator could already be used to show the boundedness of large classes of multipliers. In particular, the classical multiplier theorem of Mihlin was generalized (by F. Zimmermann [20] to the full generality on $\mathbf{R}^n$, based on the deep results of Bourgain [3] in the one-dimensional case) to the setting of scalar-valued multipliers acting on UMD-valued functions.

However, the general situation of operator-valued kernels or multipliers, which is of interest in the theory of evolution equations, remained open until the turn of the millennium. As the naïve generalization of the classical Mihlin condition by means of replacing absolute values by norms was found, by G. Pisier (unpublished), to imply the desired boundedness only in the Hilbert space setting, a new idea was required to build a condition strong enough to get the desired conclusion but reasonable enough to cover a wide range of relevant applications. This idea turned out to be the notion of R-boundedness, already implicit in the work of Bourgain and later Zimmermann and explicitly formulated by Ph. Clément, B. de Pagter, F. A. Sukochev and H. Witvliet [5] and by the second author [19] who first generalized the Mihlin theorem to allow for operator-valued multipliers but requiring R-boundedness instead of norm boundedness in reformulating Mihlin's conditions. Clément and J. Prüss [6] showed that the R-boundedness of the range of the multiplier is also necessary.

The realization of R-boundedness as the right notion for operator-valued multiplier theorems has spurred significant activity in the field, leading to several generalizations and improvements of the first results in this direction, as well as to applications in differential equations (see [7], [16] for a survey). In the present paper, we make use of these modern ideas to attack the operator-valued versions of the problems originally treated by Calderón and Zygmund, i.e., to search for conditions on the operator-valued singular kernel k to yield a bounded operator $f \mapsto k * f$ from $L^p(\mathbf{R}^n; X)$ to $L^p(\mathbf{R}^n; Y)$.

In the scalar-valued context it follows from Plancherel's theorem that $k*$ is bounded on $L^2(\mathbf{R}^n)$ if and only if $\hat{k}$ is bounded, and in the general situation we know from Clément and J. Prüss [6] that the range of $\hat{k}$ must even be R-bounded. Thus it is natural to impose the condition

$$\mathcal{R}(\{\hat{k}(\xi) |\ \xi \in \mathbf{R}^n\}) \leq A < \infty, \tag{1.2}$$

where $\mathcal{R}(\mathcal{T})$ denotes the R-bound (cf. Def. 3.2) of the set $\mathcal{T}$.

In the context of multiplier theorems, appropriate additional conditions are obtained by imposing Mihlin-type bounds, but replaced by R-bounds, for the derivatives of $\hat{k}$ (see [1], [11], [13], [17], [19]). However, we now search for conditions directly on the convolution kernel k, and it will be shown (Theorem 4.1) that sufficient conditions are obtained by incorporating the notion of R-boundedness into the classical Hörmander conditions so as to require that

$$\int_{|t|>2|s|} \mathcal{R}(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t)) |\ j \in \mathbf{Z}\}) \log(2+|t|) \,\mathrm{d}t \leq A \log(2+|s|), \tag{1.3}$$

Besides the R-bound, the new feature compared to the classical situation is the additional logarithmic factor, which arises from the use of a deep result of Bourgain concerning UMD-spaces. Nevertheless, this condition is still satisfied by large classes of singular kernels (cf. Theorem 5.10), and it also gives new information about collective properties (the R-boundedness) of families of singular integral operators (Theorem 6.4).

Besides being of interest on their own right, the results for the convolution operators can also be used to derive some recent operator-valued multiplier theorems (e.g. from [11], cf. Theorem 7.9). This is not surprising in view of the historical fact that Hörmander used his results on singular integrals to improve the theorem of Mihlin on Fourier multipliers. As a general remark, which will be given more quantitative content in the body of the paper, it seems that the understanding of the multipliers and convolution integrals greatly benefits from the interaction of the two different points of view. As an additional illustration for its usefulness, we give an alternative proof of the characterization (from [19]) of maximal regularity in terms of R-boundedness (Example 5.13).

2 A framework for vector-valued singular integrals

In this section we set up a convenient framework for vector-valued singular integrals of the form

$$Kf(t) = \int_{\mathbf{R}^n} k(t-s)f(s)\,\mathrm{d}s, \qquad t \in \mathbf{R}^n, \tag{2.1}$$

that will allow us to use the basic tools of harmonic analysis.

In the scalar case it is customary to assume that k is a tempered distribution which agrees on $\mathbf{R}^n \setminus \{0\}$ with a locally integrable function. For $\phi \in \mathcal{S}(\mathbf{R}^n)$, one interprets (2.1) as

$$K\phi(t) := (k * \phi)(t) = \langle k, \phi(t-\cdot)\rangle, \qquad t \in \mathbf{R}^n.$$

If one can prove an L^p-estimate $\|K\phi\|_{L^p} \le C\,\|\phi\|_{L^p}$ for all $\phi \in \mathcal{S}(\mathbf{R}^n)$, then by the density of $\mathcal{S}(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$, $1 \le p < \infty$, the operator K can be extended to a bounded operator on $L^p(\mathbf{R}^n)$, and we can think of this operator as formally given by (2.1).

In this paper we are typically interested in the case where k is an operator-valued function, say $t \in \mathbf{R}^n \setminus \{0\} \mapsto k(t) \in \mathcal{L}(X,Y)$, and f is in $L^p(\mathbf{R}^n; X)$. To give a meaning to (2.1), we therefore assume that k is an operator-valued distribution in

$$\mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X;Y)) := \mathcal{L}(\mathcal{S}(\mathbf{R}^n); \mathcal{L}(X;Y)). \tag{2.2}$$

But to avoid annoying technicalities about the convolutions of vector-valued distributions, we choose a special class of test-functions, namely $X \otimes \mathcal{S}(\mathbf{R}^n)$: for $x \in X$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$ we define a linear functional $x \otimes \phi$ on $\mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X,Y))$ by

$$[x \otimes \phi](k) := \langle k, \phi\rangle\, x,$$

and extend this definition by linearity from $X \times \mathcal{S}(\mathbf{R}^n)$ to the algebraic tensor product $X \otimes \mathcal{S}(\mathbf{R}^n)$. In particular, for $f = x \otimes \phi$, we can now interpret (2.1) as $\langle k, \phi(t-\cdot)\rangle\, x$, which we may also write as $k * \phi(t)x$ or $k * x\phi(t)$ or even $k(\cdot)x * \phi(t)$, which ever seems convenient in a particular context. Recall that the convolution $k * \phi(t) := \langle k, \phi(t-\cdot)\rangle$ of a tempered distribution k with a Schwartz function $\phi \in \mathcal{S}(\mathbf{R}^n)$ is an infinitely differentiable function with polynomially bounded derivatives of all orders; the vector-valued situation does not bring any complications at this point, and one can simply repeat the standard proofs from the scalar-valued theory.

Note that $X \otimes \mathcal{S}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n; X)$ for $1 \le p < \infty$, so that the class $X \otimes \mathcal{S}(\mathbf{R}^n)$ is sufficient to prove the boundedness of the operator K from (2.1) from $L^p(\mathbf{R}^n; X)$ to $L^p(\mathbf{R}^n; Y)$. For that matter, it will be enough to consider the even smaller class $X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n)$, where

$$\hat{\mathcal{D}}_0(\mathbf{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbf{R}^n) \,\middle|\, \hat{\psi} \in \mathcal{D}(\mathbf{R}^n),\ 0 \notin \operatorname{supp}\hat{\psi} \right\}.$$

This leads us to the following basic assumption for the kernel of a singular integral operator as in (2.1):

Assumption 2.3. For every $x \in X$, the distribution $k(\cdot)x \in \mathcal{S}'(\mathbf{R}^n; Y)$ (defined by $\langle k(\cdot)x, \phi\rangle := \langle k, \phi\rangle\, x$) agrees away from the origin with a locally integrable Y-valued function, which we denote by the same symbol $k(\cdot)x$. That is, we have

$$\langle k(\cdot)x, \phi\rangle = \int_{\mathbf{R}^n} k(t)x\,\phi(t)\,\mathrm{d}t \qquad \text{for } \phi \in \mathcal{S}(\mathbf{R}^n),\ 0 \notin \operatorname{supp}\phi.$$

Remark 2.4. By the definition of the convolution, and linearity, this gives

$$k * f(t) = \int_{\mathbf{R}^n} k(s)f(t-s)\,\mathrm{d}s \qquad \text{for } f \in X \otimes \mathcal{S}(\mathbf{R}^n),\ t \notin \operatorname{supp} f.$$

It is easy to see that, for $t \notin \operatorname{supp} f$, the representation $\sum x_j \otimes \phi_j$ of $f \in X \otimes \mathcal{S}(\mathbf{R}^n)$ can be chosen in such a way that $t \notin \operatorname{supp}\phi_j$ for any j.

We will often use duality arguments in which the adjoint distribution $k' \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(Y', X'))$ plays a rôle. This is defined by $\langle k', \phi \rangle := \langle k, \phi \rangle'$.

Let us now look at some examples.

Example 2.5. Repeating the argument for the scalar-valued situation, e.g. pp. 193–4 of [10], one can show that the following prominent class of operators provide singular integrals in the sense of the above definition: Let $t \mapsto k(t) \in \mathcal{L}(X;Y)$ be strongly locally integrable on $\mathbf{R}^n \setminus \{0\}$ and satisfy the conditions

$$\int_{r<|t|<2r} |k(t)x|_Y \,\mathrm{d}t \leq A_1 |x|_X \qquad \text{for all } r > 0,\ x \in X, \tag{2.6}$$

$$\left| \int_{r<|t|<R} k(t)x \,\mathrm{d}t \right|_Y \leq A_2 |x|_X \qquad \text{for all } R > r > 0,\ x \in X, \qquad \text{and} \tag{2.7}$$

$$\lim_{r \downarrow 0} \int_{r<|t|<1} k(t)x \,\mathrm{d}t \qquad \text{exists as a weak limit in } Y \text{ for all } x \in X. \tag{2.8}$$

Then the operator p.v.-k defined on $\phi \in \mathcal{S}(\mathbf{R}^n)$ by

$$\langle \text{p.v.-}k, \phi \rangle\, x := \lim_{\epsilon \downarrow 0} \int_{|t|>\epsilon} k(t)x\, \phi(t) \,\mathrm{d}t \tag{2.9}$$

gives a well-defined tempered distribution p.v.-$k \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X;Y))$; actually

$$|\langle \text{p.v.-}k, \phi \rangle\, x|_Y \leq \Big(2A_1 (\|\nabla \phi(t)\|_{L^\infty(\mathrm{d}t)} + \| |t|\, \phi(t) \|_{L^\infty(\mathrm{d}t)}) + A_2 |\phi(0)| \Big) |x|_X\,. \tag{2.10}$$

It is obvious from the definition that this satisfies Assumption 2.3.

While the previous example showed that certain results simply carry over to the operator-valued situation with essentially no modifications, the purpose of the next one is to illustrate the new phenomena not present in the scalar-valued context.

Example 2.11. We show that the integrability conditions for $t \mapsto k(t)x$ of the previous example do not imply anything similar for $t \mapsto k(t)f(t)$, where $f \in L^\infty(\mathbf{R}^n; X)$, even compactly supported away from the origin. This fact motivates the procedure adopted above first to define our operators on the rather restricted algebraic tensor products, where they make sense without any further assumptions. It is then a different matter to search for conditions guaranteeing the boundedness of these operators; it seems wise to do the hard work with the theorems and not the definitions.

Consider $X := \ell^p(\mathbf{Z})$, $1 \leq p < \infty$, which we identify with $L^p(\mathbf{R}, \sigma([0,1)+\mathbf{Z}), \mathrm{d}s)$ in the obvious way, and let $Y := \mathbf{K}$, the field of scalars. Note in particular that the example includes $\ell^2(\mathbf{Z})$, the prototype of all separable Hilbert spaces, so that there is certainly nothing pathological in the geometry of the Banach spaces in question.

For $t > 0$, $\log_2 t \notin \mathbf{Z} + 1/2$, we set $\alpha(t) := \tan(\pi \log_2(t))$; this map restricted to any of the intervals $(2^{j-1/2}, 2^{j+1/2})$ with $j \in \mathbf{Z}$ is an increasing bijection onto $(-\infty, \infty)$. Let further $g \in \ell^{p'}(\mathbf{Z}) \setminus \ell^1(\mathbf{Z})$. We can then define the operators $k(t) : X \to Y$ by

$$k(t)x := \operatorname{sgn}(t) \cdot x(\alpha(|t|)) \cdot g(\alpha(|t|)) \cdot \alpha'(|t|)$$

for $t \neq 0$, $\log_2(|t|) \notin \mathbf{Z} + 1/2$, and $k(t)x := 0$, say, for the countably many values of t just mentioned. Clearly these operators are linear, and moreover $\|k(t)\|_{X \to Y} = |g(\alpha(|t|))|\, \alpha'(|t|)$ (or 0 for the countably many special cases).

The kernel $k(\cdot)$ is manifestly odd, so that it satisfies (2.7) and (2.8) rather trivially, and moreover

$$\begin{aligned}\int_r^{2r} |k(t)x|_Y \,\mathrm{d}t &= \left(\int_r^{2^{j+1/2}} + \int_{2^{j+1/2}}^{2r} \right) |x(\alpha(t))| \cdot |g(\alpha(t))|\, \alpha'(t) \,\mathrm{d}t \\ &= \left(\int_{\alpha(r)}^{\infty} + \int_{-\infty}^{\alpha(2r)} \right) |x(s)| \cdot |g(s)| \,\mathrm{d}s = \int_{-\infty}^{\infty} |x(s)| \cdot |g(s)| \,\mathrm{d}s \leq \|x\|_{L^p} \|g\|_{L^{p'}} = c\, |x|_X\,,\end{aligned}$$

where j is the unique integer such that $\log_2 r \leq j + 1/2 < \log_2(2r) = \log_2 r + 1$, and we have taken into account that $\alpha(r) = \alpha(2r)$ by the π-periodicity of the tangent.

Now we define our function $f \in L^\infty(\mathbf{R}^n; X)$. Let $\eta \in \mathcal{D}(\mathbf{R})$ be $= 1$ in $[-1, 1]$, have range $[0, 1]$ and vanish outside $[-2, 2]$, and define

$$f(t)(s) := f(t, s) := \begin{cases} \eta(\alpha(|t|) - \lfloor s \rfloor) & \text{if } t \neq 0,\ \log_2(|t|) \notin \mathbf{Z} + 1/2, \\ 0 & \text{else.} \end{cases}$$

This f is actually not only bounded, but it is $\mathcal{C}^\infty$ in the regions $(2^{j-1/2}, 2^{j+1/2})$, $j \in \mathbf{Z}$.

