

R -BOUNDEDNESS OF SMOOTH OPERATOR-VALUED FUNCTIONS

TUOMAS HYTÖNEN AND MARK VERAAR

ABSTRACT. In this paper we study R -boundedness of operator families $\mathcal{T} \subset \mathcal{B}(X, Y)$, where X and Y are Banach spaces. Under cotype and type assumptions on X and Y we give sufficient conditions for R -boundedness. In the first part we show that certain integral operators are R -bounded. This will be used to obtain R -boundedness in the case that \mathcal{T} is the range of an operator-valued function $T : \mathbb{R}^d \rightarrow \mathcal{B}(X, Y)$ which is in a certain Besov space $B_{r,1}^{d/r}(\mathbb{R}^d; \mathcal{B}(X, Y))$. The results will be applied to obtain R -boundedness of semigroups and evolution families, and to obtain sufficient conditions for existence of solutions for stochastic Cauchy problems.

1. INTRODUCTION

The notion of R -boundedness (see Section 2.3 for definition) appeared implicitly in the work of Bourgain [6] and was formalized by Berkson and Gillespie [5]. Clément, de Pagter, Suckochev and Witvliet [7] studied it in more detail in relation to vector-valued Schauder decompositions, and shortly after Weis [40] found a characterization of maximal regularity for the Cauchy problem $u' = Au + f$, $u(0) = 0$, in terms of R -boundedness of the resolvent of A or the associated semigroup. After this, many authors have used R -boundedness techniques in the theory of Fourier multipliers and Cauchy problems (cf. [8, 16, 21] and references therein).

For Hilbert space operators, R -boundedness is equivalent to uniform boundedness. The basic philosophy underlying much of the work cited above is that many results for Hilbert spaces remain true in certain Banach spaces if one replaces boundedness by R -boundedness. Thus it is useful to be able to recognize R -bounded sets of operators.

Let X and Y be Banach spaces. In this paper we will study R -boundedness of some subsets of $\mathcal{B}(X, Y)$ under type and cotype assumptions. Although the definition of R -boundedness suggests connections with type and cotype, there are only few results on this in the literature. Arendt and Bu [3, Proposition 1.13] pointed out that uniform boundedness already implies R -boundedness if (and only if) X has cotype 2 and Y has type 2. Recently, van Gaans [12] showed that a countable union of R -bounded sets remains R -bounded if the individual R -bounds are ℓ^r summable

2000 *Mathematics Subject Classification.* 47B99 (Primary); 46B09, 46E35, 46E40, 60B05 (Secondary).

Key words and phrases. R -boundedness, operator theory, type and cotype, Besov space, semigroup theory, evolution family, stochastic Cauchy problem.

Tuomas Hytönen is supported by the Academy of Finland (grant 114374).

Mark Veraar is supported by the Alexander von Humboldt foundation. His visit to Helsinki, which started this project, was funded by the Finnish Centre of Excellence in Analysis and Dynamics Research.

for an appropriate r depending on the type and cotype assumptions, improving on the trivial result with $r = 1$ (the triangle inequality!) valid for any Banach spaces. Implicitly, one can find similar ideas already in Figiel [11].

In [13, Theorem 5.1], Girardi and Weis have found criteria for R -boundedness of the range of operator-valued functions $T : \mathbb{R}^d \rightarrow \mathcal{B}(X, Y)$ in terms of their smoothness and the Fourier type of the Banach space Y . Their result states that if Y has Fourier type $p \in [1, 2]$ and T is in the Besov space $B_{p,1}^{\frac{d}{p}}(\mathbb{R}^d; \mathcal{B}(X, Y))$, then $\{T(t) : t \in \mathbb{R}^d\}$ is R -bounded.

