VECTOR-VALUED MULTIPLIER THEOREMS OF COIFMAN–RUBIO DE FRANCIA–SEMMES TYPE

T. P. HYTÖNEN AND D. POTAPOV

Abstract. We improve the vector-valued Marcinkiewicz multiplier theorem in a subclass of UMD spaces (introduced by Berkson, Gillespie and Torrea), where Rubio de Francia's generalized Littlewood–Paley inequality is valid.

The aim of this note is to provide sufficient conditions, in terms of the s-variation of the symbol m on dyadic intervals, for boundedness of the Fourier multiplier operator $T_m = \mathscr{F}^{-1} m \mathscr{F}$ acting on Banach space valued functions. The underlying space X is required to have the so-called Littlewood–Paley–Rubio property.

We recall this notion and the basic related facts in Sec. 1. Our main results for multipliers in the mentioned class of spaces are stated and proved in Sec. 2, and in Sec. 3 we further improve these multiplier theorems in certain interpolation spaces.

1. The Littlewood–Paley–Rubio property of a Banach space

Let $\mathscr{F}$ be the Fourier transform on $\mathbf{R}$, let $I \subset \mathbf{R}$ be an interval, and 1_I its indicator function. We denote by $\Delta[I]$ the spectral projection

$$\Delta[I]f := \mathscr{F}^{-1}(1_I \mathscr{F} f),$$

where $f \in L^p_X(\mathbf{R})$ and X is some Banach space. These are known to be bounded operators if and only if $1 < p < \infty$ and X is a UMD space (cf. [4]).

By ε_i (with various index sets), we designate the Rademacher functions on some probability space (Ω, Σ, P). These are random variables with $P(\varepsilon_i = +1) = P(\varepsilon_i = -1) = 1/2$, and two such functions with different indices are always assumed independent. $E := \int_\Omega (\cdot) dP$ is the mathematical expectation.

E. Berkson, T. A. Gillespie and J. L. Torrea [2] (cf. also [7]) have introduced the following Banach space notion:

1.1. **Definition.** Let X be a Banach space and $2 \le p < \infty$. The space X is said to have the LPR_p (Littlewood–Paley–Rubio) property if there is a constant $C < \infty$ so that for every collection $\mathscr{I}$ of disjoint intervals $I \subset \mathbf{R}$ and every $f \in L^p_X(\mathbf{R})$ the following inequality holds:

$$E\Big\| \sum_{I \in \mathscr{I}} \varepsilon_I \Delta[I] f \Big\|_{L^p_X} \le C \, \|f\|_{L^p_X} . \tag{1.2}$$

For $X = \mathbf{C}$ the previous estimate is the remarkable Littlewood–Paley inequality for arbitrary intervals proved by J. L. Rubio de Francia [11]; hence the name. An alternative approach to this inequality is due to J. Bourgain [3].

Several results concerning the classes LPR_p were established in [2]: A space X having any LPR_p is necessarily UMD, and also satisfies

$$p(X) := \sup\{t : X \text{ has Rademacher-type } t\} = 2.$$

Each LPR_p is a super-property, i.e., every Banach space Y which is finitely representable in X satisfies LPR_p if X does. In [7] it is shown that (1.2) is equivalent to a similar inequality for periodic functions $f \in L^p_X(\mathbf{T})$.

Date: January 30, 2009.

2000 *Mathematics Subject Classification.* 42A45.

Key words and phrases. Fourier multiplier, Littlewood-Paley inequality, s-variation, UMD space, R-boundedness.

At the present, the LPR_p property is only known for a fairly restricted class of spaces, for which it is deduced with standard techniques from Rubio de Francia's original scalar-valued inequality: An application of Fubini's theorem shows that $L^p_X(\mu)$ inherits the LPR_p property from X whenever μ is a σ-finite measure; in particular the usual Lebesgue spaces $L^p(\mu)$ have LPR_p when $2 \leq p < \infty$. On the other hand, the original proof of Rubio de Francia is perfectly valid in arbitrary Hilbert spaces, showing that each Hilbert space H has LPR_p for every $2 \leq p < \infty$. Interpolation between the LPR_p estimates for $L^2(\mu)$ and $L^p(\mu)$ proves the LPR_p property of $L^q(\mu)$ for $2 \leq q \leq p < \infty$.

These are essentially all the examples known today; no general sufficient geometric condition is known for checking the LPR_p property, and also the dependence of the classes LPR_p on the parameter p is unclear.