Since our integrability conditions concern compact subsets of $\mathbf{R} \setminus \{0\}$, we can take f to be compactly supported away from 0 by simply making a cut-off outside our domain of integration. We then have

$$\begin{aligned} \int_{1/\sqrt{2}}^{\sqrt{2}} |k(t)f(t)|_Y \, \mathrm{d}t &= \int_{1/\sqrt{2}}^{\sqrt{2}} |f(t, \alpha(t))| \cdot |g(\alpha(t))| \, \alpha'(t) \, \mathrm{d}t \\ = \int_{1/\sqrt{2}}^{\sqrt{2}} \eta(\alpha(t) - \lfloor \alpha(t) \rfloor) \, |g(\alpha(t))| \, \alpha'(t) \, \mathrm{d}t &= \int_{-\infty}^{\infty} \eta(s - \lfloor s \rfloor) \, |g(s)| \, \mathrm{d}s = \int_{-\infty}^{\infty} |g(s)| \, \mathrm{d}s = \infty, \end{aligned}$$

since $g \notin L^1(\mathbf{R}, \mathrm{d}s)$, and this shows quite explicitly that $k(\cdot)f(\cdot)$ is not integrable.

Note that the failure of integrability in the last computation in no way depended on the singularity of $k(\cdot)$ at the origin. In fact, we could have defined k as above only in the annulus $1/\sqrt{2} < |t| < \sqrt{2}$, say, and set $k(t) := 0$ elsewhere. Then we would have even global integrability $\int_{-\infty}^{\infty} |k(t)x|_Y \, \mathrm{d}t \leq c \, |x|_X$ for every fixed $x \in X$, and yet a blow-up of even the local integrals for a function $f \in L^\infty(\mathbf{R}; X)$ in place of x, as above.

3 Some estimates for random series

In this section we review some techniques related to vector-valued random series that have proved to be fundamental for the vector-valued extension of classical results of harmonic analysis.

Denote by ε_j, $j \in \mathbf{Z}$, the Rademacher system of independent random variables on a probability space $(\Omega, \Sigma, \mathbf{P})$ verifying $\mathbf{P}(\varepsilon_j = 1) = \mathbf{P}(\varepsilon_j = -1) = 1/2$. Let $\mathbf{E} := \int (\cdot) \, \mathrm{d}\mathbf{P}$ be the corresponding expectation.

For a Banach space X, let $\mathrm{Rad}(X)$ be the closure in $L^2(\Omega; X)$ of the algebraic tensor product $X \otimes (\varepsilon_j)_{-\infty}^{\infty}$ equipped with the norm of $L^2(\Omega; X)$. By the Khintchine–Kahane inequality, any $p \in [1, \infty)$ in place of 2 gives the same space (as a set) with an equivalent norm.

If X is B-convex, then various useful properties of $\mathrm{Rad}(X)$ follow readily from the boundedness of the Rademacher projection

$$(Rf)(\omega) := \sum_{-\infty}^{\infty} \mathbf{E}[\varepsilon_j f] \varepsilon_j(\omega); \tag{3.1}$$

we recall that one possible characterization of B-convexity is to say that the operator R above is well-defined and bounded on $L^2(\Omega; X)$. Also recall that every UMD-space is B-convex, so that we do not get any new geometric restrictions, since the places where we exploit serious analytic (as opposed to algebraic) properties of $\mathrm{Rad}(X)$ are such that the UMD-condition is required anyway. Note that the boundedness of R implies the uniform boundedness of its partial sum projections by the Banach–Steinhaus theorem.

Denoting by $\mathcal{R}(R)$ the range of R, it is obvious that $X \otimes (\varepsilon_j)_{-\infty}^{\infty} \subset \mathcal{R}(R)$. On the other hand, the fact that the partial sums of the series in (3.1) (which are in $X \otimes (\varepsilon_j)_{-\infty}^{\infty}$ by definition) converge to Rf for every $f \in L^2(\Omega; X)$ shows that $\mathcal{R}(R) \subset \overline{X \otimes (\varepsilon_j)_{-\infty}^{\infty}}$. Finally, since R as a bounded projection has a closed range, we conclude that $\mathrm{Rad}(X) = \mathcal{R}(R : L^2(\Omega; X) \to L^2(\Omega; X))$ whenever X is a B-convex space. This allows us to identify $f = Rf \in \mathrm{Rad}(X)$ with the sequence appearing in (3.1),

$$f = Rf \approx (\mathbf{E}[\varepsilon_j f])_{-\infty}^{\infty} \in X^{\mathbf{Z}}.$$

The density of finitely non-zero sequences in $\mathrm{Rad}(X)$ follows from the very definition of $\mathrm{Rad}(X)$ as the closure of $X \otimes (\varepsilon_j)_{-\infty}^{\infty}$.

Let us make a useful observation concerning the dual of $\mathrm{Rad}(X)$. Since the unit ball of $L^2(\Omega; X')$ is norming for $L^2(\Omega; X) \supset \mathrm{Rad}(X)$, we have, for $f = Rf \in \mathrm{Rad}(X)$,

$$\|f\|_{\mathrm{Rad}(X)} = \sup_{\|g\|_{L^2(\Omega;X')} \leq 1} \left| \langle g, Rf \rangle_{\langle L^2(\Omega;X'), L^2(\Omega;X) \rangle} \right| = \sup |\langle Rg, f\rangle| \leq \sup_{\substack{h \in \mathrm{Rad}(X') \\ \|h\|_{L^2(\Omega;X')} \leq C}} |\langle h, f\rangle|$$

where the easily verified self-adjointness of R was used, and C is the operator norm of R on $L^2(\Omega; X)$, thus also the norm of its adjoint. This shows that the unit ball of $\mathrm{Rad}(X')$ is equivalently norming for $\mathrm{Rad}(X)$.

As a consequence of Fubini's theorem and the equivalence of the definitions of $\mathrm{Rad}(X)$ in terms of different exponents we also have

$$L^p(\Gamma; \mathrm{Rad}(X)) \approx \mathrm{Rad}(L^p(\Gamma; X))$$

whenever Γ is a σ-finite measure space. (We really need this only for $\Gamma = \mathbf{R}^n$.)

The Rademacher classes $\mathrm{Rad}(X)$, $\mathrm{Rad}(Y)$ provide a straightforward but occationally useful reformulation of the concept of *R-boundedness*, whose definition we recall:

Definition 3.2. A collection $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called R-bounded if, for some $C < \infty$, the inequality

$$\left(\mathbf{E} \left| \sum_{j=-N}^{N} \varepsilon_j T_j x_j \right|_Y^p \right)^{\frac{1}{p}} \leq C \left(\mathbf{E} \left| \sum_{j=-N}^{N} \varepsilon_j x_j \right|_X^p \right)^{\frac{1}{p}} \tag{3.3}$$

holds for all $N \in \mathbf{N}$ and all $x_j \in X$, $T_j \in \mathcal{T}$ and some [equivalently, all] $p \in [1, \infty[$. The smallest constant C [when $p = 1$, say] is called the *R-bound* of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

With the understanding that $x_j = 0$ for $|j| > N$,we can write (3.3) as

$$\left\| (T_j x_j)_{-\infty}^{\infty} \right\|_{\mathrm{Rad}(Y)} \leq C \left\| (x_j)_{-\infty}^{\infty} \right\|_{\mathrm{Rad}(X)},$$

and by the density of finitely non-zero sequences $(x_j)_{-\infty}^{\infty} \in \mathrm{Rad}(X)$, the condition is simply that of boundedness of the diagonal operators $(T_j)_{-\infty}^{\infty}$ from $\mathrm{Rad}(X)$ to $\mathrm{Rad}(Y)$.

The following permanence property of R-boundedness will be useful.

Lemma 3.4. *Let X be a B-convex space and $\mathcal{T} \subset \mathcal{L}(X; Y)$ be R-bounded. Then $\mathcal{T}' := \{T' | \, T \in \mathcal{T}\} \subset \mathcal{L}(Y'; X')$ is also R-bounded, and more precisely $\mathcal{R}(\mathcal{T}') \leq C\mathcal{R}(\mathcal{T})$, where C is a geometric constant.*

Proof. For $g \in L^2(\Omega; X)$, we have

$$\begin{aligned} \mathbf{E} \left\langle \sum_{-N}^{N} \varepsilon_j T_j' y_j', g \right\rangle &= \mathbf{E} \sum_{-N}^{N} \langle \varepsilon_j T_j' y_j', \varepsilon_j \mathbf{E}[\varepsilon_j g] \rangle = \mathbf{E} \left\langle \sum_{-N}^{N} \varepsilon_j y_j', \sum_{-N}^{N} \varepsilon_i T_i \mathbf{E}[\varepsilon_i g] \right\rangle \\ &\leq \left(\mathbf{E} \left| \sum_{-N}^{N} \varepsilon_j y_j' \right|_{X'}^2 \right)^{\frac{1}{2}} \mathcal{R}(\mathcal{T}) \left(\mathbf{E} \left| \sum_{-N}^{N} \varepsilon_j \mathbf{E}[\varepsilon_j g] \right|_X^2 \right)^{\frac{1}{2}} \leq \left(\mathbf{E} \left| \sum_{-N}^{N} \varepsilon_i y_i' \right|_{X'}^2 \right)^{\frac{1}{2}} \mathcal{R}(\mathcal{T}) C \|g\|_{L^2(\Omega;X)}, \end{aligned}$$

recalling the uniform boundedness of the partial sum projections of the Rademacher projection R. Taking supremum over $g \in L^2(\Omega; X)$ of unit norm, we find that $\mathcal{R}(\mathcal{T}') \leq C\mathcal{R}(\mathcal{T})$, where C is the same constant as above. ☐

We also recall (e.g. from [19]) that the family $\tilde{\mathcal{T}}$ of canonical extensions $(\tilde{T}f)(t) := T[f(t)]$ of $T \in \mathcal{T} \subset \mathcal{L}(X;Y)$ to $L^p(\Gamma;X) \to L^p(\Gamma;Y)$ is R-bounded whenever $\mathcal{T}$ is, with the same R-bound and without any geometric assumptions. (This is easy to see.)

An R-bounded collection is always uniformly bounded, but the converse is not true in general. Perhaps the simplest example of a uniformly bounded, non-R-bounded family of operators is the group of translations acting on $L^p(\mathbf{R}^n)$, $p \neq 2$. However, there is a remarkable result due to Bourgain [3] providing a partial substitute of this boundedness under appropriate restrictions on the support of the Fourier transforms of the functions involved. This result plays an important rôle in Bourgain's paper [3], as well as in the present one. The difficult part of the proof, the case $n = 1$ for the unit-circle $\mathbf{T}$ in place of $\mathbf{R}^n$, is given in [3], Lemma 10. The transference to $\mathbf{R}^n$ uses standard methods and is detailed in [11], Lemma 3.5.

Lemma 3.5. *Let X be a UMD-space and $(f_j)_{-\infty}^{\infty} \subset L^p(\mathbf{R}^n;X)$ a finitely non-zero sequence such that* $\operatorname{supp}\hat{f}_j \subset \bar{B}(0,2^j)$*. Let $(h_j)_{-\infty}^{\infty} \subset \mathbf{R}^n$ be a sequence, lying on the same line through the origin and such that $|h_j| < K2^{-j}$ for some constant K. Then*

$$\mathbf{E}\left\|\sum \varepsilon_j f_j(\cdot - h_j)\right\|_{L^p(\mathbf{R}^n;X)} \leq C\log(2+K)\mathbf{E}\left\|\sum \varepsilon_j f_j\right\|_{L^p(\mathbf{R}^n;X)}.$$

Remark 3.6. Although we do not need it, we mention that one can get away from the assumption that the h_j lie on the same line, with the cost of getting $\log^n(2+K)$ in place of $\log(2+K)$. While the case $n = 1$ is obviously handled already, the case of $n > 1$ dimensions can be reached by induction on n.

To ensure the support condition of the Fourier transforms for the application of Bourgain's lemma, we will exploit (a smooth version of) a Littlewood–Paley-type dyadic decomposition. Let $\eta \in \mathcal{D}(\mathbf{R}^n)$ have range $[0,1]$, equal 1 for $|\xi| < 1/4$ and vanish for $|\xi| > 1/2$. Let then $\hat{\varphi}_0(\xi) := \eta(\xi) - \eta(2\xi)$, and $\hat{\varphi}_j(\xi) := \hat{\varphi}(2^{-j}\xi)$. Then $\sum_{-\infty}^{\infty}\hat{\varphi}_j(\xi) = 1$ for $\xi \neq 0$ and $\hat{\varphi}_j$ is supported in the annulus $2^{j-3} \leq |\xi| \leq 2^{j-1}$. Moreover, $\hat{\Phi}_j := \hat{\varphi}_{j-1} + \hat{\varphi}_j + \hat{\varphi}_{j+1}$ is equal to unity on the support of $\hat{\varphi}_j$, and is supported in the annulus $2^{j-4} \leq |\xi| \leq 2^j$. Our indices are slightly shifted from the usual choice, the sole purpose of which being to ensure the condition $\operatorname{supp}\hat{\Phi}_j \subset [-2^j, 2^j]^n$ so as to avoid playing with indices when applying Lemma 3.5.

The next lemma allows us to estimate deterministic L^p-norms with randomized ones, i.e., to incorporate the Rademacher functions ε_j into our equations. Slight variants of this lemma and the next one appear in several papers, cf. e.g. Girardi and Weis [11], Cor. 3.3.

Lemma 3.7. *Let X be a UMD-space, $1 < p < \infty$, and $(g_j)_{-\infty}^{\infty} \subset (\mathcal{S}' \cap L^{1,\mathrm{loc}})(\mathbf{R}^n;X)$ be a finitely non-zero sequence. Assume further that $\hat{g}_j$ is supported in the annulus $|\xi| \in 2^j[a,b]$ for some $0 < a < b$. Then*

$$\left\|\sum g_j\right\|_{L^p(\mathbf{R}^n;X)} \leq C\mathbf{E}\left\|\sum \varepsilon_j g_j\right\|_{L^p(\mathbf{R}^n;X)}, \tag{3.8}$$

where the constant depends only on a and b (and the geometry of X).

Proof. Let us first observe that we can assume that $g_j \in L^p(\mathbf{R}^n;X)$ for all j, since otherwise the right-hand side is ∞. Indeed, let $E_m := \{t \in \mathbf{R}^n |\ |t| \leq m,\ |g_j(t)|_X \leq m \text{ for all } j\}$. Then

$$\|g_i 1_{E_m}\|_{L^p(\mathbf{R}^n;X)} \leq \frac{1}{2}\left\|\left(g_i + \sum_{j\neq i} g_j\right)1_{E_m}\right\|_{L^p(\mathbf{R}^n;X)} + \frac{1}{2}\left\|\left(g_i - \sum_{j\neq i} g_j\right)1_{E_m}\right\|_{L^p(\mathbf{R}^n;X)}.$$

As $m \to \infty$, the left-hand side becomes the $L^p(\mathbf{R}^n;X)$-norm of g_i, whereas on the right-hand side we have two terms appearing on the right-hand side of (3.8). Should we have $\|g_i\|_{L^p(\mathbf{R}^n;X)} = \infty$, the right-hand side of (3.8) would also be ∞, and there is nothing to prove.