We will prove a similar result as [13, Theorem 5.1] under assumptions on the cotype of X and the type of Y . More precisely, if X has cotype q and Y has type p and if $T \in B_{r,1}^{\frac{d}{r}}(\mathbb{R}^d; \mathcal{B}(X, Y))$ for some $r \in [1, \infty]$ such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, then $\{T(t) : t \in \mathbb{R}^d\}$ is R -bounded (see Theorem 5.1 below). Our result improves [13, Theorem 5.1]. This follows from the fact that every space with Fourier type p has type p . Furthermore, we note that the only spaces which have Fourier type 2 are spaces which are isomorphic to a Hilbert space. However, there are many Banach spaces with type 2, e.g., all L^p spaces with $p \in [2, \infty)$. In the limit case that X has cotype 2 and Y has type 2 our assumption on T becomes $T \in B_{\infty,1}^0(\mathbb{R}^d; \mathcal{B}(X, Y))$. This condition is quite close to uniform boundedness of $\{T(t) : t \in \mathbb{R}^d\}$ which under these assumption on X and Y is equivalent to R -boundedness.

Following [13, Section 5], we apply the sufficient condition for R -boundedness to strongly continuous semigroups. Furthermore, we show that our results are sharp in the case of the translation semigroup on $L^p(\mathbb{R})$. The R -boundedness result for semigroups leads to existence, uniqueness and regularity results for stochastic equations with additive noise. As a second application we present an R -boundedness result for evolution families, assuming the conditions of Acquistapace and Terreni [1].

We will write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leq Cb$, and $a \approx b$ if $a \lesssim b \lesssim a$. If the constant C is allowed to depend on some parameter t , we write $a \lesssim_t b$ and $a \approx_t b$ instead.

2. PRELIMINARIES

Throughout this paper $(\Omega, \mathcal{A}, \mathbb{P})$ denotes a probability space, and \mathbb{E} is the expectation. Let X and Y be Banach spaces. Let $(r_n)_{n \geq 1}$ be a Rademacher sequence on Ω , i.e. an independent sequence with

$$\mathbb{P}(r_n = 1) = \mathbb{P}(r_n = -1) = \frac{1}{2}.$$

For $N \geq 1$ and $x_1, \dots, x_N \in E$, recall the Kahane-Khinchine inequalities (cf. [10, Section 11.1] or [23, Proposition 3.4.1]): for all $p, q \in [1, \infty)$, we have

$$(2.1) \quad \left(\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^p \right)^{\frac{1}{p}} \approx_{p,q} \left(\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^q \right)^{\frac{1}{q}}.$$

These inequalities will be often applied without referring to it explicitly.

For each integer N , the space $\text{Rad}_N(X) \subset L^2(\Omega; X)$ is defined as all elements of the form $\sum_{n=1}^N r_n x_n$, where $(x_n)_{n=1}^N$ are in X .

2.1. Type and cotype. Let $p \in [1, 2]$ and $q \in [2, \infty]$. The space X is said to have *type* p if there exists a constant $C \geq 0$ such that for all $(x_n)_{n=1}^N$ in X we have

$$\left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|^2\right)^{\frac{1}{2}} \leq C \left(\sum_{n=1}^N \|x_n\|^p\right)^{\frac{1}{p}}.$$

The space X is said to have *cotype* q if there exists a constant $C \geq 0$ such that for all $(x_n)_{n=1}^N$ in X we have

$$\left(\sum_{n=1}^N \|x_n\|^q\right)^{\frac{1}{q}} \leq C \left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|^2\right)^{\frac{1}{2}},$$

with the obvious modification in the case $q = \infty$.

For a detailed study of type and cotype we refer to [10]. Every Banach space has type 1 and cotype ∞ with constant 1. Therefore, we say that X has *non-trivial type* (*non-trivial cotype*) if X has type p for some $p \in (1, 2]$ (cotype q for some $2 \leq q < \infty$). If the space X has non-trivial type, it has non-trivial cotype. Hilbert spaces have type 2 and cotype 2 with constants 1. For $p \in [1, \infty)$ the L^p -spaces have type $p \wedge 2$ and cotype $p \vee 2$.