2. Multiplier theorems from the Littlewood–Paley–Rubio estimate

R. Coifman, Rubio de Francia and S. Semmes [5] observed that the inequality (1.2) for $X = \mathbf{C}$ can be exploited to yield a substantial improvement of the classical Marcinkiewicz multiplier theorem on $L^p(\mathbf{R})$, $1 < p < \infty$. Our purpose here is to obtain an extension of their result to the spaces $L^p_X(\mathbf{R})$ where X has LPR_p (or X' has $LPR_{p'}$).

The extensions of Marcinkiewicz' theorem to $L^p_X(\mathbf{R})$ by J. Bourgain [4] (for scalar multipliers) and L. Weis [12] (for operator multipliers) have provided powerful tools e.g. in the theory of evolution equations (see [9] for recent survey lectures). These results are valid under the condition that X have UMD. While the present assumptions on the space are rather more restrictive, they still cover a range (depending on p) of the reflexive Lebesgue spaces, which are the most important concrete spaces in the typical applications like L^p-L^q-type estimates for solutions of differential equations.

The following result is central in extending the Coifman–Rubio de Francia–Semmes theorem to the vector-valued situation:

2.1. **Proposition.** *Let X be a Banach space with finite cotype. Let K and J_k, for all $k \in K$, be disjoint finite index sets, and let $J := \bigcup_{k\in K} J_k$. For all $j \in J$, let $x_j \in X$ and $\lambda_j \in \mathbf{C}$. If the scalars satisfy*

$$\max_{k\in K} \sum_{j\in J_k} |\lambda_j|^2 \leq M^2,$$

then the following estimate holds with some finite C depending only on the space X:

$$E\Big\| \sum_{k\in K} \varepsilon_k \sum_{j\in J_k} \lambda_j x_j \Big\|_X \leq CME \Big\| \sum_{j\in J} \varepsilon_j x_j \Big\|_X .$$

Proof. Under the assumption that X have finite cotype, the randomized norms with Rademacher functions ε_j are bounded from both sides by similar norms, where the ε_j's are replaced by an independent sequence γ_j of standard Gaussian random variables (see [6], 12.27). Thus it suffices to show that

$$E\Big\| \sum_{k\in K} \gamma_k y_k \Big\|_X \leq ME \Big\| \sum_{j\in J} \gamma_j x_j \Big\|_X, \qquad y_k := \sum_{j\in J_k} \lambda_j x_j.$$

This follows from

$$\begin{aligned} \sum_{k\in K} |\langle y_k, x'\rangle|^2 &= \sum_{k\in K} \Big| \sum_{j\in J_k} \lambda_j \langle x_j, x'\rangle \Big|^2 \leq \sum_{k\in K} \Big(\sum_{j\in J_k} |\lambda_j|^2 \Big) \Big(\sum_{j\in J_k} |\langle x_j, x'\rangle|^2 \Big) \\ &\leq M^2 \sum_{k\in K} \sum_{j\in J_k} |\langle x_j, x'\rangle|^2 = M^2 \sum_{j\in J} |\langle x_j, x'\rangle|^2 , \qquad x' \in X', \end{aligned}$$

where X' is the dual of X, and [10], Proposition 3.7. □

We next introduce a vector-valued version of a function class considered by Coifman, Rubio de Francia and Semmes [5]:

2.2. **Definition** (The space R^2_Y)**.** Let $J \subset \mathbf{R}$ be an interval and Y a Banach space. We say that a function a is in $\mathscr{R}^2_Y(J)$, if there is a collection $\mathscr{I}$ of disjoint subintervals $I \subseteq J$ and elements $c_I \in Y$ such that

$$a = \sum_{I \in \mathscr{I}} c_I 1_I, \qquad \Big(\sum_{I \in \mathscr{I}} |c_I|_Y^2 \Big)^{1/2} \leq 1.$$

We define $R^2_Y(J)$ as the atomic space generated by these atoms, i.e., $f \in R^2_Y(J)$ if there are $a_k \in \mathscr{R}^2_Y(J)$ and $\lambda_k \in \mathbf{C}$ so that

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \qquad \sum_{k=1}^{\infty} |\lambda_k| < \infty.$$

The $R^2_Y(J)$ norm of f is defined as the infimum of these sums taken over all such representations.