Let us hence assume that $g_j \in L^p(\mathbf{R}^n;X)$ for all j. We choose $N \in \mathbf{N}$ large enough so that $2^N > b/a$. Then, by the triangle inequality,

$$\left\|\sum_{j=-\infty}^{\infty} g_j\right\|_{L^p(\mathbf{R}^n;X)} \leq \sum_{k=0}^{N-1}\left\|\sum_{j\equiv k \bmod N} g_j\right\|_{L^p(\mathbf{R}^n;X)}. \tag{3.9}$$

The motivation for this rearrangement is the fact that the supports of $\hat{g}_j$ for $j \equiv k \mod N$ are disjoint for any fixed k.

Choose $\phi \in \mathcal{D}(\mathbf{R}^n)$ with range $[0,1]$, equal to unity in $[a,b]$ and with support in $[2^{-m}b, 2^m a]$. Then $\hat{g}_j = \phi(2^{-j}\cdot)\hat{g}_j$ and $\phi(2^{-i}\cdot)\hat{g}_j = 0$ for $i \neq j$. Thus, for $(\epsilon_j)_{-\infty}^{\infty} \in \{-1,1\}^{\mathbf{Z}}$,

$$\sum_{j\equiv k} \epsilon_j \hat{g}_j = \sum_{j\equiv k} \epsilon_j \phi(2^{-j}\cdot)\hat{g}_j = \left(\sum_{i\equiv k} \epsilon_i \phi(2^{-i}\cdot) \right) \sum_{j\equiv k} \hat{g}_j =: m \sum_{j\equiv k} \hat{g}_j,$$

and the Fourier multiplier m satisfies infinitely many of the Mihlin-type conditions

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq \sum_{j\equiv k} |\xi|^{|\alpha|} 2^{-j|\alpha|} \left|(D^\alpha \phi)(2^{-j}\xi)\right| \leq 2 \sup_\xi |\xi|^{|\alpha|} |D^\alpha \phi(\xi)| < \infty,$$

where the factor 2 follows from the fact that at most two of the functions $\phi(2^{-j}\cdot)$, $j \equiv k$, are supported at any given point.

The UMD-space version of Mihlin's multiplier theorem (which is due to Zimmermann [20]) implies that

$$\left\| \sum_{j\equiv k} \epsilon_j g_j \right\|_{L^p(\mathbf{R}^n;X)} \leq K \left\| \sum_{j\equiv k} g_j \right\|_{L^p(\mathbf{R}^n;X)},$$

and the inequality is readily seen to be two-sided by taking $\epsilon_j g_j$ in place of g_j. Then, taking $\epsilon_j := \varepsilon_j(\omega)$ and integrating over $\omega \in \Omega$, we have

$$\left\| \sum_{j\equiv k} g_j \right\|_{L^p(\mathbf{R}^n;X)} \leq K\mathbf{E} \left\| \sum_{j\equiv k} \varepsilon_j g_j \right\|_{L^p(\mathbf{R}^n;X)} \leq K\mathbf{E} \left\| \sum_{j=-\infty}^{\infty} \varepsilon_j g_j \right\|_{L^p(\mathbf{R}^n;X)},$$

where the last inequality follows from the the fact that $(\varepsilon_j y_j)$, with $y_j \in Y$, is a monotone basic sequence in $L^1(\Omega;Y)$ for any Banach space Y. (Here, of course, $Y = L^p(\mathbf{R}^n;X)$.)

Combining this with (3.9), we have the assertion with $C = NK$. ☐

We also need to be able to get rid of the randomization, and for this we have the following:

Lemma 3.10. *For $f \in L^p(\mathbf{R}^n;X)$ we have*

$$\mathbf{E} \left\| \sum \varepsilon_j \Phi_j * f \right\|_{L^p(\mathbf{R}^n;X)} \leq C \|f\|_{L^p(\mathbf{R}^n;X)}.$$

Proof. Since $\mathcal{F} \sum \varepsilon_j \Phi_j * f = \sum \varepsilon_j \hat{\Phi}_j \hat{f} =: m\hat{f}$, and

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq \sum |\xi|^{|\alpha|} 2^{-j|\alpha|} \left|(D^\alpha \hat{\Phi}_0)(2^{-j}\xi)\right| \leq 3 \sup_\xi |\xi|^{|\alpha|} \left|D^\alpha \hat{\Phi}_0(\xi)\right| < \infty,$$

even a stonger result with the expectation replaced by the supremum norm over the random variables ε_j is an immediate consequence of the Mihlin–Zimmermann theorem. ☐

4 A Hörmander-type condition for singular integrals

The classical result for scalar-valued singular integral operators (see Hörmander [14]) states that the formal convolution (2.1), interpreted as explained in Sect. 2, defines a bounded operator on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, if $\hat{k}$ is bounded and k satisfies the Hörmander condition (1.1). Our main result in this section is the following version of this theorem for operator-valued kernel functions:

Theorem 4.1. *Let X, Y be UMD-spaces. Assume that $k \in \mathcal{S}'(\mathbf{R}^n;\mathcal{L}(X;Y))$ and k' satisfy Assumption 2.3 and that the Fourier-transform $\hat{k}$ coincides with a bounded function such that $\hat{k}(\cdot)$ and $\hat{k}(\cdot)'$ are strongly measurable. Moreover, assume the following conditions:*

$$\mathcal{R}\left(\{\hat{k}(\xi)|\ \xi \in \mathbf{R}^n\}\right) \le A_0, \tag{4.2}$$

and

$$\int_{|t|>2|s|} \mathcal{R}\left(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t))|\ j \in \mathbf{Z}\}\right) w(t)\,\mathrm{d}t \le A_1 w(s), \tag{4.3}$$

where $w(t) = \log(2+|t|)$.

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n) \mapsto k * f$ extends to a bounded linear operator

$$f \in L^p(\mathbf{R}^n;X) \mapsto k * f \in L^p(\mathbf{R}^n;Y)$$

with norm at most $C(A_0 + A_1)$, where C is a geometric constant.

Remark 4.4. (*i*) For $X = Y$ and $n = 1$, the Hilbert transform $H = (\text{p.v.-}1/\pi t)*$, with $k(t) = 1/\pi t$, $\hat{k}(\xi) = -\mathbf{i}\,\mathrm{sgn}(\xi)$, is easily seen to verify the conditions of the theorem – note in particular that the R-boundedness reduces to uniform boundedness for a scalar kernel – which shows that the UMD-assumption is necessary in this case. On the other hand, if Z is a UMD-space, $A \in \mathcal{L}(X,Z)$ and $B \in \mathcal{L}(Z,Y)$, then the singular integral operator of the special form $BHA : L^p(\mathbf{R};X) \to L^p(\mathbf{R};Y)$ satisfies the conclusion of the theorem for arbitrary Banach spaces X and Y.

(*ii*) The operator $f \mapsto k * f$ can also be interpreted as a Fourier multiplier transformation $\hat{f} \mapsto \hat{k}\hat{f}$, with operator-valued multiplier $\hat{k} \in L^\infty(\mathbf{R}^n;\mathcal{L}(X;Y))$. Thus a result of Clément and Prüss [6] shows that the R-boundedness condition (4.2) of the operators $\hat{k}(\xi)$ is necessary.

(*iii*) The R-boundedness assumption in our version of the Hörmander condition enables us to use the Littlewood–Paley decomposition, whereas the logarithmic factor is forced on us by Lemma 3.5. Note that while the usual Hörmander condition (1.1) is sufficient (also in the vector-valued context) to obtain the boundedness on the whole scale $p \in]1,\infty[$ as soon as the boundedness is known for one $L^{\tilde{p}}$, we *do not* assume any *a priori* boundedness.

(*iv*) For the verification of the weighted Hörmander condition (4.3) in concrete situations, it is useful to note the estimate

$$\int_r^\infty t^{-(1+\delta)} \log(2+t)\,\mathrm{d}t \le C(\delta) r^{-\delta} \log(2+r) \qquad \text{for all } r, \delta > 0, \tag{4.5}$$

whose verification is elementary calculus.

Theorem 4.1 will be a special case of Theorem 4.21 below. As a preparation for the proof, we first give a somewhat technical condition for the boundedness of singular integral operators. It is a version of Proposition 3.7 in Girardi and Weis [11].

Proposition 4.6. *Let X, Y be UMD-spaces and $k \in \mathcal{S}'(\mathbf{R}^n;\mathcal{L}(X;Y))$ and k' satisfy Assumption 2.3. Define, for every $t \in \mathbf{R}^n$, an operator from $\mathrm{Rad}(X)$ to $\mathrm{Rad}(Y)$ by*

$$K(t) := ((\varphi_0 * 2^{-nj} k(2^{-j}\cdot))(t))_{j=-\infty}^{\infty}, \tag{4.7}$$

and assume that the Banach adjoints $K(t)' : \mathrm{Rad}(Y') \to \mathrm{Rad}(X')$, canonically extended to $L^{p'}(\mathbf{R}^n;\mathrm{Rad}(Y')) \to L^{p'}(\mathbf{R}^n;\mathrm{Rad}(X'))$, satisfy the condition

$$\int_{\mathbf{R}^n} \|K(t)'g\|_{L^{p'}(\mathbf{R}^n;\mathrm{Rad}(X'))}\, w(t)\,\mathrm{d}t \le A\,\|g\|_{L^{p'}(\mathbf{R}^n;\mathrm{Rad}(Y'))}\,, \tag{4.8}$$

with $w(t) := \log(2+|t|)$, for every $g \in L^{p'}(\mathbf{R}^n;X) \otimes (\varepsilon_j)_{-\infty}^{\infty}$.

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n) \mapsto k * f$ extends to a bounded linear operator $L^p(\mathbf{R}^n;X) \to L^p(\mathbf{R}^n;Y)$, of norm at most CA, where C is a geometric constant.

Proof. We have $\mathcal{F}[k * f] = \hat{k}\hat{f} = \sum_{-\infty}^{\infty} \hat{\varphi}_j \hat{k}\hat{f}$, where the sum contains only finitely many non-zero terms for $f \in \hat{\mathcal{D}}_0$. Moreover, we have $\hat{\varphi}_j \hat{k}\hat{f} = \hat{\varphi}_j \hat{k}\,\hat{\Phi}_j \hat{f} = \mathcal{F}[(\varphi_j * k) * (\Phi_j * f)]$. Denoting $f_j := \Phi_j * f$, we have the decomposition $k * f = \sum_{-\infty}^{\infty} (\varphi_j * k) * f_j$.

As a last preparatory manipulation, we write

$$(\varphi_j * k) * f_j(t) = \int_{\mathbf{R}^n} (\varphi_j * k)(2^{-j}s) f_j(t - 2^{-j}s) 2^{-jn}\,\mathrm{d}s = \int_{\mathbf{R}^n} 2^{-jn} (\varphi_0 * k(2^{-j}\cdot))(s) f_j(t - 2^{-j}s)\,\mathrm{d}s,$$

where a simple change of variable was performed, recalling that $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\cdot)$, whence $\varphi_j = 2^{jn}\varphi(2^j\cdot)$. A functional notation is used to denote the dilation of the distribution k for simplicity, but this is defined by the duality $\langle k(\delta\cdot), \phi\rangle := \langle k, \delta^{-n}\phi(\delta^{-1}\cdot)\rangle$.

We now invoke the UMD-property by means of the Littlewood–Paley decomposition (more precisely, Lemma 3.7), which allows us to write

$$\begin{aligned}
\Big\| \sum_{-\infty}^{\infty} (\varphi_j * k) * f_j \Big\|_{L^p(\mathbf{R}^n;X)} &\leq C\mathbf{E} \Big\| \sum_{-\infty}^{\infty} \varepsilon_j (\varphi_j * k) * f_j \Big\|_{L^p(\mathbf{R}^n;Y)} \\
&= C\mathbf{E} \Big\| \sum_{-\infty}^{\infty} \varepsilon_j \int_{\mathbf{R}^n} 2^{-jn} [\varphi_0 * k(2^{-j}\cdot)](s) f_j(\cdot - 2^{-j}s)\,\mathrm{d}s \Big\|_{L^p(\mathbf{R}^n;Y)} \\
&= C \Big\| \int_{\mathbf{R}^n} K(s) \big(f_j(\cdot - 2^{-j}s)\big)_{-\infty}^{\infty}\,\mathrm{d}s \Big\|_{\mathrm{Rad}(L^p(\mathbf{R}^n;Y))},
\end{aligned}$$

where we recalled the definition of our auxiliary sequence-valued kernel K from (4.7).

To estimate the norm on the right of the previous inequality, we pick an arbitrary $g \in \mathrm{Rad}(L^{p'}(\mathbf{R}^n;Y'))$. We have

$$\begin{aligned}
&\Big\langle g, \int_{\mathbf{R}^n} K(s) \big(f_j(\cdot - 2^{-j}s)\big)_{-\infty}^{\infty}\,\mathrm{d}s \Big\rangle_{\langle L^{p'}(\mathbf{R}^n;\mathrm{Rad}(Y')), L^p(\mathbf{R}^n;\mathrm{Rad}(Y))\rangle} \\
&\quad = \int_{\mathbf{R}^n} \mathrm{d}s \Big\langle K(s)'g, \big(f_j(\cdot - 2^{-j}s)\big)_{-\infty}^{\infty} \Big\rangle_{\langle L^{p'}(\mathbf{R}^n;\mathrm{Rad}(X')), L^p(\mathbf{R}^n;\mathrm{Rad}(X))\rangle} \\
&\quad \leq \int_{\mathbf{R}^n} \mathrm{d}s \, \|K(s)'g\|_{L^{p'}(\mathbf{R}^n;\mathrm{Rad}(X'))} \Big\| \big(f_j(\cdot - 2^{-j}s)\big)_{-\infty}^{\infty} \Big\|_{\mathrm{Rad}(L^p(\mathbf{R}^n;X))}. \qquad (4.9)
\end{aligned}$$

The second factor can be estimated with the help of Bourgain's lemma to the result

$$\mathbf{E} \Big\| \sum_{-\infty}^{\infty} \varepsilon_j f_j(\cdot - 2^{-j}s) \Big\|_{L^p(\mathbf{R}^n;X)} \leq C \log(2 + |s|) \Big\| \sum_{-\infty}^{\infty} \varepsilon_j f_j \Big\|_{L^p(\mathbf{R}^n;X)} \leq \tilde{C} \log(2 + |s|) \|f\|_{L^p(\mathbf{R}^n;X)},$$

the last step being again a consequence of the Littlewood–Paley decomposition for UMD-valued functions (more precisely, Lemma 3.10).