Recall the following duality result for $\text{Rad}_N(X)$ (cf. [34] or [10, Chapter 13]). If X has non-trivial type then

$$(2.2) \quad \text{Rad}_N(X)^* = \text{Rad}_N(X^*)$$

isomorphically with constants independent of N .

2.2. Fourier type. The Fourier transform $\hat{f} = \mathcal{F}f$ of a function $f \in L^1(\mathbb{R}^d; X)$ will be normalized as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

Let $p \in [1, 2]$ and p' be the conjugate exponent, $\frac{1}{p} + \frac{1}{p'} = 1$. The space X has *Fourier type* p , if \mathcal{F} defines a bounded linear operator for some (and then for all) $d = 1, 2, \dots$ from $L^p(\mathbb{R}^d; X)$ to $L^{p'}(\mathbb{R}^d; X)$.

If X has Fourier type p , then it has both type p and cotype p' . In particular, spaces isomorphic to a Hilbert space are the only ones with Fourier type 2 (see [22]). The L^p -spaces have Fourier type $p \wedge p'$ (see [30]), while every Banach space has Fourier type 1. The notion becomes more restrictive with increasing p .

2.3. R -boundedness. A collection $\mathcal{T} \subset \mathcal{B}(X, Y)$ is said to be *R -bounded* if there exists a constant $M \geq 0$ such that

$$\left(\mathbb{E}\left\|\sum_{n=1}^N r_n T_n x_n\right\|_Y^2\right)^{\frac{1}{2}} \leq M \left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|_X^2\right)^{\frac{1}{2}},$$

for all $N \geq 1$ and all sequences $(T_n)_{n=1}^N$ in \mathcal{T} and $(x_n)_{n=1}^N$ in X . The least constant M for which this estimate holds is called the *R -bound* of \mathcal{T} , notation $R(\mathcal{T})$. By (2.1), the role of the exponent 2 may be replaced by any exponent $1 \leq p < \infty$ (at the expense of a possibly different constant).

The notion of R -boundedness has played an important role in recent progress in Fourier multiplier theory and this has applications to regularity theory of parabolic evolution equations. For details on the subject we refer to [8, 21] and references therein.

A property which we will need later on is the following. If $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R -bounded and X has non-trivial type, then it follows from (2.2) that the set of adjoint operators $\mathcal{T}^* = \{T^* \in \mathcal{B}(Y^*, X^*) : T \in \mathcal{T}\}$ is R -bounded as well.

2.4. Lorentz spaces. We recall the definition of the Lorentz space (cf. [14, 38]). Let (S, Σ, μ) be a σ -finite measure space. For $f \in L^1(S) + L^\infty(S)$ define the non-increasing rearrangement of f as

$$f^*(s) = \inf\{t > 0 : \mu(|f| > t) \leq s\}, \quad s > 0.$$

For $p, q \in [1, \infty]$ define

$$L^{p,q}(S) = \{f \in L^1(S) + L^\infty(S) : \|f\|_{L^{p,q}(S)} < \infty\},$$

where

$$\|f\|_{L^{p,q}(S)} = \begin{cases} \left(\int_0^\infty t^{q/p} f^*(t)^q \frac{dt}{t} \right)^{1/q} & \text{if } q \in [1, \infty), \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty. \end{cases}$$

For $p \in [1, \infty]$ and $q_1 < q_2$ one has

$$\|f\|_{L^{p,p}(S)} = \|f\|_{L^p(S)}, \quad \|f\|_{L^{p,q_2}(S)} \leq c_{p,q_1,q_2} \|f\|_{L^{p,q_1}(S)}.$$

Also recall (e.g. [35, pp. 331–2]) that for $p, q \in [1, \infty]$

$$(2.3) \quad \|f\|_{L^{p,q}(S)} = \left(\int_0^\infty p t^q (\mu(|f| > t))^{q/p} \frac{dt}{t} \right)^{\frac{1}{q}};$$

indeed, just compare the two iterated integrals of $s^{q-1} t^{q/p-1}$ over the subset $\{f^*(t) > s\} = \{\mu(|f| > s) > t\}$ of $(0, \infty)^2$.