If $B \subset Z$ is an absolutely convex, bounded subset of a Banach space Z and $Y = \operatorname{span} B$, the linear span of B, is normed by the Minkowski functional $\|\cdot\|_B$ of B (which makes it a Banach space), we also denote $R^2_B := R^2_Y$.

We denote the standard dyadic partition of $\mathbf{R}$ by

$$\Delta := \{\eta\, [2^k, 2^{k+1}[\,: k \in \mathbf{Z}, \eta \in \{-1, +1\}\},$$

and recall Bourgain's vector-valued Littlewood–Paley inequality [4], valid for all UMD spaces X:

$$A_p(X) \|f\|_{L^p_X} \leq E \Big\| \sum_{J \in \Delta} \varepsilon_J \Delta[J] f \Big\|_{L^p_X} \leq B_p(X) \|f\|_{L^p_X}, \qquad p \in \,]1, \infty[\,.$$

For the following multiplier theorem, let us further recall the notion of R-boundedness: A subset $\mathscr{T} \subseteq \mathscr{L}(X)$ is called R-bounded if for all $n = 1, 2, \ldots$, all $x_i \in X$ and $T_i \in \mathscr{T}$ there holds

$$E \Big| \sum_{i=1}^{n} \varepsilon_i T_i x_i \Big|_X \leq C E \Big| \sum_{i=1}^{n} \varepsilon_i x_i \Big|_X.$$

The smallest admissible constant C above is denoted by $\mathscr{R}(\mathscr{T})$, or $\mathscr{R}(\mathscr{T}|\mathscr{L}(X))$ if the space needs to be specified. Note that $\mathscr{T} = \bar{B}_{\mathbf{C}} \cdot \mathrm{id}_X$, where $\bar{B}_{\mathbf{C}}$ is the unit-ball of $\mathbf{C}$, has this property for every Banach space X, so that all the results below apply in particular to scalar multipliers m.

2.3. **Theorem.** *Let X be a Banach space, and let $\mathscr{T} \subset \mathscr{L}(X)$ be an absolutely convex, R-bounded set. Let $m : \mathbf{R} \to \operatorname{span} \mathscr{T}$ be a function such that*

$$\sup_{J \in \Delta} \|m|_J\|_{R^2_{\mathscr{T}}} < \infty.$$

Suppose that either $2 \leq p < \infty$ and X has LPR_p, or else that $1 < p \leq 2$ and X' has $LPR_{p'}$.

Then m is an $L^p_X(\mathbf{R})$ Fourier multiplier, and more precisely

$$\|T_m\|_{\mathscr{L}(L^p_X)} \leq C_p(X) \mathscr{R}(\mathscr{T}) \sup_{J \in \Delta} \|m|_J\|_{R^2_{\mathscr{T}}},$$

where $T_m f := \mathscr{F}^{-1}(m \mathscr{F} f)$ is the multiplier operator with symbol m.

Proof. Let us prove the case that X has LPR_p. The other case follows from this by a standard argument: The collection $\mathscr{T}' \subset \mathscr{L}(X')$ of adjoint operators is also R-bounded, so that we get $T_{m'} \in \mathscr{L}(L^{p'}_{X'}(\mathbf{R}))$ by the first case, and we can use duality.

Let $f \in L^p_X(\mathbf{R})$, and assume without loss of generality that its (standard, dyadic) Littlewood–Paley decomposition has only finitely many non-zero terms. We may also assume by approximation that the $m|_J$, for these intervals J, admit finite atomic decompositions. Further, using the fact that $\operatorname{conv}(E_1) \times \cdots \times \operatorname{conv}(E_n) = \operatorname{conv}(E_1 \times \cdots \times E_n)$, we may take

$$m|_J = \sum_{k=1}^{N} \lambda_k a_k^J, \qquad a_k^J = \sum_{I \in \mathscr{I}_k^J} c_I^{J,k} 1_I,$$

where the sequence of coefficients λ_k does not depend on J. Each $\mathscr{I}_k^J$ is a collection of disjoint subintervals $I \subseteq J$, and by further approximation we may assume that the $\mathscr{I}_k^J$ are finite. Then

$$\|T_m f\|_{L^p_X} = \Big\| \sum_{J\in\Delta} \Delta[J] \sum_{k=1}^N \lambda_k \sum_{I\in\mathscr{I}_k^J} c_I^{J,k} \Delta[I] f \Big\|_{L^p_X} \leq C \sum_{k=1}^N |\lambda_k|\, E \Big\| \sum_{J\in\Delta} \varepsilon_J \sum_{I\in\mathscr{I}_k^J} c_I^{J,k} \Delta[I]\Delta[J] f \Big\|_{L^p_X},$$