It remains to integrate over s in (4.9), invoke the assumption (4.8), and condider the supremum over all appropriate $g \in \mathrm{Rad}(L^{p'}(\mathbf{R}^n;X))$ of norm at most unity, to conclude that

$$\|k * f\|_{L^p(\mathbf{R}^n;X)} \leq CA \|f\|_{L^p(\mathbf{R}^n;X)}$$

for all f in the dense subspace considered. Thus the proposition is proved. □

In the previous proposition, the boundedness of a singular integral operator acting on the space $L^p(\mathbf{R}^n;X)$ was related to a boundedness condition of another operator acting on the Rademacher class $\mathrm{Rad}(X)$ and related spaces. The new kernel $K(t)$ in (4.7) has some special structure, in particular, the convolution with a nice test function φ_0. To be able to exploit this particular structure, so as to find a sufficient condition more explicitly in terms of the original kernel k, we need the following decomposition lemma, which is a variant of a similar result used by the first author (in a rather different context) in [15].

Lemma 4.10. *Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ have a vanishing integral. Then there exists a decomposition $\varphi = \sum_{m=0}^{\infty} \psi_m$ with the following properties:*

$$\psi_m \in \mathcal{D}(\mathbf{R}^n), \qquad \operatorname{supp} \psi_m \subset \bar{B}(0, 2^m), \qquad \int \psi_m(y)\, \mathrm{d}y = 0,$$

and for every pair of multi-indices $\alpha, \beta \in \mathbf{N}^n$ and every $M > 0$ the sequence of Schwartz norms $\|\psi_m\|_{\alpha,\beta}$, as well as $\left\|\hat{\psi}_m\right\|_{\alpha,\beta}$, is $\mathcal{O}(2^{-mM})$ as $m \to \infty$.

In particular, for every $p \in [1, \infty]$ and every $M > 0$, the sequence of Lebesgue norms $\|\psi_m\|_{L^p}$, as well as $\left\|\hat{\psi}_m\right\|_{L^p}$, is $\mathcal{O}(2^{-mM})$ as $m \to \infty$.

Proof. Fix $\eta \in \mathcal{D}(\mathbf{R}^n)$, with range $[0, 1]$, equal to 1 for $|x| \le 1/2$ and vanishing for $|x| \ge 1$. We set, for $r > 0$,

$$\varphi_r(x) := \eta(x/r) \left(\phi(x) + \frac{1}{r^n \int \eta(y)\, \mathrm{d}y} \int \varphi(y)(1 - \eta(y/r))\, \mathrm{d}y \right).$$

Then φ_r has a vanishing integral since φ does, and φ_r is supported in $\bar{B}(0, r)$ by the support condition imposed on η. Moreover,

$$\begin{aligned} |\varphi_r(x) - \varphi(x)| &\le |\eta(x/r) - 1| \cdot |\varphi(x)| + \frac{\eta(x/r)}{r^n \int \eta(y)\, \mathrm{d}y} \int |\varphi(y)|\, (1 - \eta(y/r))\, \mathrm{d}y \\ &\le \max_{|y| \ge r} |\phi(y)| + c r^{-n} \int_{|y| \ge r} |\varphi(y)|\, \mathrm{d}y, \end{aligned}$$

which tends to zero as $r \to \infty$; thus $\varphi_r \to \varphi$ uniformly.

We next set $\psi_0 := \varphi_1$ and $\psi_m := \varphi_{2^m} - \varphi_{2^{m-1}}$ for $m > 0$; whence $\sum_{m=0}^{\infty} \psi_m = \lim_{m\to\infty} \varphi_{2^m} = \varphi$, uniformly. Explicitly, for $m > 0$, we have

$$\begin{aligned} \psi_m(x) := \varphi(x) \left(\eta(2^{-m}x) - \eta(2^{-(m-1)}x) \right) &+ \frac{\eta(2^{-m}x)}{2^{nm} \int \eta(y)\, \mathrm{d}y} \int \varphi(y)(1 - \eta(2^{-m}y))\, \mathrm{d}y \\ &- \frac{\eta(2^{-(m-1)}x)}{2^{n(m-1)} \int \eta(y)\, \mathrm{d}y} \int \varphi(y)(1 - \eta(2^{-(m-1)}y))\, \mathrm{d}y. \end{aligned} \tag{4.11}$$

It remains to estimate the order of the size of the Schwartz norms of the terms appearing here.

Let us first have a look at the last two terms in (4.11). We have, by a simple change of variable, $\|\eta(2^{-m}\cdot)\|_{\alpha,\beta} = 2^{m(|\beta|-|\alpha|)} \|\eta\|_{\alpha,\beta}$, which looks a little bad for $|\beta| > |\alpha|$. However, the thing that settles the matters is the constant factor, whose size is estimated by

$$\int |\varphi(y)|\, (1 - \eta(2^{-m}y))\, \mathrm{d}y \le \int_{|y| > 2^m} |\varphi(y)|\, \mathrm{d}y \le C(M, n, \varphi) \int_{|y| > 2^m} |y|^{-M-n}\, \mathrm{d}y = \tilde{C} 2^{-Mm}.$$

We then turn to estimate the first term in (4.11) and denote for simplicity $\phi(x) := \eta(x) - \eta(2x)$, so that this term is $\varphi(x)\phi(2^{-m}x)$. Note that $\phi \in \mathcal{D}(\mathbf{R}^n)$ is supported away from the origin. By Leibniz' rule we have

$$x^\beta D_x^\alpha(\varphi(x)\phi(2^{-m}x)) = \sum_{\theta \le \alpha} \binom{\alpha}{\theta} x^\beta D^{\alpha-\theta} \varphi(x) 2^{-m|\theta|} D^\theta \phi(2^{-m}x). \tag{4.12}$$

Let us make a Taylor expansion of $D^\theta \phi(2^{-m}x)$ at the origin; since $D^\vartheta \phi(0) = 0$ for all $\vartheta \in \mathbf{N}^n$, all we get is the error term:

$$D^\theta \phi(2^{-m}x) = \sum_{|\vartheta| = M} \frac{2^{-mM} x^\vartheta}{\vartheta!} \int_0^1 D^{\theta+\vartheta} \phi(u 2^{-m} x) M (1 - u)^{M-1}\, \mathrm{d}u.$$

Now a typical term in the sum in (4.12) is estimated as

$$\sum_{|\vartheta|=M} 2^{-mM} \frac{1}{\vartheta!} \left| x^{\beta+\vartheta} D^{\alpha-\theta} \varphi(x) \right| \int_0^1 \left| D^{\theta+\vartheta} \phi(u 2^{-m} x) \right| M(1-u)^{M-1} \, \mathrm{d}u$$
$$\leq 2^{-mM} \sum_{|\vartheta|=M} \frac{1}{\vartheta!} \|\varphi\|_{\alpha-\theta,\beta+\vartheta} \|\phi\|_{\theta+\vartheta,0} \,.$$

Summing over the finite number of bounds of this type, we have established the desired rate of convergence of the sequence $\|\psi_m\|_{\alpha,\beta}$ as $m \to \infty$.

The assertion concerning the Schwartz norms of the Fourier transforms $\hat{\psi}_m$ follows from the continuity of the Fourier transform on $\mathcal{S}(\mathbf{R}^n)$. The assertion concerning the Lebesgue norms follows by estimating $\|\psi_m\|_{L^p}$ by a finite number of Schwartz norms. □

Remark 4.13. (*i*) The result is equally valid for vector-valued functions, with the same proof, but we only need it here for scalar-valued ones.

(*ii*) The decomposition established is an *atomic decomposition* of φ, and shows the well-known fact that $\varphi \in \mathcal{S}(\mathbf{R}^n)$ belongs to the Hardy space $H^1(\mathbf{R}^n)$ provided it has a vanishing integral. However, more than this we are interested in the particular type of the decomposition, with the rapid rate of convergence.

Lemma 4.14. *Consider a mapping $t \in \mathbf{R}^n \mapsto \mathcal{L}(X;Y)$ having the form*

$$K(t) := k * \varphi(t)$$

where $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \varphi(y)\,\mathrm{d}y = 0$, and $k \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X;Y))$ is an operator-valued tempered distribution satisfying Assumption 2.3, and moreover $\hat{k}$ agrees with a function in $L^\infty_{\mathrm{str}}(\mathbf{R}^n; \mathcal{L}(X;Y))$. In addition, suppose that

$$\left\| \hat{k}(\xi) \right\|_{X \to Y} \leq A_0 \qquad \text{for a.e. } \xi \in \mathbf{R}^n, \tag{4.15}$$

$$\int_{|t|>2|s|} |(k(t-s) - k(t))x|_Y \, w_0(t) \, \mathrm{d}t \leq A_1 w_1(s) \, |x|_X \qquad \text{for all } s \in \mathbf{R}^n \setminus \{0\} \text{ and } x \in X_1, \tag{4.16}$$

where w_0 and w_1 are positive, measurable and polynomially bounded functions, and $X_1 \subset X$.

Then, for every $x \in X_1$,

$$\int |K(t)x|_Y \, w(t) \, \mathrm{d}t \leq (A_0 C(\varphi, w_0) + A_1 C(\varphi, w_1)) \, |x|_X \,,$$

where the C's are finite quantities depending only on the objects indicated.

Proof. We apply Lemma 4.10 to write $\varphi = \sum_{m=0}^\infty \psi_m$, where the ψ_m have the properties stated in that lemma. Then we devide $K(t)$ into the pieces

$$K_m(t) := k * \psi_m(t),$$

and investigate each of them separately.

Recall that ψ_m is supported in the ball $\bar{B}_m := \bar{B}(0, 2^m)$. We first investigate the integral of $|K_m(t)x|_Y$, with $x \in X_1$, well away from this ball, i.e., outside the larger ball $\bar{B}_{m+1}$:

$$\int_{\bar{B}^c_{m+1}} |K_m(t)x|_Y \, w_0(t) \, \mathrm{d}t = \int_{\bar{B}^c_{m+1}} \mathrm{d}t \, w_0(t) \left| \int_{\bar{B}_m} k(t-s)x \, \psi_m(s) \, \mathrm{d}s \right|_Y \,.$$

Since the integral of ψ_m vanishes, we can continue with

$$
\begin{aligned}
&= \int_{\bar{B}^c_{m+1}} \mathrm{d}t\, w_0(t) \left| \int_{\bar{B}_m} (k(t-s) - k(t))x\, \psi_m(s)\,\mathrm{d}s \right|_Y \\
&\leq \int_{\bar{B}_m} \mathrm{d}s\, |\psi_m(s)| \int_{|t|>2|s|} |(k(t-s)-k(t))x|_Y\; w_0(t)\,\mathrm{d}t \\
&\leq \int_{\bar{B}_m} \mathrm{d}s\, |\psi_m(s)|\, A_1 w_1(s)\, |x|_X \leq A_1 \|\psi_m\|_{L^\infty} \nu_1(\bar{B}_m)\, |x|_X\,,
\end{aligned}
$$

where we have denoted $\mathrm{d}\nu_1(t) := w_1(t)\,\mathrm{d}t$.

Inside the ball $\bar{B}_{m+1}$ we argue as follows, with the obvious definition of ν_0:

$$
\begin{aligned}
\int_{\bar{B}_{m+1}} &|K_m(t)x|_Y\, w_0(t)\,\mathrm{d}t \leq \nu_0(\bar{B}_{m+1})\, \|K_m(\cdot)x\|_{L^\infty(\mathbf{R}^n;Y)} \\
&\leq \nu_0(\bar{B}_{m+1}) \left\| \hat{K}_m(\cdot)x \right\|_{L^1(\mathbf{R}^n;Y)} = \nu_0(\bar{B}_{m+1}) \int_{\mathbf{R}^n} \left| \hat{k}(\xi)x\, \hat{\psi}_m(\xi) \right|_Y \mathrm{d}\xi \\
&\leq A_0 \nu_0(\bar{B}_{m+1}) \left\| \hat{\psi}_m \right\|_{L^1} |x|_X\,.
\end{aligned}
$$

Summing over the estimates, we have

$$
\int_{\mathbf{R}^n} |K(t)x|_Y\, w_0(t)\,\mathrm{d}t \leq \sum_{m=0}^{\infty} \left(A_1 \nu_1(\bar{B}_m) \|\psi_m\|_{L^\infty} + A_0 \nu_0(\bar{B}_{m+1}) \left\| \hat{\psi}_m \right\|_{L^1} \right) |x|_X\,.
$$

The convergence of the series follows from the properties of the decomposition $\varphi = \sum_{m=0}^{\infty} \psi_m$. Indeed, the at most polynomial growth of w_i guarantees that $\nu_i(\bar{B}(0,r)) \leq Cr^N$ as $r \to \infty$; hence $\nu_0(\bar{B}_{m+1}), \nu_1(\bar{B}_m) \leq C2^{mN}$, but we have $\|\psi_m\|_{L^\infty}, \left\|\hat{\psi}_m\right\|_{L^1} \leq C_M 2^{-mM}$ for any $M > 0$, so it suffices to take $M > N$. This completes the proof. □

Remark 4.17. (*i*) The result of the lemma is rather general, since no conditions on the Banach space geometry are required.

(*ii*) Although our application of the lemma is to the boundedness of operators acting on the usual Bôchner spaces with respect to the plain Lebesgue measure, where a specific choice of the weight w is relevant, the above result itself has some taste of a more general weighted norm inequality. The assumption that the weights w_i be polynomially bounded is exploited via the growth condition of the size of the balls $\bar{B}(0,r)$ in terms of the measures $\mathrm{d}\nu_i(t) := w_i(t)\,\mathrm{d}t$. Such a growth estimate would also follow from the *doubling condition* $\nu(\bar{B}(x,2r)) \leq C\nu(\bar{B}(x,r))$, which is the usual regularity assumption when dealing with more general measure spaces.

The following corollary simply specializes Lemma 4.14 to the spaces $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$ in place of X and Y. The subset $X_1 \subset X$ that appeared in Lemma 4.14 will now be the set $X \otimes (\varepsilon_j)_{-\infty}^{\infty}$ of finitely non-zero elements of $\mathrm{Rad}(X)$.

Corollary 4.18. *Consider a mapping $t \in \mathbf{R}^n \mapsto \mathcal{L}(\mathrm{Rad}(X); \mathrm{Rad}(Y))$ having the form*

$$
K(t) := (k_j * \varphi(t))_{-\infty}^{\infty}
$$

where $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \varphi(y)\,\mathrm{d}y = 0$, and $k_j \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X;Y))$ are operator-valued tempered distributions satisfying Assumption 2.3, and moreover $\hat{k}_j$ agrees with a function in $L^\infty_{\mathrm{str}}(\mathbf{R}^n; \mathcal{L}(X;Y))$ for every $j \in \mathbf{Z}$. In addition, suppose that

$$
\left\| (\hat{k}_j(\xi))_{-\infty}^{\infty} \right\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)} \leq A_0 \qquad \textit{for a.e. } \xi \in \mathbf{R}^n, \tag{4.19}
$$

$$
\int_{|t|>2|s|} \left\| ((k_j(t-s) - k_j(t))x_j)_{-\infty}^{\infty} \right\|_{\mathrm{Rad}(Y)} w_o(t)\,\mathrm{d}t \leq A_1 w_1(s)\, \|x\|_{\mathrm{Rad}(X)} \tag{4.20}
$$

for all $s \in \mathbf{R}^n \setminus \{0\}$ *and* $x \in X \otimes (\varepsilon_j)_{\infty}^{\infty}$, *where* w_0 *and* w_1 *are positive, measurable and polynomially bounded.*

Then

$$\int \|K(t)x\|_{\mathrm{Rad}(Y)}\, w_0(t)\,\mathrm{d}t \leq (A_0 C(\varphi, w_0) + A_1 C(\varphi, w_1))\,\|x\|_{\mathrm{Rad}(X)}$$

for all $x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$.