2.5. Besov spaces. We recall the definition of Besov spaces using the so-called Littlewood-Paley decomposition (cf. [4, 38]). Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be a fixed Schwartz function whose Fourier transform $\widehat{\phi}$ is nonnegative and has support in $\{\xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2\}$ and which satisfies

$$\sum_{k \in \mathbb{Z}} \widehat{\phi}(2^{-k}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Such a function can easily be constructed (cf. [4, Lemma 6.1.7]). Define the sequence $(\varphi_k)_{k \geq 0}$ in $\mathcal{S}(\mathbb{R}^d)$ by

$$\widehat{\varphi}_k(\xi) = \widehat{\phi}(2^{-k}\xi) \quad \text{for } k = 1, 2, \dots \quad \text{and} \quad \widehat{\varphi}_0(\xi) = 1 - \sum_{k \geq 1} \widehat{\varphi}_k(\xi), \quad \xi \in \mathbb{R}^d.$$

Similar as in the real case one can define $\mathcal{S}(\mathbb{R}^d; X)$ as the usual *Schwartz space* of rapidly decreasing X -valued smooth functions on \mathbb{R}^d . As in the real case this is a Fréchet space. Let the space of X -valued *tempered distributions* $\mathcal{S}'(\mathbb{R}^d; X)$ be defined as the continuous linear operators from $\mathcal{S}(\mathbb{R}^d)$ into X .

For $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ the *Besov space* $B_{p,q}^s(\mathbb{R}^d; X)$ is defined as the space of all X -valued tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{B_{p,q}^s(\mathbb{R}^d; X)} := \left\| \left(2^{ks} \varphi_k * f \right)_{k \geq 0} \right\|_{l^q(L^p(\mathbb{R}^d; X))}$$

is finite. Endowed with this norm, $B_{p,q}^s(\mathbb{R}^d; X)$ is a Banach space, and up to an equivalent norm this space is independent of the choice of the initial function ϕ . The sequence $(\varphi_k * f)_{k \geq 0}$ is called the *Littlewood-Paley decomposition* of f associated with the function ϕ .

If $1 \leq p, q < \infty$, then $B_{p,q}^s(\mathbb{R}^d; X)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d; X)$ as a dense subspace.

For $1 \leq p_1 \leq p_2 \leq \infty$, $q \in [1, \infty]$ and $s_1, s_2 \in \mathbb{R}$ with $s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$ the following continuous inclusion holds (cf. [38, Theorem 2.8.1(a)])

$$B_{p_1,q}^{s_1}(\mathbb{R}^d; X) \hookrightarrow B_{p_2,q}^{s_2}(\mathbb{R}^d; X).$$

Next we give an alternative definition of Besov spaces. Let $I = (a, b)$ with $-\infty \leq a < b \leq \infty$. For $h \in \mathbb{R}$ and a function $f : I \rightarrow X$ we define the function $T(h)f : \mathbb{R} \rightarrow X$ as the translate of f by h , i.e.

$$(T(h)\phi)(t) := \begin{cases} \phi(t+h) & \text{if } t+h \in I, \\ 0 & \text{otherwise.} \end{cases}$$

For $h \in \mathbb{R}$ put

$$I[h] := \{r \in I : r+h \in I\}.$$

For a strongly measurable function $f \in L^p(I; X)$ and $t > 0$ let

$$\varrho_p(f, t) := \sup_{|h| \leq t} \left(\int_I \|T(h)f(r) - f(r)\|^p dr \right)^{\frac{1}{p}}.$$