where we used the triangle inequality, the UMD property of X and the commutativity of $\Delta[I]$ and $\Delta[J]$. Writing $c_I^{J,k} = \|c_I^{J,k}\|_{\mathscr{T}} \times c_I^{J,k}/\|c_I^{J,k}\|_{\mathscr{T}}$ (the quotient is interpreted as zero if $c_I^{J,k}$ vanishes), and applying Proposition 2.1 (recalling that LPR_p implies UMD which implies finite cotype, cf. [2]) we get

$$\leq C \sum_{k=1}^N |\lambda_k|\, E \Big\| \sum_{J\in\Delta} \sum_{I\in\mathscr{I}_k^J} \varepsilon_I^{J,k} \frac{c_I^{J,k}}{\|c_I^{J,k}\|_{\mathscr{T}}} \Delta[I]\Delta[J] f \Big\|_{L^p_X}.$$

The operators $c_I^{J,k}/\|c_I^{J,k}\|_{\mathscr{T}}$ belong by definition to the R-bounded set $\mathscr{T}$, and so we may continue

$$\leq C\mathscr{R}(\mathscr{T}) \sum_{k=1}^N |\lambda_k|\, E \Big\| \sum_{J\in\Delta} \sum_{I\in\mathscr{I}_k^J} \varepsilon_I^{J,k} \Delta[I]\Delta[J] f \Big\|_{L^p_X} = C \sum_{k=1}^N |\lambda_k|\, E \Big\| \sum_{I\in\mathscr{I}_k} \varepsilon_I \Delta[I] f \Big\|_{L^p_X},$$

where $\mathscr{I}_k := \bigcup_{J\in\Delta} \mathscr{I}_k^J$ is a collection of disjoint intervals $I \subset \mathbf{R}$ for every k, and we used the fact that $\Delta[I]\Delta[J] = \Delta[I]$ for $I \in \mathscr{I}_k^J$. By the assumed inequality (1.2) we finally get

$$\leq C\mathscr{R}(\mathscr{T}) \sum_{k=1}^N |\lambda_k|\, \|f\|_{L^p_X} \leq C\mathscr{R}(\mathscr{T}) \sup_{J\in\Delta} \|m|_J\|_{R^2_{\mathscr{T}}} \|f\|_{L^p_X},$$

provided the atomic decompositions of $m|_J$ were chosen close to optimal. □

2.4. **Definition** (The space V^s_Y)**.** Let $J = [a,b] \subset \mathbf{R}$ be an interval. We say that a function $f : J \to Y$ has bounded s-variation, $1 \leq s < \infty$, if the supremum of the quantities

$$\Big(|f(t_0)|_Y^s + \sum_{i=1}^N |f(t_{i-1}) - f(t_i)|_Y^s \Big)^{1/s}$$

is bounded when $a = t_0 < t_1 < \ldots < t_N = b$ ranges over all finite partitions of J into smaller intervals. We denote this supremum by $\|f\|_{V^s_Y}$. We employ a similar short-hand $V^s_B = V^s_Y$ as with the space R^2.

Now we can state a consequence of Theorem 2.3:

2.5. **Corollary.** *Let X be a Banach space, let $1 \leq s < 2$, and let $\mathscr{T} \subset \mathscr{L}(X)$ be an R-bounded set like in Theorem 2.3. Let $m : \mathbf{R} \to \operatorname{span}\mathscr{T}$ be of bounded s-variation with respect to the Minkowski functional $\|\cdot\|_{\mathscr{T}}$, uniformly on the dyadic intervals. Suppose that either $2 \leq p < \infty$ and X has LPR_p, or else that $1 < p \leq 2$ and X' has $LPR_{p'}$.*

Then m is an $L^p_X(\mathbf{R})$ Fourier multiplier; more precisely

$$\|T_m\|_{\mathscr{L}(L^p_X)} \leq C_s C_p(X) \mathscr{R}(\mathscr{T}) \sup_{J\in\Delta} \|m|_J\|_{V^s_{\mathscr{T}}}.$$

Proof. It is shown in [5] that $V^s \subset R^2$ for $1 \leq s < 2$, and the same argument extends to our vector-valued situation. □

Let us recall that the conclusions of Corollary 2.5 with $s = 1$ are true for all UMD spaces X and all $p \in]1,\infty[$. This corresponds to the classical Marcinkiewicz multiplier theorem which was extended to UMD spaces for scalar-multipliers (i.e., $\mathscr{T} = \bar{B}_{\mathbf{C}} \cdot \mathrm{id}_X$) by Bourgain [4], and for operator-multipliers by L. Weis [12].