Combining Corollary 4.18 with Proposition 4.6, we have the following result, which contains Theorem 4.1 as a special case, as shown below:

Theorem 4.21. *Let* X, Y *be UMD-spaces,* $p \in]1, \infty[$, *and* $k \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X;Y))$ *and* k' *satisfy Assumption 2.3, and moreover* $\hat{k}$ *coincide with a bounded function such that* $\hat{k}(\cdot)'$, *is strongly measurable. Let the following conditions be satisfied:*

$$\left\|(\hat{k}(2^j\xi))_{j=-\infty}^{\infty}\right\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)} \leq A_0 \qquad \text{for a.e. } \xi \in \mathbf{R}^n, \tag{4.22}$$

and

$$\int_{|t|>2|s|} \left\|(2^{-nj}(k(2^{-j}(t-s))' - k(2^{-j}t)')g_j)_{j=-\infty}^{\infty}\right\|_{\mathrm{Rad}(L^{p'}(\mathbf{R}^n;X'))} w(t)\,\mathrm{d}t \\ \leq A_1 w(s)\,\|g\|_{\mathrm{Rad}(L^{p'}(\mathbf{R}^n;Y'))} \tag{4.23}$$

for all $s \in \mathbf{R}^n \setminus \{0\}$ *and* $g \in L^{p'}(\mathbf{R}^n; Y') \otimes (\varepsilon_j)_{-\infty}^{\infty}$, *where* $w(t) := \log(2 + |t|)$.

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n) \mapsto k * f$ *extends to a bounded linear operator*

$$f \in L^p(\mathbf{R}^n; X) \mapsto k * f \in L^p(\mathbf{R}^n; Y)$$

with norm at most $C(A_0 + A_1)$, *where* C *is a geometric constant.*

Proof. By the permanence properties of R-bounds (see Lemma 3.4 and the paragraph after it), we also have analogue of conditions (4.22) valid for the extensions of the adjoint operators $\hat{k}(\xi)'$ to $L^{p'}(\mathbf{R}^n; Y') \to L^{p'}(\mathbf{R}^n; X')$. Let us then define $k_j(t) := 2^{-nj}k(2^{-j}t)'$, whence (4.23) is the same as (4.20) with $L^{p'}(\mathbf{R}^n; Y')$ in place of X and $L^{p'}(\mathbf{R}^n; X')$ in place of Y. Moreover, $\hat{k}_j(\xi) = \hat{k}(2^j\xi)'$, so that (4.22) implies the analogue of (4.19) with the same substitutions. Thus Corollary 4.18 shows that the kernel $K(t)$ defined in (4.7) satisfies the assumption (4.8) of Proposition 4.6, and hence that proposition implies the assertion of the theorem. □

Now we can also give

Proof of Theorem 4.1. Clearly the assumption (4.2) implies (4.22). As for the condition (4.3), we can again use the permanence properties of R-bounds to obtain the same condition first for $k(\cdot)' : Y' \to X'$ in place of $k(\cdot)$, and finally for the extension $k(\cdot)' : L^{p'}(\mathbf{R}^n; Y') \to L^{p'}(\mathbf{R}^n; X')$. Thus (4.3) implies the operator norm version of the strong condition (4.23), and hence Theorem 4.1 follows as a special case of Theorem 4.21. □

Remark 4.24. As we saw in the proof of Theorem 4.21, the Hörmander condition (4.3) with operator norms could be used to deduce the more technical condition (4.23). While it is nice to have the sufficiency of the strong condition (4.23) for its own sake, the sufficiency of strong estimates becomes essential when applying Theorem 4.21 to prove multiplier theorems. Namely, as soon as our Banach space X has a Fourier-type $q > 1$ (and a UMD-space always has), the Hausdorff–Young inequality allows us to pass from estimates for $\|\hat{f}\|_{L^q(\mathbf{R}^n;X)}$ to those for $\|f\|_{L^{q'}(\mathbf{R}^n;X)}$, i.e., we are able to transform strong estimates for the q-norm in the frequency domain to strong estimates for the q'-norm in the spatial domain. However, the operator spaces $\mathcal{L}(X)$ only have trivial Fourier-type, and thus the transference of norm conditions does not work.

5 Application to special classes of singular integrals

For the application of Theorem 4.21 to classical operator-valued kernels (see Theorem 5.10, Cor. 5.12), we first provide criteria for checking the condition (4.22) without the need to know the Fourier tranform $\hat{k}$ of the distribution of interest. This is done in the following lemma, the core of whose proof is simply a repetition of the classical argument. Nevertheless, we need to consider several technical points to reduce the considerations to this classical situation.

Lemma 5.1. *Let Y be a UMD space, and consider a distribution* p.v.-$k \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(X;Y))$ *as in* (2.6)–(2.9)*, whose related sequence valued kernel $K(t) := (2^{-nj}k(2^{-j}t))_{-\infty}^{\infty}$ verifies the analogues of the properties* (2.6)–(2.7)*. More precisely, assume that, for every finitely non-zero $x = (x_j)_{-\infty}^{\infty} \in \mathrm{Rad}(X)$ we have*

$$\int_{r<|t|<2r} \|K(t)x\|_{\mathrm{Rad}(Y)} \,\mathrm{d}t \leq A\,\|x\|_{\mathrm{Rad}(X)} \qquad \textit{for all } r > 0, \tag{5.2}$$

$$\left\| \int_{r<|t|<R} K(t)x \,\mathrm{d}t \right\|_{\mathrm{Rad}(Y)} \leq A\,\|x\|_{\mathrm{Rad}(X)} \qquad \textit{for all } R > r > 0, \tag{5.3}$$

and moreover

$$\int_{|t|>2|s|} \|(K(t-s) - K(t))x\|_{\mathrm{Rad}(Y)} \,\mathrm{d}t \leq A\,\|x\|_{\mathrm{Rad}(X)}\,. \tag{5.4}$$

Then the Fourier transform $\hat{k}$ (taken in the sense of distributions) is identified with a bounded strongly measurable function, and actually

$$\hat{K}(\xi) := (\hat{k}(2^j\xi))_{-\infty}^{\infty} \qquad \textit{satisfies} \qquad \left\|\hat{K}(\xi)\right\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)} \leq cA \qquad \textit{for } \xi \in \mathbf{R}^n, \tag{5.5}$$

where c is a numerical constant.

Remark 5.6. (*i*) Of course, the assumption of the conditions (2.6)–(2.7), which are related to the existence of the principal value integral (2.9), is superfluous, since they follow from the stronger estimates (5.2)–(5.3). On the other hand, the analogue of (2.8),

$$\lim_{r\downarrow 0} \int_{r<|t|<1} K(t)x \,\mathrm{d}t \qquad \text{exists as in } \mathrm{Rad}(Y) \text{ for finitely non-zero } x \in \mathrm{Rad}(X), \tag{5.7}$$

already follows from (2.8), since we have the existence of the finite number of non-zero limits

$$\lim_{r\downarrow 0} \int_{r<|t|<1} 2^{-nj}k(2^{-j}t)x_j \,\mathrm{d}t = \lim_{r\downarrow 0} \int_{2^{-j}r<|t|<2^{-j}} k(t)x_j \,\mathrm{d}t,$$

and we just add them up.

(*ii*) The assumption (5.4) obviously follows if we have (4.23). The conditions (5.3) and (5.7) are trivial if k is odd, or slightly more generally, if its strong integral vanishes over almost every origin-centered sphere rS^{n-1}.

(*iii*) It is not exactly UMD but some weaker geometric properties implied by it that are relevant in the present context; however, we will only use the lemma for UMD spaces.

Proof of Lemma 5.1. The same classical argument, which could be repeated to show that the conditions (2.6)–(2.8) imply that (2.9) gives a well-defined tempered distribution p.v.-$k : X \times \mathcal{S}(\mathbf{R}^n) \to Y$ verifying the estimate (2.10), can equally well be used to give from (5.2), (5.3) and (5.7) the analogous estimates with X and Y replaced by $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$. Thus we have

$$\text{p.v.-}K \in \mathcal{S}'(\mathbf{R}^n; \mathcal{L}(\mathrm{Rad}(X); \mathrm{Rad}(Y))).$$

We then make a cut-off to define

$$K^{\epsilon,R}(t) := K(t)1_{\epsilon<|t|<R} \qquad \text{for } R > \epsilon > 0.$$

We claim that

$$\left\langle K^{\epsilon,R}, \phi\right\rangle x \underset{\epsilon\downarrow 0, R\uparrow\infty}{\longrightarrow} \langle K, \phi\rangle x \qquad \text{in } \operatorname{Rad}(Y) \text{ for all } \phi \in \mathcal{S}(\mathbf{R}^n) \text{ and } x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}. \tag{5.8}$$

Indeed, for a finitely non-zero $x \in \operatorname{Rad}(X)$,

$$\begin{aligned}\left\langle K^{\epsilon,R}, \phi\right\rangle x &= \sum \varepsilon_j \int_{\epsilon<|t|<R} 2^{-nj} k(2^{-j}t) x_j\, \phi(t)\, \mathrm{d}t = \sum \varepsilon_j \int_{2^{-j}\epsilon<|s|<2^{-j}R} k(s) x_j\, \phi(2^j s)\, \mathrm{d}s \\ &\to \sum \varepsilon_j \,\text{p.v.-}\int_{\mathbf{R}^n} k(s) x_j\, \phi(2^j s)\, \mathrm{d}s = \sum \varepsilon_j \,\text{p.v.-}\int_{\mathbf{R}^n} 2^{-nj} k(2^{-j}t) x_j\, \phi(t)\, \mathrm{d}t = \langle K, \phi\rangle x,\end{aligned}$$

since we can separately take each of the finite number of limits whose existence we know, and add them up.

With these technicalities out of the way, we are effectively in the classical situation, and the proof of [10], pp. 206–7, can merely be reproduced to estimate the integrals defining the Fourier transform of $K^{\epsilon,R}(\cdot)x$ with $x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$, to the result

$$\left\| \hat{K}^{\epsilon,R}(\cdot)x \right\|_{L^\infty(\mathbf{R}^n;\operatorname{Rad}(Y))} \le cA \|x\|_{\operatorname{Rad}(X)}.$$

By the obvious estimate, this implies for $\phi \in \mathcal{S}(\mathbf{R}^n)$

$$\left\| \left\langle K^{\epsilon,R}, \hat{\phi}\right\rangle x \right\|_{\operatorname{Rad}(Y)} = \left\| \left\langle \hat{K}^{\epsilon,R}, \phi\right\rangle x \right\|_{\operatorname{Rad}(Y)} \le cA \|x\|_{\operatorname{Rad}(X)} \|\phi\|_{L^1(\mathbf{R}^n)},$$

and it follows from (5.8) that the same inequality holds with $K^{\epsilon,R}$ replaced by K. But this means that $\phi \in \mathcal{S}(\mathbf{R}^n) \mapsto \langle K, \phi\rangle x \in \operatorname{Rad}(Y)$ extends to a bounded linear operator $\phi \in L^1(\mathbf{R}^n) \mapsto \left\langle \hat{K}, \phi\right\rangle x \in \operatorname{Rad}(Y)$, of norm at most $cA \|x\|_{\operatorname{Rad}(X)}$.

Now we use the fact that $\operatorname{Rad}(Y)$, when Y is UMD, is also UMD, hence reflexive, thus has the Radon–Nikodým property. We exploit this by means of the equivalent condition of validity of the vector-valued Riesz Representation Theorem; see [8], Theorem III.1.5. (It is easy to see that the finiteness of the measure space, assumed in the theorem cited, can be replaced by σ-finiteness.) This gives, for every $x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$, an essentially unique $\hat{K}_x(\cdot) \in L^\infty(\mathbf{R}^n; \operatorname{Rad}(Y))$ such that $\left\langle \hat{K}, \phi\right\rangle x = \int \hat{K}_x(\xi)\phi(\xi)\,\mathrm{d}\xi$. It is clear that $x \mapsto \hat{K}_x(\cdot)$ is linear from $X \otimes (\varepsilon_j)_{-\infty}^{\infty}$ to $L^\infty(\mathbf{R}^n; \operatorname{Rad}(Y))$ and maps $X \otimes \varepsilon_\ell$ to $Y \otimes \varepsilon_\ell$ for every ℓ.

Let then $\phi_m = m^n \phi(m\cdot) \in \mathcal{D}(\mathbf{R}^n)$ be a usual approximation of the identity, and $\Lambda \in (\ell^\infty)'$ a Banach limit. For $\xi \in \mathbf{R}^n$ and $x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$, we define an element $\hat{K}(\xi)x \in \operatorname{Rad}(Y)'' = \operatorname{Rad}(Y)$ by

$$\left\langle y', \hat{K}(\xi)x \right\rangle := \Lambda\left(y' \left\langle \hat{K}, \phi_m(\cdot - \xi) \right\rangle x \right)_{m=1}^{\infty} = \Lambda\left(y' \int \hat{K}_x(\eta+\xi)\phi_m(\eta)\,\mathrm{d}\eta \right)_{m=1}^{\infty}.$$

It follows at once that $\left| \left\langle y', \hat{K}(\xi)x \right\rangle \right| \le cA \|y'\|_{\operatorname{Rad}(Y)'} \|x\|_{\operatorname{Rad}(X)}$, and hence by density there is a unique $\hat{K}(\xi) \in \mathcal{L}(\operatorname{Rad}(X), \operatorname{Rad}(Y))$ which still satisfies the above equality. Moreover, if ξ is a Lebesgue point of $\hat{K}_x$, then $\left\langle y', \hat{K}(\xi)x \right\rangle = \left\langle y', \hat{K}_x(\xi) \right\rangle$, thus $\hat{K}(\xi)x = \hat{K}_x(\xi)$. This shows that for any $x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$, $\hat{K}(\xi)x$ agrees for a.e. ξ with the measurable function $\hat{K}_x(\xi)$. In particular $\left\langle \hat{K}, \phi\right\rangle x = \int \hat{K}(\xi)x\, \phi(\xi)\,\mathrm{d}\xi$. With $x = x_0\varepsilon_0$, this gives the coincidence of $\hat{k}$ with a bounded, strongly measurable function. □

With Theorem 4.21 and Lemma 5.1 at our disposal, it becomes a routine task to obtain operator-valued generalizations of classical results on the boundedness of singular integrals, with the recipe "replace any boundedness assumption by R-boundedness". In this spirit, we have the following:

Lemma 5.9. *Suppose that for $k \in \mathcal{C}(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$ and some $\delta > 0$, the collection*

$$\mathcal{T} := \{|t|^{n+\delta} |s|^{-\delta} (k(t-s) - k(t)) : |t| > 2|s| > 0\}$$

is R-bounded. Then the Hörmander condition (4.23) holds with a constant $c(n,\delta)\mathcal{R}(\mathcal{T})$.