We use the obvious modification if $p = \infty$. For $p, q \in [1, \infty]$ and $s \in (0, 1)$ define

$$\Lambda_{p,q}^s(I; X) := \{f \in L^p(I; X) : \|f\|_{\Lambda_{p,q}^s(I; X)} < \infty\},$$

where

$$(2.4) \quad \|f\|_{\Lambda_{p,q}^s(I; X)} = \left(\int_I \|f(r)\|^p dr \right)^{\frac{1}{p}} + \left(\int_0^1 (t^{-s} \varrho_p(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

with the obvious modification if $p = \infty$ or $q = \infty$. Endowed with the norm $\|\cdot\|_{\Lambda_{p,q}^s(I; X)}$, $\Lambda_{p,q}^s(I; X)$ is a Banach space. Moreover, if $I = \mathbb{R}$, then $\Lambda_{p,q}^s(\mathbb{R}; X) = B_{p,q}^s(\mathbb{R}; X)$ with equivalent norms (cf. [31, Proposition 3.1] and [36, Theorem 4.3.3]). Similarly, if $I \neq \mathbb{R}$, then for every $f \in \Lambda_{p,q}^s(I; X)$ there exists a function $g \in \Lambda_{p,q}^s(\mathbb{R}; X)$ such that $g|_I = f$ and there exists a constant $C > 0$ independent of f and g such that

$$(2.5) \quad C^{-1} \|g\|_{\Lambda_{p,q}^s(\mathbb{R}; X)} \leq \|f\|_{\Lambda_{p,q}^s(I; X)} \leq \|g\|_{\Lambda_{p,q}^s(\mathbb{R}; X)}.$$

3. TENSOR PRODUCTS

We start with a basic lemma, which can be viewed as a generalization of the Kahane-contraction principle.

Lemma 3.1. *Let X be a Banach space and let (S, Σ, μ) be a σ -finite measure space and let $q \in [2, \infty)$. The following assertions hold:*

- (1) *If X has cotype q , then there exists a constant C such that for all $(f_n)_{n=1}^N$ in $L^{q,1}(S)$ and $(x_n)_{n=1}^N$ in X*

$$(3.1) \quad \begin{aligned} & \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^q(S; L^2(\Omega; X))} \\ & \leq C \int_0^\infty \max_{1 \leq n \leq N} \mu(|f_n| > t)^{1/q} dt \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}. \end{aligned}$$

(2) If X has cotype q , then for all $\tilde{q} \in (q, \infty]$ there exists a constant C such that for all $(f_n)_{n=1}^N$ in $L^{\tilde{q}}(S)$ and $(x_n)_{n=1}^N$ in X

$$(3.2) \quad \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^{\tilde{q}}(S; L^2(\Omega; X))} \leq C \sup_{1 \leq n \leq N} \|f_n\|_{L^{\tilde{q}}(S)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}.$$

Moreover, if $q \in \{2, \infty\}$, then (3.2) holds with $\tilde{q} = q$.

(3) Assume (S, Σ, μ) contains N disjoint sets of equal finite positive measure for every $N \in \mathbb{Z}_+$. If there exists a constant C such that (3.1) holds for all $(f_n)_{n=1}^N$ in $L^{q,1}(S)$ and $(x_n)_{n=1}^N$ in X , then X has cotype q .

Remark 3.2. Note that by (2.3) for $q \in [1, \infty)$ it holds that

$$\int_0^\infty \max_{1 \leq n \leq N} \mu(|f_n| > t)^{1/q} dt \leq \int_0^\infty \lambda\left(\max_{1 \leq n \leq N} (f_n^*) > t\right)^{1/q} dt = \frac{1}{q} \left\| \max_{1 \leq n \leq N} (f_n^*) \right\|_{L^{q,1}},$$

where λ denotes the Lebesgue measure on $(0, \infty)$. Moreover, if the f_1, \dots, f_N are identically distributed, then one has

$$\int_0^\infty \max_{1 \leq n \leq N} \mu(|f_n| > t)^{1/q} dt = \frac{1}{q} \|f_1\|_{L^{q,1}(S)}.$$

With this in mind, one can also view (3.1) as an extension of [24, Proposition 9.14] and [33, Proposition 3.2(ii)]. There it is shown that (3.1) holds for the case that $\mu(S) = 1$, $(f_n)_{n \geq 1}$ are i.i.d. and symmetric.