3. Multiplier theorems in interpolation spaces

In the scalar-valued case treated by Coifman, Rubio de Francia and Semmes in [5], the above stated results may further be improved by interpolating with the trivial estimate $\|T_m\|_{\mathscr{L}(L^2)} = \|m\|_{L^\infty}$ provided by Plancherel's theorem. This idea can be partly carried over to the vector-valued situation under the additional assumption that our space X with LPR_p (or whose dual has $LPR_{p'}$) is a complex interpolation space between another similar space and a Hilbert space, which is the case, e.g., for $X = L^q(\mu)$ where $2 \le q < p$ (or $p < q \le 2$ in the dual case). Before formulating the theorem precisely, let us state an interpolation result concerning the s-variation multiplier classes. We introduce the notation

$$\ell^\infty(V_Y^s) := \{m : (\|m|_J\|_{V_Y^s})_{J\in\Delta} \in \ell^\infty\}, \qquad 1 \le s \le \infty,$$

where we let $V_Y^\infty := L_Y^\infty$.

3.1. **Lemma.** *Let $\alpha, \theta \in \,]0,1[$, $\beta \in \,]0,1]$, Y be any Banach space, and $\epsilon > 0$ be small. Then*

$$\ell^\infty(V_Y^{(1-(1-\theta)\alpha-\theta\beta+\epsilon)^{-1}}) \subseteq [\ell^\infty(V_Y^{(1-\alpha)^{-1}}), \ell^\infty(V_Y^{(1-\beta)^{-1}})]_\theta \subseteq \ell^\infty(V_Y^{(1-(1-\theta)\alpha-\theta\beta-\epsilon)^{-1}}).$$

Proof. Everything is based on [8], formula (5):

$$\ell^\infty(V^{(1-\eta+\epsilon)^{-1}}) \subseteq [\ell^\infty(V^1), \ell^\infty(L^\infty)]_\eta \subseteq \ell^\infty(V^{(1-\eta)^{-1}}), \tag{3.2}$$

which also holds in the vector-valued setting by a straightforward modification of the same argument. We also omit the reference to the Banach space Y in the rest of the proof to simplify notation. Using (3.2) and some reiterations (cf. [1], Theorem 4.6.1) we get:

$$\begin{aligned}
\ell^\infty(V^{(1-(1-\theta)\alpha-\theta\beta+\epsilon)^{-1}}) &\subseteq [\ell^\infty(V^1), \ell^\infty(L^\infty)]_{(1-\theta)\alpha+\theta\beta} \\
&= [[\ell^\infty(V^1), \ell^\infty(L^\infty)]_\alpha, [\ell^\infty(V^1), \ell^\infty(L^\infty)]_\beta]_\theta \\
&\subseteq [\ell^\infty(V^{(1-\alpha)^{-1}}), \ell^\infty(V^{(1-\beta)^{-1}})]_\theta \\
&\subseteq [[\ell^\infty(V^1), \ell^\infty(L^\infty)]_{\alpha+\epsilon}, [\ell^\infty(V^1), \ell^\infty(L^\infty)]_{\beta+\epsilon}]_\theta \\
&= [\ell^\infty(V^1), \ell^\infty(L^\infty)]_{(1-\theta)\alpha+\theta\beta+\epsilon} \\
&\subseteq \ell^\infty(V^{(1-(1-\theta)\alpha-\theta\beta-\epsilon)^{-1}}).
\end{aligned}$$

□

The following result is simply a Hilbert space version of the Coifman–Rubio de Francia–Semmes multiplier theorem:

3.3. **Proposition.** *Let H be a Hilbert space, $s \ge 2$, and $m : \mathbf{R} \to \mathscr{L}(H)$. Then*

$$\|T_m\|_{\mathscr{L}(L_H^p)} \le C_{s,p} \sup_{J\in\Delta} \|m|_J\|_{V^s_{\mathscr{L}(H)}}, \qquad |p^{-1} - 2^{-1}| < s^{-1}.$$