Proof. We have

$$\begin{aligned}&\mathcal{R}(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t))\}_{j=-\infty}^{\infty})\\ &= \mathcal{R}(\{(2^{-j}|t|)^{n+\delta}(2^{-j}|s|)^{-\delta}(k(2^{-j}(t-s)) - k(2^{-j}t))\}_{-\infty}^{\infty}) |t|^{-(n+\delta)} |s|^{\delta} \leq \mathcal{R}(\mathcal{T}) |t|^{-(n+\delta)} |s|^{\delta},\end{aligned}$$

and with this estimate, (4.23) follows by integrating in the polar coordinates and using (4.5). □

Theorem 5.10. *Let X and Y be UMD-spaces and suppose $k \in \mathcal{C}(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$ is a kernel satisfying the existence of the limit (2.8), as well as*

$$\begin{aligned}&\mathcal{R}(\{|t|^n k(t),\ |t|^{n+\delta} |s|^{-\delta} (k(t-s) - k(t)) : |t| > 2|s| > 0\})\\ &\quad + \mathcal{R}\Big(\Big\{\int_{r<|t|<R} k(t)\,\mathrm{d}t : R > r > 0\Big\}\Big) =: A < \infty.\end{aligned}$$

Then k gives rise to a tempered distribution p.v.-k *in the sense of (2.9), and $f \in X \otimes \mathcal{S}(\mathbf{R}^n) \mapsto$* p.v.-$k * f$ *extends to a bounded mapping from $L^p(\mathbf{R}^n;X)$ to $L^p(\mathbf{R}^n;Y)$, for all $p \in]1,\infty[$, of norm at most CA with C geometric.*

Proof. By Lemma 5.9, k satisfies the condition (4.23) of Theorem 4.21. It is also not difficult to verify the assumptions of Lemma 5.1, which then yields the desired bound (4.22) for the Fourier transform. Concerning the strong measurability of $\hat{k}(\cdot)'$, just note that the assumptions of the theorem imply their analogues for k'. □

Remark 5.11. (*i*) With $X = Y = \mathbf{C}$, Theorem 5.10 is classical. Observe that, despite its general geometric setting and the complications on the way here, our theorem is strong enough to recover the classical result, since R-boundedness then reduces to uniform boundedness.

(*ii*) A generalization to the vector-valued situation, with $Y = X$ a UMD-space, but with a scalar-valued kernel, is first due to Bourgain [3], who considers the periodic domain $\mathbf{T}$.

As in the classical case, Theorem 5.10 has the following immediate corollary which is sufficient for many concrete examples of kernels.

Corollary 5.12. *Let X and Y be UMD-spaces, and $k \in \mathcal{C}^1(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$ satisfy the limit condition (2.8) as well as*

$$\mathcal{R}(\{|t|^n k(t),\ |t|^{n+1} \nabla k(t) : t \neq 0\}) + \mathcal{R}\Big(\Big\{\int_{r<|t|<R} k(t)\,\mathrm{d}t : R > r > 0\Big\}\Big) =: A < \infty.$$

Then k satisfies the conclusion of Theorem 5.10

Example 5.13 (R-bounded semigroups). The maximal regularity question for the abstract Cauchy problem (cf. [9, 19])

$$\dot{u}(t) = Au(t) + f(t) \quad \text{for } t \geq 0, \qquad u(0) = 0,$$

with A the generator of a bounded analytic semigroup and f a given function, leads one to consider the mild solution given by the variation-of-constants formula

$$Au(t) = \int_0^t AT^{t-s} f(s)\,\mathrm{d}s. \tag{5.14}$$

This is obviously a (singular) convolution integral with the kernel

$$k(t) := \begin{cases} AT^t & t > 0, \\ 0 & t \leq 0, \end{cases}$$

When (T^t) is bounded and analytic, $\mathcal{R}(T^t) \subset \mathcal{D}(A)$ for $t > 0$, and the AT^t are bounded operators whose norm behaves like $1/t$; thus tAT^t are uniformly bounded operators for $t > 0$. If we assume a little more, i.e., R-boundedness of T^t and tAT^t instead of uniform boundedness, then $|t|\, k(t)$ is obviously R-bounded, and in this special case, this already implies that $|t|^2 k'(t) = t^2A^2T^t = 4(t/2)^2A^2(T^{t/2})^2 = 4(t/2\,AT^{t/2})^2$ is also R-bounded. Moreover,

$$\int_r^R AT^t \,\mathrm{d}t = \int_r^R \frac{\mathrm{d}T^t}{\mathrm{d}t}\,\mathrm{d}t = T^R - T^r,$$

from which we get the remaining R-boundedness condition required in Cor. 5.12 by the R-boundedness of T^t, as well as the existence of the limit (2.8), noting that $T^r x \to x$ as $r \downarrow 0$.

Thus we have shown that, on a UMD-space, the mapping $f \mapsto Au$ defined by (5.14) maps $L^p(\bar{\mathbf{R}}_+; X)$ to $L^p(\bar{\mathbf{R}}_+; X)$ [and hence the Cauchy problem has maximal L^p-regularity] whenever A is the generator of the analytic semigroup (T^t) for which the sets $\{T^t|\, t > 0\}$ and $\{tAT^t|\, t > 0\}$ are R-bounded. Thus our results on singular integrals provide a direct approach to the recent maximal regularity results, allowing one to work with the variation-of-constants formula (5.14) instead of Fourier multipliers.

6 R-boundedness of families of singular integral operators

An interesting general phenomenon in the world of vector-valued inequalities is that they almost immediately self-improve to give related statements for large families of kernels; cf. e.g. [10], p. 493. In the more specific context of R-boundedness, this was observed by Girardi and the second author [12], and following these ideas, we next show how Theorem 5.10 can in fact be used to derive not only boundedness of certain singular integrals but in fact R-boundedness of families of singular integral operators which satisfy the assumptions of the theorem in such a way that the ranges of the kernels belong to the *same* R-bounded set $\mathcal{T}$.

The precise formulation of the result vaguely described above requires Pisier's notion of the *property* (α) from the geometry of Banach spaces. We exploit this notion via the following lemma which is essentially in [5], Lemma 3.13. The "traditional" definition of the property (α) can also be found in this same article (Def. 3.11), but actually, for $Y = X$, one could (equivalently) take the assertion of the lemma as the definition of X having the property (α). While the property (α) is independent of the UMD-condition, it is also satisfied by the most common reflexive spaces appearing in analysis; cf. [5], [12].

Lemma 6.1. *Let X and Y be Banach spaces with property (α). Then*

$$\mathbf{E}\,\mathbf{E}' \left| \sum_{i,j=-N}^{N} \varepsilon_i \varepsilon'_j T_{ij} x_{ij} \right|_Y \leq \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})\mathbf{E}\,\mathbf{E}' \left| \sum_{i,j=-N}^{N} \varepsilon_i \varepsilon'_j x_{ij} \right|_X$$

whenever $N \in \mathbf{N}$, $x_{ij} \in X$, $T_{ij} \in \mathcal{T} \subset \mathcal{L}(X;Y)$, and the (ε_i) and (ε'_j) are two independent systems of Rademacher functions, the related expectation operators of which are denoted by $\mathbf{E}$ and $\mathbf{E}'$, respectively. Here $\alpha(X), \alpha(Y) < \infty$ are geometric constants.

Corollary 6.2. *Let $\mathcal{T} \subset \mathcal{L}(X;Y)$ be R-bounded, where the Banach spaces X and Y have the property (α). Then*

$$\tilde{\mathcal{T}} := \{(T_j)_{-\infty}^{\infty} \text{ finitely non-zero, } T_j \in \mathcal{T}\} \subset \mathcal{L}(\mathrm{Rad}(X); \mathrm{Rad}(Y)) \tag{6.3}$$

is R-bounded, and in fact $\mathcal{R}(\tilde{\mathcal{T}}) \leq \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})$.

Proof. For $N \in \mathbf{N}$, $\tilde{T}_i = (T_{ij})_{j=-\infty}^{\infty} \in \tilde{\mathcal{T}}$ and $\tilde{x}_i = (x_{ij})_{j=-\infty}^{\infty} \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$ we have

$$\begin{aligned}\mathbf{E}\left\|\sum_{i=-N}^{N}\varepsilon_i\tilde{T}_i\tilde{x}_i\right\|_{\mathrm{Rad}(Y)} &= \mathbf{E}\,\mathbf{E}'\left|\sum_{i,j}\varepsilon_i\varepsilon_j'T_{ij}x_{ij}\right|_Y \\ &\leq \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})\mathbf{E}\,\mathbf{E}'\left|\sum_{i,j}\varepsilon_i\varepsilon_j'x_{ij}\right|_X = \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})\mathbf{E}\left\|\sum_{i=-N}^{N}\varepsilon_i\tilde{x}_i\right\|_{\mathrm{Rad}(X)},\end{aligned}$$

as a direct consequence of Lemma 6.1. □

Now we are ready for the theorem. For simplicity of statement, we restrict ourselves to odd kernels, but the reader will find no difficulty in replacing this by a more general cancellation condition in the spirit of Theorem 5.10.

Theorem 6.4. *Let X and Y be UMD-spaces with property (α), and $k_\lambda \in \mathcal{C}(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$, where $\lambda \in \Lambda$ (any index set), be odd kernels which satisfy*

$$\{|t|^n\, k_\lambda(t),\ |t|^{n+\delta}\,|s|^{-\delta}\,(k_\lambda(t-s)-k_\lambda(t)))\,|\ |t| > 2\,|s| > 0\} \subset \mathcal{T},$$

where $\mathcal{T} \subset \mathcal{L}(X;Y)$ is R-bounded. Then the family

$$\{k_\lambda * : L^p(\mathbf{R}^n;X) \to L^p(\mathbf{R}^n;Y)|\ \lambda \in \Lambda\}$$

is R-bounded for all $p \in \,]1,\infty[$.

Proof. Let $k_j := k_{\lambda_j}$ for some $\lambda_j \in \Lambda$ when $|j| \leq N$, and $k_j := 0$ otherwise. Consider the sequence-valued kernel $K(t) := (k_j(t))_{j=-\infty}^{\infty}$. With $\tilde{\mathcal{T}}$ defined as in (6.3), it is clear that

$$\{|t|^n\, K(t),\ |t|^{n+\delta}\,|s|^{-\delta}\,(K(t-s)-K(t))\} \subset \tilde{\mathcal{T}}.$$

But then by Corollary 6.2 and Theorem 5.10, the operator $f \in \mathrm{Rad}(X) \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n) \mapsto K * f$ extends to a bounded linear operator from $L^p(\mathbf{R}^n;\mathrm{Rad}(X)) \approx \mathrm{Rad}(\mathbf{R}^n;X)$ to $L^p(\mathbf{R}^n;\mathrm{Rad}(Y)) \approx \mathrm{Rad}(L^p(\mathbf{R}^n;Y))$, of norm at most $C\mathcal{R}(\mathcal{T})$. But this boundedness means, by definition, that

$$\mathbf{E}\left\|\sum_{j=-N}^{N}\varepsilon_j k_{\lambda_j} * f_j\right\|_{L^p(\mathbf{R}^n;Y)} \leq C\mathcal{R}(\mathcal{T})\mathbf{E}\left\|\sum_{j=-N}^{N}\varepsilon_j f_j\right\|_{L^p(\mathbf{R}^n;X)}$$

for all $f_j \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n)$. In the above argument, the $N \in \mathbf{N}$ and the $\lambda_j \in \Lambda$ were fixed but arbitrary, and hence the result obtained is exactly the asserted R-boundedness of the collection $\{k_\lambda * \mid \lambda \in \Lambda\}$. □

Let us note some immediate consequences of this theorem:

Remark 6.5. *(i)* The conclusion of the theorem follows in particular if

$$\{|t|^n\, k_\lambda(t),\ |t|^{n+1}\,\nabla k_\lambda(t)|\ t \neq 0\} \subset \mathcal{T}.$$

(ii) If X and Y are Hilbert spaces, the R-boundedness assumptions reduce to the norm-boundedness of $\mathcal{T}$.

(iii) If all the kernels k_λ are scalar-valued (but the spaces X and Y are any UMD-spaces with property (α)), then again the R-boundedness means just norm-boundedness of $\mathcal{T}$.

If $X = Y = \mathbf{C}$, then it is well-known that R-boundedness of operators on $L^p(\mathbf{R}^n)$ is equivalent to square-function estimates (see [7], Prop. 3.3, or [16], Sect. 2, for details). Therefore Theorem 6.4 implies classical results of the following kind (cf. [10], pp. 494–5):

Corollary 6.6. *For all $\lambda \in \Lambda$ (any index set), let $k_\lambda \in \mathcal{C}(\mathbf{R}^n \setminus \{0\})$ be odd kernels satisfying*

$$|k_\lambda(t)| \leq \frac{A}{|t|^n}, \qquad |k_\lambda(t-s) - k_\lambda(t)| \leq A\frac{|s|^\delta}{|t|^{n+\delta}}$$

for all $|t| > 2|s| > 0$. Then the family

$$\{k_\lambda * : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) |\ \lambda \in \Lambda\}$$

is R-bounded for all $p \in]1, \infty[$; equivalently, we have the square-function inequality

$$\left\| \Big(\sum |k_{\lambda_j} * f_j(\cdot)|^2 \Big)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \leq CA \left\| \Big(\sum |f_j(\cdot)|^2 \Big)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \tag{6.7}$$

for all $f_j \in L^p(\mathbf{R}^n; X)$ and $\lambda_j \in \Lambda$.

7 Application to Fourier multipliers

We can also use Theorem 4.21 to obtain sufficient conditions for the L^p-boundedness of an operator $f \mapsto k * f$ entirely in terms of the symbol $\hat{k} =: m$. We present a Hörmander-type multiplier theorem in a rather general form, with a continuous smoothness parameter ℓ. The Hölder continuity assumptions (7.12) and (7.13) of the highest derivatives, which can be used to relax by one the number of classical derivatives required, is introduced in the classical context by Strömberg and Torchinsky [18]. An operator-valued multiplier theorem with the slightly stronger assumptions (7.3) and (7.4) for all $|\alpha| \leq \lfloor n/q \rfloor + 1$ is proved by Girardi and the second author [11] as a consequence of a general multiplier theorem assuming Besov norm estimates for the multiplier function. Instead of using this result, we follow here an alternative approach which is closer to the classical proof of these theorems in the scalar setting, as found e.g. in [10], and which sheds some light on the interplay of multiplier theorems and singular integrals.

We first formulate a somewhat technical result, nevertheless containing the essential flavour of the actual theorem which is then readily derived from this intermediate result.