Remark 3.3. By (2.1), one could rephrase (3.2) as follows when $\tilde{q} < \infty$. In the natural embedding $L^{\tilde{q}}(S) \hookrightarrow \mathcal{B}(X, L^{\tilde{q}}(S; X))$, $f \mapsto f \otimes (\cdot)$, the unit ball $B_{L^{\tilde{q}}(S)}$ becomes an R -bounded subset of $\mathcal{B}(X, L^{\tilde{q}}(S; X))$.

Proof of Lemma 3.1. (1) and (2): Define the operator $T : \ell_N^\infty \rightarrow L^2(\Omega; X)$ by $T(a) = \sum_{n=1}^N r_n a_n x_n$. By the Kahane contraction principle there holds

$$\|T\|_{\mathcal{B}(\ell_N^\infty, L^2(\Omega; X))} \leq 2 \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}.$$

Since $L^2(\Omega; X)$ has cotype q it follows from [10, Theorem 11.14] that T is $(q, 1)$ -summing with $\pi_{q,1}(T) \leq C_{X,q} \|T\|$. Then [32, Theorem 2.1] (also see [10, Theorem 10.9]) implies that there is a probability measure ν on $\{1, 2, \dots, N\}$ such that $T = \tilde{T}j$, where $j : \ell_N^\infty \rightarrow \ell_N^{q,1}(\nu)$ is the embedding and $\tilde{T} \in \mathcal{B}(\ell_N^{q,1}(\nu), L^2(\Omega; X))$ satisfies $\|\tilde{T}\|_{\mathcal{B}(\ell_N^{q,1}(\nu), L^2(\Omega; X))} \leq q^{-1 + \frac{1}{q}} \pi_{q,1}(T)$. It follows that for all scalars $(a_n)_{n=1}^N$,

$$\begin{aligned} \left\| \sum_{n=1}^N r_n a_n x_n \right\|_{L^2(\Omega; X)} &\leq C_{X,q} \|(a_n)_{n=1}^N\|_{\ell_N^{q,1}(\nu)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)} \\ &\leq C_{X,q,\tilde{q}} \|(a_n)_{n=1}^N\|_{\ell_N^{\tilde{q}}(\nu)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}, \end{aligned}$$

where $\ell_N^{q,1}(\nu)$ denotes the Lorentz space $L^{q,1}$ defined on $\{1, \dots, N\}$ with measure ν , and the second step follows from the embedding of $\ell_N^{q,1}(\nu)$ into $\ell_N^{\tilde{q}}(\nu)$ for $\tilde{q} \in (q, \infty]$.

If we apply this with $a_n = f_n(s)$ and take the $L^q(\mu)$ -norms, then it follows from (2.3) and Minkowski's inequality that

$$\begin{aligned}
& \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^q(S; L^2(\Omega; X))} \times \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}^{-1} \\
& \leq C_{X,q} \left(\int_S \left(\int_0^\infty \left(\sum_{n=1}^N \nu(n) \mathbf{1}_{\{|f_n(s)| > t\}} \right)^{1/q} dt \right)^q d\mu(s) \right)^{\frac{1}{q}} \\
& \leq C_{X,q} \int_0^\infty \left(\int_S \sum_{n=1}^N \nu(n) \mathbf{1}_{\{|f_n(s)| > t\}} d\mu(s) \right)^{1/q} dt \\
& = C_{X,q} \int_0^\infty \left(\sum_{n=1}^N \nu(n) \mu(|f_n| > t) \right)^{1/q} dt \\
& \leq C_{X,q} \int_0^\infty \left(\max_{1 \leq n \leq N} \mu(|f_n| > t) \right)^{1/q} dt.
\end{aligned}$$