Proof. From Plancherel's theorem one has

$$\|T_m f\|_{L_H^2} \le \|m\|_{\ell^\infty(L^\infty_{\mathscr{L}(H)})} \|f\|_{L_H^2} \tag{3.4}$$

It follows from the hypothesis $\left|p^{-1} - 2^{-1}\right| < s^{-1}$ that there are numbers $q \in]1,\infty[$ and $\beta \in [0, 2^{-1}[$ such that

$$(1-\theta)(1-\beta) = \left|p^{-1} - 2^{-1}\right| \cdot \frac{1-\beta}{|q^{-1} - 2^{-1}|} < s^{-1},$$

where $\theta \in [0,1]$ satisfies $p^{-1} = (1-\theta)/q + \theta/2$. We then apply Corollary 2.5 with H in place of X and q in place of p; this is legitimate since both H and its dual have $LPR_{\max\{q,q'\}}$. With $\mathscr{T} = \bar{B}_{\mathscr{L}(H)}$ we get

$$\|T_m f\|_{L_H^q} \le C \|m\|_{\ell^\infty(V^{(1-\beta)^{-1}}_{\mathscr{L}(H)})} \|f\|_{L_H^q}. \tag{3.5}$$

Using bilinear interpolation ([1], Theorem 4.4.1) between (3.4) and (3.5) together with Lemma 3.1 yields

$$\|T_m f\|_{L_H^p} \le C \|m\|_{\ell^\infty(V^{((1-\theta)(1-\beta)+\epsilon)^{-1}}_{\mathscr{L}(H)})} \|f\|_{L_H^p}, \qquad \epsilon > 0.$$

With $\epsilon := s^{-1} - (1-\theta)(1-\beta)$, the claim follows. $\square$

By a further interpolation argument we obtain the following result for complex interpolation spaces X:

3.6. **Theorem.** *Let X and Y be Banach spaces. Suppose that either $2 \leq p < \infty$ and Y has LPR_p, or else that $1 < p \leq 2$ and Y' has $LPR_{p'}$. Let H be a Hilbert space, and let $X = [Y, H]_\theta$ for some $\theta \in]0, 1[$. Let $\mathscr{T} \subset \mathscr{L}(Y) \cap \mathscr{L}(H)$ be an absolutely convex set, which is R-bounded in $\mathscr{L}(Y)$ and in $\mathscr{L}(H)$ (the latter being equivalent to uniform boundedness).*

Then for $s \geq 2$, there holds

$$\|T_m\|_{\mathscr{L}(L^p_X)} \leq C_{s,p,\theta}(Y)\mathscr{R}(\mathscr{T}|\mathscr{L}(Y))^{1-\theta}\mathscr{R}(\mathscr{T}|\mathscr{L}(H))^\theta \sup_{J\in\Delta} \|m|_J\|_{V^s_{\mathscr{T}}},$$

provided that

$$\theta\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{s} - \frac{1-\theta}{2}.$$

In particular (with $s = 2$), we have

$$\|T_m\|_{\mathscr{L}(L^p_X)} \leq C_{p,\theta}(Y)\mathscr{R}(\mathscr{T}|\mathscr{L}(Y))^{1-\theta}\mathscr{R}(\mathscr{T}|\mathscr{L}(H))^\theta \sup_{J\in\Delta} \|m|_J\|_{V^2_{\mathscr{T}}}.$$

Proof. We make use of the estimates provided by Corollary 2.5,

$$\|T_m f\|_{L^p_Y} \leq C\mathscr{R}(\mathscr{T}|\mathscr{L}(Y))\,\|m\|_{\ell^\infty(V^{(1-\alpha)^{-1}}_{\mathscr{T}})}\,\|f\|_{L^p_Y}, \qquad 1-\alpha = \frac{1}{2} + \epsilon,$$

and Proposition 3.3,

$$\begin{aligned}\|T_m f\|_{L^p_H} &\leq C\,\|m\|_{\ell^\infty(V^{(1-\beta)^{-1}}_{\mathscr{L}(H)})}\,\|f\|_{L^p_H}\\ &\leq C\sup\{\|T\|_{\mathscr{L}(H)} : T \in \mathscr{T}\}\,\|m\|_{\ell^\infty(V^{(1-\beta)^{-1}}_{\mathscr{T}})}\,\|f\|_{L^p_H}, \qquad 1-\beta = \left|\frac{1}{p} - \frac{1}{2}\right| + \epsilon.\end{aligned}$$