Proposition 7.1. *Let X and Y be UMD-spaces and Y have Fourier type $q \in]1, 2]$. Let $\ell > n/q$,*

$$m \in \mathcal{C}^{\lfloor \ell \rfloor}_{\mathrm{str}}(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(X;Y)) \qquad \text{and} \qquad M(\xi) := (m(2^j\xi))_{j=-\infty}^{\infty}. \tag{7.2}$$

Suppose further that

$$\|M(\xi)\|_{\mathrm{Rad}(X) \to \mathrm{Rad}(Y)} \leq A \qquad \text{for a.e. } \xi \in \mathbf{R}^n, \tag{7.3}$$

$$\left(\frac{1}{r^n} \int_{r<|\xi|<2r} \|D^\alpha M(\xi)' g\|^q_{\mathrm{Rad}(L^q(\mathbf{R}^n;X'))} \,\mathrm{d}\xi \right)^{1/q} \leq A r^{-|\alpha|} \|g\|_{\mathrm{Rad}(L^q(\mathbf{R}^n;Y'))} \qquad \text{for } |\alpha| \leq \lfloor \ell \rfloor, \tag{7.4}$$

and finally

$$\left(\frac{1}{r^n} \int_{r<|\xi|<2r} \|(D^\alpha M(\xi - \zeta)' - D^\alpha M(\xi)')g\|^q_{\mathrm{Rad}(L^q(\mathbf{R}^n;X'))} \,\mathrm{d}\xi \right)^{1/q}$$
$$\leq A r^{-\ell} |\zeta|^{\ell - \lfloor \ell \rfloor} \|g\|_{\mathrm{Rad}(L^q(\mathbf{R}^n;Y'))} \qquad \text{for } |\alpha| = \lfloor \ell \rfloor,\ |\zeta| \leq r/2, \tag{7.5}$$

where (7.4)–(7.5) are assumed for all $r \in]0, \infty[$ and all finitely non-zero $g \in \mathrm{Rad}(L^q(\mathbf{R}^n; Y'))$.

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n) \mapsto \mathcal{F}^{-1}[m\hat{f}]$ extends to a bounded linear mapping

$$f \in L^p(\mathbf{R}^n; X) \mapsto \mathcal{F}^{-1}[m\hat{f}] \in L^p(\mathbf{R}^n; Y)$$

for all $p \in [q', \infty[$, with norm at most C_pA, where C_p is a geometric constant.

Remark 7.6. (*i*) From the "periodicity" of the sequence-valued multiplier M [in the sense that the sequence $M(2^i\xi) = (m(2^{i+j}\xi))_{j=-\infty}^{\infty}$ is just the sequence $M(\xi) = (m(2^j\xi))_{j=-\infty}^{\infty}$ with indexing shifted by i steps], it follows easily that the conditions (7.4) and (7.5) for a general $r \in]0,\infty[$ are already implied by the corresponding conditions for (say) $r = 1$.

(*ii*) Using, as in the proof of Theorem 4.21, the permanence properties of R-bounds, it is immediate that the conditions (7.4) and (7.5) are verified if instead of (7.4) we assume

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)\|^q_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)}\,\mathrm{d}\xi\right)^{1/q} \le Ar^{-|\alpha|} \qquad \text{for } |\alpha| \le \lfloor \ell \rfloor,$$

and instead of (7.5) a similar modification obtained in the obvious way.

(*iii*) Recall that UMD-spaces automatically have some Forier-type $q \in]1,2]$.

It is also possible to verify (7.4) and (7.5) by strong integral conditions instead of operator norm conditions, yet avoiding considerations of the extended operators acting on $L^q(\mathbf{R}^n; Y')$. Indeed, assume

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)'y'\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi\right)^{1/q} \le A\,r^{-|\alpha|}\,\|y'\|_{\mathrm{Rad}(Y')}\,. \tag{7.7}$$

Then

$$\begin{aligned}
&\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)'g(\cdot)\|^q_{\mathrm{Rad}(L^q(\mathbf{R}^n;X'))}\,\mathrm{d}\xi\right)^{1/q}\\
&\qquad \le C\left(\frac{1}{r^n}\int_{r<|\xi|<2r}\left(\int_{\mathbf{R}^n}\|D^\alpha M(\xi)'g(t)\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}t\right)\mathrm{d}\xi\right)^{1/q}\\
&\qquad = C\left(\int_{\mathbf{R}^n}\left(\frac{1}{r^n}\int_{r<|\xi|<2r}\|D^\alpha M(\xi)'g(t)\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi\right)\mathrm{d}t\right)^{1/q}\\
&\qquad \le C\left(\int_{\mathbf{R}^n}(A\,r^{-|\alpha|}\,\|g(t)\|_{\mathrm{Rad}(Y')})^q\,\mathrm{d}t\right)^{1/q} \le \tilde{C}A\,r^{-|\alpha|}\,\|g\|_{\mathrm{Rad}(L^{p'}(\mathbf{R}^n;Y'))}\,,
\end{aligned}$$

where we used the isomorphism of $\mathrm{Rad}(L^{p'}(\mathbf{R}^n; Z))$ and $L^{p'}(\mathbf{R}^n; \mathrm{Rad}(Z))$ in the first and last steps, Fubini's theorem in the second, and the assumption (7.7) in the third. What we have proved is that (7.7) (for all $|\alpha| \le \lfloor \ell \rfloor$) implies (7.4), and exactly the same reasoning yields out of

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|(D^\alpha M(\xi-\zeta)' - D^\alpha M(\xi)')y'\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi\right)^{1/q} \le Ar^{-\ell}\,|\zeta|^{\ell-\lfloor\ell\rfloor}\,\|y'\|_{\mathrm{Rad}(Y')} \tag{7.8}$$

(for appropriate α and ζ) the condition (7.5).

These remarks lead us to the following refinement of Corollaries 4.9 and 4.10 in Girardi and Weis [11], where one takes $\ell = \lfloor n/q \rfloor + 1$ so that the difference estimates below are replaced by having some more derivatives, and moreover the pair of strong conditions as in (7.10), (7.11) is replaced by a single norm condition.

Theorem 7.9. *Let X and Y be UMD-spaces with Fourier-type $q \in]1,2]$. Let $\ell > n/q$, and assume (7.2), (7.3), and moreover the conditions [for all $x \in X \otimes (\varepsilon_j)_{-\infty}^{\infty}$, $y' \in Y' \otimes (\varepsilon_j)_{-\infty}^{\infty}$]*

$$\int_{1<|\xi|<2} \|D^\alpha M(\xi)x\|_{\mathrm{Rad}(Y)}^q \,\mathrm{d}\xi \leq A^q \|x\|_{\mathrm{Rad}(X)}^q \qquad \text{for } |\alpha| \leq \lfloor \ell \rfloor \tag{7.10}$$

$$\int_{1<|\xi|<2} \|D^\alpha M(\xi)'y'\|_{\mathrm{Rad}(X')}^q \,\mathrm{d}\xi \leq A^q \|y'\|_{\mathrm{Rad}(Y')}^q \qquad \text{"} \quad \text{"} \tag{7.11}$$

$$\int_{1<|\xi|<2} \|(D^\alpha M(\xi-\zeta) - D^\alpha M(\xi))x\|_{\mathrm{Rad}(Y)}^q \,\mathrm{d}\xi \leq A^q \|x\|_{\mathrm{Rad}(X)}^q |\zeta|^{(\ell-\lfloor\ell\rfloor)q} \text{ for } |\alpha| = \lfloor \ell \rfloor,\ |\zeta| < \frac{1}{2} \tag{7.12}$$

$$\int_{1<|\xi|<2} \|(D^\alpha M(\xi-\zeta)' - D^\alpha M(\xi)')y'\|_{\mathrm{Rad}(X')}^q \,\mathrm{d}\xi \leq A^q \|y'\|_{\mathrm{Rad}(Y')}^q |\zeta|^{(\ell-\lfloor\ell\rfloor)q} \qquad \text{"} \quad \text{"} \tag{7.13}$$

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n) \mapsto \mathcal{F}^{-1}[m\hat{f}]$ extends to a bounded linear mapping

$$f \in L^p(\mathbf{R}^n; X) \mapsto \mathcal{F}^{-1}[m\hat{f}] \in L^p(\mathbf{R}^n; Y) \qquad \textit{for all } p \in]1,\infty[$$

with norm at most $C_p A$, where C_p is a geometric constant.

Proof. By the computations before the statement of the theorem, (7.11) implies (7.4) and (7.13) implies (7.5). Using Remark 7.6(i), Proposition 7.1 yields the assertion for $p \in [q', \infty[$. On the other hand, the conditions (7.10) and (7.12) are the analogues, respectively, of (7.11) and (7.13) for the dual multiplier $\xi \mapsto m(\xi)' \in \mathcal{L}(Y', X')$. Moreover, the condition (7.3) already implies its analogue for $m(\cdot)'$ by the permanence properties of R-bounds. Thus we also obtain the boundedness of

$$g \in L^p(\mathbf{R}^n; Y') \mapsto \mathcal{F}^{-1}[m(\cdot)'\hat{f}] \in L^p(\mathbf{R}^n; X') \qquad \text{for } p \in [q', \infty[.$$

By a well-known duality argument, the boundedness of the operator corresponding to the multiplier $m(\cdot)'$ from $L^{p'}(\mathbf{R}^n; Y')$ to $L^{p'}(\mathbf{R}^n; X')$ is equivalent to the boundedness of the operator with multiplier m from $L^p(\mathbf{R}^n; X)$ to $L^p(\mathbf{R}^n; Y)$. Thus we also obtain the assertion of the theorem for $p \in]1, q]$. If $q = 2$, we have already covered all $p \in]1, \infty[$, and otherwise the boundedness for the remaining exponents $p \in]q, q'[$ is obtained by interpolation. □

Remark 7.14. Combining Theorem 7.9 with results by the first author [15] shows that the same assumptions already imply the boundedness also from the Hardy spaces $H^p(\mathbf{R}^n; X)$ to $H^p(\mathbf{R}^n; Y)$ for all $p \in](1/q' + \ell/n)^{-1}, 1]$, in particular, from $H^1(\mathbf{R}^n; X)$ to $H^1(\mathbf{R}^n; Y)$ since $\ell > n/q \Rightarrow \ell/n + 1/q' > 1/q + 1/q' = 1$. Namely, it is shown in [15] that a multiplier operator satisfying (7.10) and (7.12) [somewhat weaker conditions without randomization will do], and which is bounded from $L^{\tilde{p}}(\mathbf{R}^n; X)$ to $L^{\tilde{p}}(\mathbf{R}^n; Y)$ for some $\tilde{p} \in]1, \infty[$, extends boundedly to the scale of the Hardy spaces mentioned. See also [11], Cor. 4.6.

As a very particular case of Theorem 7.9, we state the following corollary which was already proved in [11].

Corollary 7.15. *Let X, Y be UMD-spaces and Y have Fourier type $q > 1$. If $m \in \mathcal{C}^{\lfloor n/q \rfloor + 1}(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(X; Y))$ satisfies*

$$\mathcal{R}(\{|\xi|^{|\alpha|} D^\alpha m(\xi) | \ \xi \in \mathbf{R}^n \setminus \{0\}) \leq A \qquad \textit{for all } |\alpha| \leq \lfloor n/q \rfloor + 1,$$

then m is a Fourier multiplier from $L^p(\mathbf{R}^n; X)$ to $L^p(\mathbf{R}^n; Y)$ with norm at most $C_p A$.

We then return to prove our Proposition 7.1 [which was already used to prove Theorem 7.9]. The proof becomes a simple modification of the reasoning in the scalar-valued context (cf. [14] or [10], §II.6), as soon as one realizes the right way to make these modifications. Let us elaborate a little on this.

In the scalar-valued case, the R-boundedness-type assumptions (7.3)–(7.5) are unnecessary, and one simply assumes the same conditions with m in place of M. The idea of the proof is to smoothly cut the multiplier m into pieces, say m_j, which are well-behaved enough so that they correspond to Fourier transforms of integrable functions k_j. It remains to investigate how the multiplier conditions (7.3)–(7.5) transform to the properties of the kernels k_j, so that results on singular integral operators (classical analogues of Theorem 4.21) can be applied.

In the present situation, the assumptions involve the sequence-valued multiplier M, and also in the case of singular integral, the sequence-valued kernel K. Yet the actual operators of interest are defined in terms of the multiplier m and the kernel k. To make sense of our passing from the Fourier domain to the non-transformed domain, some truncations are to be first performed, as in the scalar-case. However, it is not at all the same whether we first truncate m and then form the corresponding sequence-valued multiplier, or if we first form the sequence M, and perform a cut-off (in the variable ξ) on this sequence. In fact, we shall need to apply both types of truncations mentioned, in the appropriate order.

In the following lemma, the new features compared to the classical situation are the Fourier-type condition required to use the Hausdorff–Young inequality (which is, of course, a mere additional statement), and the weight function $\log(2+t)$ arising from Bourgain's lemma (which is also easily dealt with).

Lemma 7.16. *Let X have Fourier type $q \geq 1$ and let $\ell > n/q$. Let $k \in (L^{1,\mathrm{loc}} \cap \mathcal{S}')(\mathbf{R}^n; X)$ and let its Fourier transform be $\mathcal{C}^{\lfloor \ell \rfloor}$ and satisfy*

$$\Big(\int_{\mathbf{R}^n} \Big|D^\alpha \hat{k}(\xi-\zeta) - D^\alpha \hat{k}(\xi)\Big|_X^q \,\mathrm{d}\xi\Big)^{1/q} \leq A\,|\zeta|^{\ell-\lfloor \ell \rfloor} \qquad \text{for } |\alpha| = \lfloor \ell \rfloor,\ |\zeta| \leq \delta.$$

Then, with $w(t) := w(|t|) := \log(2+|t|)$, we have

$$\int_{|t|>r} |k(t)|_X\, w(t)\,\mathrm{d}t \leq CA\,r^{n/q-\ell} w(r) \qquad \text{for } r \geq \frac{1}{4\delta}.$$

Proof. Observe that $\sum_{i=1}^n |\sin(\pi t_i)| = 0$ if and only if $t \in \mathbf{Z}^n$. Thus, for $0 < a \leq |t| \leq b < 1$, we have, by compactness, $\sum_{i=1}^n |\sin(\pi t_i)| \geq c(a,b) > 0$. Thus, when the variable t is appropriately restricted, we can majorize unity by the sum of sines, and we use this idea to estimate

$$\int_{2^j r<|t|\leq 2^{j+1}r} |k(t)|_X\, w(t)\,\mathrm{d}t \leq C \sum_{i=1}^n \int_{2^j r<|t|\leq 2^{j+1}r} \big|k(t)\sin(\pi t_i/2^{j+2}r)\big|_X\, w(t)\,\mathrm{d}t$$
$$\leq C \sum_{i=1}^n \sum_{|\alpha|=\lfloor \ell \rfloor} \Big(\int_{2^j r<|t|\leq 2^{j+1}r} \Big|t^\alpha k(t)(e^{2\pi t\cdot e_i/2^{j+2}r}-1)\Big|_X^{q'} \,\mathrm{d}t\Big)^{1/q'} \Big(\int_{2^j r}^{2^{j+1}r} w^q(\rho)\rho^{-\ell q+n-1}\,\mathrm{d}\rho\Big)^{1/q}.$$

Using the assumptions and the Hausdorff–Young inequality, the first factor is estimated by

$$\Big(\int_{\mathbf{R}^n} \Big|D^\alpha \hat{k}(\xi - e_i/2^{j+2}r) - D^\alpha \hat{k}(\xi)\Big|_X^q \,\mathrm{d}\xi\Big)^{1/q} \leq A\,2^{-j-2}r^{-1},$$

provided $2^{-j-2}r^{-1} \leq \delta$, which holds for $j \in \mathbf{N}$, since $r \geq 1/4\delta$, and the second factor is easily seen to be bounded by $c(1+j)w(r)2^{j(n/q-\ell)}r^{n/q-\ell}$.