Bilinear interpolation ([1], Theorem 4.4.1) and an application of Lemma 3.1 now give

$$\|T_m f\|_{L^p_X} \leq C\mathscr{R}(\mathscr{T}|\mathscr{L}(Y))^{1-\theta}\mathscr{R}(\mathscr{T}|\mathscr{L}(H))^\theta\,\|m\|_{\ell^\infty(V^s_{\mathscr{T}})}\,\|f\|_{L^p_X},$$

where

$$\frac{1}{s} = 1 - (1-\theta)\alpha - \theta\beta + \epsilon = 1 - (1-\theta)\frac{1}{2} - \theta + \theta\left|\frac{1}{p} - \frac{1}{2}\right| + 2\epsilon.$$

With an appropriate choice of $\epsilon > 0$, this expression may take any value satisfying

$$\frac{1}{s} > \frac{1-\theta}{2} + \theta\left|\frac{1}{p} - \frac{1}{2}\right|.$$

For the last claim, just observe that the restriction is always satisfied when $s = 2$. $\square$

One may further derive Fourier multiplier theorems on $L^p_X(\mathbf{R})$, where neither X has LPR_p nor X' has $LPR_{p'}$, but where X is an interpolation space between a space of one of these types and an arbitrary UMD space. Here one can use the $\ell^\infty(V^1)$ condition of the Marcinkiewicz theorem as the estimate at the other end of interpolation. We leave the details to the reader.

Acknowledgement. The research was started while T.H. held a position in the European research training network HPRN-CT-2002-00281 "Evolution equations for deterministic and stochastic systems", at the Technische Universiteit Delft. It was essentially completed during T.H.'s appointment at the Flinders University as Visiting Scholar. T.H. was also sponsored by the Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation. The support of the mentioned institutions is gratefully acknowledged.

References

[1] J. Bergh, J. Löfström. *Interpolation spaces. An introduction.* Springer, Berlin–Heidelberg–New York, 1976.

[2] E. Berkson, T. A. Gillespie, J. L. Torrea. Vector valued transference. In: P. Liu (ed.), *Functional Space Theory and its applications* (Wuhan, 2003), 1–27.

[3] J. Bourgain. On square functions on the trigonometric system. *Bull. Soc. Math. Belg. Sér. B* **37** # 1 (1985), 20–26.

[4] J. Bourgain. Vector-valued singular integrals and the H^1-BMO duality. *Probability theory and harmonic analysis* (Cleveland, Ohio, 1983), 1–19, Monogr. Textbooks Pure Appl. Math., 98, Dekker, New York, 1986.

[5] R. R. Coifman, J. L. Rubio de Francia, S. Semmes. Multiplicateurs de Fourier dans $L^p(\mathbf{R})$ et estimations quadratiques. *C. R. Acad. Sci. Paris* **306** (1988), 351–354.

[6] J. Diestel, H. Jarchow, A. Tonge. *Absolutely Summing Operators.* Cambridge stud. adv. math. 43, Cambridge Univ. Press, 1995.

[7] T. A. Gillespie, J. L. Torrea. Transference of a Littlewood–Paley–Rubio inequality and dimension free estimates. *Rev. Un. Mat. Argentina* **45** # 1 (2004), 1–6.

[8] A. A. Kruglov, M. Z. Solomjak. Interpolation of operators in the spaces V_p. (Russian) *Vestnik Leningrad. Univ.* # 13 (1971), 54–60.

[9] P. C. Kunstmann, L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. *Functional analytic methods for evolution equations*, 65–311. Lecture Notes in Math., 1855, Springer, Berlin, 2004.

[10] G. Pisier. *Factorization of linear operators and geometry of Banach spaces.* CBMS Regional Conference Series in Mathematics, 60. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[11] J. L. Rubio de Francia. A Littlewood–Paley inequality for arbitrary intervals. *Rev. Mat. Iberoamericana* **1** # 2 (1985), 1–14.

[12] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.* **319** # 4 (2001), 735–758.

[13] F. Zimmermann. On vector-valued Fourier multiplier theorems. *Studia Math.* **93** # 3 (1989), 201–222.

Department of Mathematics, University of Turku, FI-20014 Turku, Finland
E-mail address: `tuomas.hytonen@utu.fi`

School of Informatics and Engineering, Flinders University of South Australia, Bedford Park, 5042 SA, Australia
E-mail address: `pota0002@infoeng.flinders.edu.au`