Summing over $j \in \mathbf{N}$ we get the desired conclusion, since the series $\sum_{j=0}^\infty (1+j)2^{j(n/q-\ell)}$ converges to a finite quanitity for $n/q - \ell < 0$. □

In the next two lemmata, we present the two kinds of cut-offs we perform on the multiplier. The proofs involve straightforward computations, and we merely mention the new features compared to the classical situation. It is convenient to adopt the abbreviations

$$U := L^q(\mathbf{R}^n; X'), \qquad V := L^q(\mathbf{R}^n; Y'), \tag{7.17}$$

since for the proof these are just two Banach spaces, whose "internal structure" is of no interest to us. Note, however, that the spaces U and V (as well as $\mathrm{Rad}(U)$ and $\mathrm{Rad}(V)$) have Fourier-type $q \in]1, 2]$ whenever X and Y have.

Lemma 7.18. *For m as in Proposition 7.1 and $\phi \in \mathcal{S}(\mathbf{R}^n)$, the multipliers $m(\cdot)\phi(\delta\cdot)$, $\delta > 0$, satisfy the assumptions of Proposition 7.1 uniformly in δ. More precisely, the inequalities (7.3)–(7.5) hold with $M(\xi) = (m(2^{-j}\xi))_{j=-\infty}^{\infty}$ replaced by $(m(2^{-j}\xi)\phi(\delta 2^{-j}\xi))_{j=-\infty}^{\infty}$ with a constant $C(\phi)A$ in place of A.*

Sketch of proof. The proof uses straighforward estimates. The only new feature related to the sequence-valuedness of the kernel is the use of Kahane's contraction principle: Leibniz' rule yields terms of the form

$$(D_\xi^\theta[m(2^j\xi)](\delta 2^j)^{|\alpha|-|\theta|}D^{\alpha-\theta}\phi(\delta 2^j\xi)x_j)_{-\infty}^{\infty}, \tag{7.19}$$

and since the scalar quantities $(\delta 2^j\,|\xi|)^{|\alpha|-|\theta|}D^{\alpha-\theta}\phi(\delta 2^{-j}\xi)$ are bounded by a constant $C(\phi)$, the contraction principle gives a bound of the form $C(\phi)\,|\xi|^{|\theta|-|\alpha|}\left\|D_\xi^\theta(m(2^j\xi)'g_j)_{-\infty}^{\infty}\right\|_{\mathrm{Rad}(U)}$ for the Rademacher norm of the quantity in (7.19). Using estimates of this type, the proof is a routine computation along entirely classical lines. □

Lemma 7.20. *For M as in Proposition 7.1, and $\sum_{\mu=-\infty}^{\infty}\hat{\varphi}_0(2^{-\mu}\xi) = 1$ the partition of unity used in the radial Littlewood–Paley decomposition, denote $M_\mu(\xi) := M(\xi)\hat{\varphi}_0(2^{-\mu}\xi)$. Then we have the inequalities*

$$\left(\int_{\mathbf{R}^n}\|D^\alpha M_\mu(\xi)'g\|_{\mathrm{Rad}(U)}^q\,\mathrm{d}\xi\right)^{1/q} \leq CA\,2^{\mu(n/q-|\alpha|)}\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| \leq \lfloor\ell\rfloor, \tag{7.21}$$

and

$$\left(\int_{\mathbf{R}^n}\|(D^\alpha M_\mu(\xi-\zeta)' - D^\alpha M_\mu(\xi)')g\|_{\mathrm{Rad}(U)}^q\,\mathrm{d}\xi\right)^{1/q} \leq CA\,2^{\mu(n/q-\ell)}\,|\zeta|^{\ell-\lfloor\ell\rfloor}\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| = \lfloor\ell\rfloor \tag{7.22}$$

as well as

$$\left(\int_{\mathbf{R}^n}\left\|D_\xi^\alpha[M_\mu(\xi)'(e^{\mathbf{i}2\pi s\cdot\xi}-1)]g\right\|_{\mathrm{Rad}(U)}^q\,\mathrm{d}\xi\right)^{1/q} \leq CA\,2^{\mu(n/q-|\alpha|+1)}\,|s|\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| \leq \lfloor\ell\rfloor,\ |s| \leq 2^{-\mu}, \tag{7.23}$$

and finally

$$\left(\int_{\mathbf{R}^n}\left\|(D^\alpha[M_\mu(\cdot)'(e^{\mathbf{i}2\pi s\cdot(\cdot)}-1)](\xi-\zeta) - D^\alpha[M_\mu(\cdot)'(e^{\mathbf{i}2\pi s\cdot(\cdot)}-1)](\xi))g\right\|_{\mathrm{Rad}(U)}^q\,\mathrm{d}\xi\right)^{1/q} \leq CA\,2^{\mu(n/q-\ell+1)}\,|s|\cdot|\zeta|^{\ell-\lfloor\ell\rfloor}\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| = \lfloor\ell\rfloor,\ |s| \leq 2^{-\mu}, \tag{7.24}$$

where C is a numerical constant, and the inequalities hold for all finitely non-zero $g \in \mathrm{Rad}(V) := \mathrm{Rad}(L^{p'}(\mathbf{R}^n; Y'))$.

Note on proof. The proof is straightforward and entirely classical. The fact that M and M_μ are sequence-valued plays no rôle here. A direct computation only gives (7.22) and (7.24) for $|\zeta| \leq c2^\mu$ [with c a numerical constant] but for $|\zeta| > c2^\mu$ one can obtain the corresponding estimates by the triangle inequality from (7.21) or (7.23), respectively. □

As the final preparatory step towards proving Proposition 7.1, we note the following reduction:

Lemma 7.25. *Without loss of generality, the multiplier m is compactly supported in $\mathbf{R}^n \setminus \{0\}$. Thus, without loss of generality, m is strongly integrable and $k := \check{m}$, taken in the strong sense, is a strongly measurable, essentially bounded function.*

Proof. To see this, let $\eta \in \mathcal{D}(\mathbf{R}^n)$, as before, have range $[0,1]$, be 1 for $|\xi| \leq 1/2$ and 0 for $|\xi| \geq 1$. Then $\eta(\cdot/R) - \eta(\cdot/\epsilon)$ will have the same range, be 1 for $\epsilon \leq |\xi| \leq R/2$ and 0 for $|\xi| < \epsilon/2$ or $|\xi| > R$. Thus, $m_\epsilon^R(\xi) := m(\xi)(\eta(\xi/R) - \eta(\xi/\epsilon))$ is compactly supported in $\mathbf{R}^n \setminus \{0\}$, and for any $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n)$, we have $m\hat{f} = m_\epsilon^R \hat{f}$ as soon as ϵ is small and R large enough. Moreover, by Lemma 7.18, the multipliers m_ϵ^R satisfy the same conditions as those assumed for m, with a constant CA in place of A. Thus, provided we can prove the assertion of Proposition 7.1 with the additional support condition on m, then for a general m and $f \in X \otimes \hat{\mathcal{D}}_0(\mathbf{R}^n)$, we have

$$\left\| \mathcal{F}^{-1}[m\hat{\phi}] \right\|_{L^p(\mathbf{R}^n;Y)} = \lim_{\epsilon\downarrow 0,\, R\uparrow\infty} \left\| \mathcal{F}^{-1}[m_\epsilon^R \hat{\phi}] \right\|_{L^p(\mathbf{R}^n;Y)} \leq CA \|\phi\|_{L^p(\mathbf{R}^n;X)},$$

and hence also the general form of the assertion follows.

That m is strongly integrable is clear, since it is strongly measurable [being even strongly continuous by (7.2)], essentially bounded [by (7.3)] and compactly supported. □

Now we are ready to prove the multiplier theorem, and with Lemma 7.25 at our disposal, it is reduced to showing that $k := \check{m}$ satisfies the appropriate conditions required for an integral kernel to give a bounded operator.

Proof of Proposition 7.1. We need to show that $k := \check{m}$ satisfies the Hörmander condition (4.23) of Theorem 4.21. Denote $K(t) := (2^{-nj}k(2^{-j}t))_{j=-\infty}^{\infty}$; i.e., $K := \check{M}$, and moreover $K_\mu := \check{M}_\mu$, where the M_μ are the pieces of M from the radial Littlewood–Paley decomposition, as in Lemma 7.20.

We derive two different estimates for

$$\int_{|t|>2|s|} \|(K_\mu(t-s)' - K_\mu(t)')g\|_{\mathrm{Rad}(U)}\, w(t)\,\mathrm{d}t, \tag{7.26}$$

which are useful for different ranges of s and μ:

As a *first case*, we can make the crude estimate by

$$2\int_{|t|>|s|} \|K_\mu(t)'g\|_{\mathrm{Rad}(U)}\, w(t)\,\mathrm{d}t.$$

The Fourier transform of $K_\mu(t)'g$ is $M_\mu(\xi)'g$, which satisfies (7.22), and so Lemma 7.16 gives the bound

$$CA\, 2^{\mu(n/q-\ell)} |s|^{n/q-\ell} w(s) \|g\|_{\mathrm{Rad}(V)}. \tag{7.27}$$

As a *second case*, we observe that the Fourier transform of $t \mapsto K_\mu(t-s)'g - K_\mu(t)'g$ is $M_\mu(\xi)'(e^{\mathrm{i}2\pi s\cdot\xi} - 1)g$, which satisfies (7.24); whence Lemma 7.16 gives the bound

$$CA\, 2^{\mu(n/q-\ell+1)} |s| \cdot |s|^{n/q-\ell} w(s) \|g\|_{\mathrm{Rad}(V)}. \tag{7.28}$$

Using one of the two estimates (7.27) or (7.28) for (7.26) when appropriate, we have

$$\begin{aligned}\sum_{\mu=-\infty}^{\infty} \int_{|t|>2|s|} &\|(K_\mu(t-s)' - K_\mu(t)')g\|_{\mathrm{Rad}(U)}\, w(t)\,\mathrm{d}t \\ &\leq CA\, w(s) \|g\|_{\mathrm{Rad}(V)} \left(\sum_{\mu:\, 2^\mu|s|\geq 1} (2^\mu |s|)^{n/q-\ell} + \sum_{\mu:\, 2^\mu|s|<1} (2^\mu |s|)^{n/q-\ell+1} \right).\end{aligned}$$

We recall that $n/q - \ell > 0$ by the assumption in Proposition 7.1. On the other hand, since the assumptions of Proposition 7.1 are the stronger the larger ℓ we have, we may assume that

$\ell < n/q + 1$, i.e., $n/q - \ell + 1 > 0$. When this is the case, the two geometric series above are bounded by finite quanities depending only on n, q and ℓ.

The estimate established shows that the sequence-valued kernels $K^\nu := \sum_{\mu=-\nu}^{\nu} K_\mu$ satisfy uniformly the weighted Hörmander condition

$$\int_{|t|>2|s|} \|(K^\nu(t-s)' - K^\nu(t)')g\|_{\mathrm{Rad}(U)}\, w(t)\,\mathrm{d}t \le CA\, w(s)\, \|g\|_{\mathrm{Rad}(V)}\,. \tag{7.29}$$

The Fourier transform of $K^\nu(t)'g$ is

$$\sum_{\mu=-\nu}^{\nu} M_\mu(\xi)'g = \sum_{\mu=-\nu}^{\nu} M(\xi)'\hat{\varphi}_0(2^{-\mu}\xi)g = \left(m(2^j\xi)' \sum_{\mu=-\nu}^{\nu} \hat{\varphi}_0(2^{-\mu}\xi)g_j \right)_{j=-\infty}^{\infty}.$$

We recall that m is compactly supported away from 0; hence also $\xi \mapsto m(2^j\xi)'$ has the same property. Thus, for any finitely non-zero $g = (g_j)_{-\infty}^{\infty} \in \mathrm{Rad}(V) := \mathrm{Rad}(L^{p'}(\mathbf{R}^n; Y'))$, we observe that $\sum_{\mu=-\nu}^{\nu} \hat{\varphi}_0(2^{-\mu}\xi) = 1$ for ξ on the union of the supports of $m(2^j\xi)'g_j$, as soon as ν is large enough. Whence for all large enough ν (depending on g), $K^\nu(\cdot)'g = K(\cdot)'g$, and we find that the weighted Hörmander condition (4.23) (with $p' = q$), which we need in order to apply Theorem 4.21, is already contained in the uniform estimate (7.29). Thus the assertion for $p = q'$ follows from Theorem 4.21.

To show the assertion for $p \in\,]q', \infty[$, we invoke the classical theory of singular integrals. We take in the estimate 4.23, which we already proved, $g = g_0\varepsilon_0$, where $g_0(\cdot) = \psi(\cdot)y'$ for some non-zero $\psi \in L^q$, and some $y' \in Y'$. In this case, (4.23) reduces to

$$\int_{|t|>2|s|} |(k(t-s)' - k(t)')y'|_{X'}\, \mathrm{d}t \le CA\, |y'|_{Y'}\,;$$

we also dropped the weight w, as we clearly can, since the weighted condition is stronger than the unweighted one. But this is just the vector-valued generalization of the usual Hörmander condition for the kernel $k(\cdot)'$. Moreover, it is well-known (from a duality argument) that the operator $k(\cdot)*$ belongs to $\mathcal{L}(L^{q'}(\mathbf{R}^n; X), L^{q'}(\mathbf{R}^n; Y))$ if and only if $k(\cdot)'*$ belongs to $\mathcal{L}(L^q(\mathbf{R}^n; Y'), L^q(\mathbf{R}^n; X'))$ and the operator-norms agree.

So we conclude, by the duality argument, that $k(\cdot)'*$ is bounded from $L^q(\mathbf{R}^n; Y')$ to $L^q(\mathbf{R}^n; X')$, and from the fact that $k(\cdot)'$ satisfies Hörmander's condition that it is bounded from $L^{p'}(\mathbf{R}^n; Y')$ to $L^{p'}(\mathbf{R}^n; X')$ for $p' \in\,]1, q]$, with a constant C_pA. Finally, again by duality, we have that $k(\cdot)*$ is bounded from $L^p(\mathbf{R}^n; X)$ to $L^p(\mathbf{R}^n; Y)$ for $p \in [q', \infty[$, and this completes the proof. □

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