Similarly with $L^{\tilde{q}}(\mu)$ -norms, it follows that

$$\begin{aligned}
& \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^{\tilde{q}}(S; L^2(\Omega; X))} \times \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}^{-1} \\
& \leq C_{X,q,\tilde{q}} \left(\int_S \left(\sum_{n=1}^N \nu(n) |f_n(s)|^{\tilde{q}} \right)^{\tilde{q}/\tilde{q}} d\mu(s) \right)^{1/\tilde{q}} \\
& = C_{X,q,\tilde{q}} \left(\sum_{n=1}^N \nu(n) \int_S |f_n(s)|^{\tilde{q}} d\mu(s) \right)^{1/\tilde{q}} \\
& \leq C_{X,q,\tilde{q}} \left(\max_{1 \leq n \leq N} \int_S |f_n(s)|^{\tilde{q}} d\mu(s) \right)^{1/\tilde{q}}.
\end{aligned}$$

(3): Let $x_1, \dots, x_N \in X$. Let $(S_n)_{n=1}^N$ be disjoint sets in Σ with $\mu(S_n) = \mu(S_1) \in (0, \infty)$ for all n . Letting $f_n = \mu(S_1)^{-1/q} \mathbf{1}_{S_n}$ for $n = 1, 2, \dots, N$, we obtain that

$$\left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^q(S; L^2(\Omega; X))}^q = \sum_{n=1}^N \|x_n\|^q.$$

On the other hand,

$$\mu(|f_n| > t) = \mu(S_1) \cdot \mathbf{1}_{[0, \mu(S_1)^{-1/q}]}(t)$$

for all $n = 1, \dots, N$. Therefore, (3.1) implies that

$$\sum_{n=1}^N \|x_n\|^q \leq C^q \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}^q,$$

which shows that X has cotype q . □

We do not know, whether we can take $\tilde{q} = q$ in Lemma 3.1(2) if $q \neq 2$. However, if X is an L^q -space with $q \geq 2$, then one may take $\tilde{q} = q$. This follows from the next remark in the case that $\tilde{X} = \mathbb{R}$.

Remark 3.4. Let (A, \mathcal{A}, ν) be a σ -finite measure space. Let $2 \leq q_1 < q < \infty$. Let \tilde{X} be a Banach space with cotype q_1 . If $X = L^q(A; \tilde{X})$, then (3.2) of Lemma 3.1(2) holds with $\tilde{q} = q$.

Proof. By Fubini's theorem, Lemma 3.1(2) applied to \tilde{X} and (2.1) we obtain that

$$\begin{aligned} \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^q(S; L^2(\Omega; X))} &\approx_q \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^q(A; L^q(S; L^2(\Omega; \tilde{X})))} \\ &\leq C \sup_{1 \leq n \leq N} \|f_n\|_{L^q(S)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^q(A; L^2(\Omega; \tilde{X}))} \\ &\approx_q C \sup_{1 \leq n \leq N} \|f_n\|_{L^q(S)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)}. \end{aligned}$$

□

Remark 3.5. Notice that a version of Lemma 3.1 also holds for quasi-Banach spaces. This can be proved in a similar way as above. Instead of [32] one has to use the factorization result of [18, Theorem 4.1]. Note that in [18] the role of the Lorentz space $L^{q,1}(S)$ is replaced by $L^{q,r}(S)$, where r is some number in $(0, 1]$ which depends on X . One can see from the above proof that this number r will also appear in the quasi-Banach space version of (3.2). The details are left to the interested reader.

The following dual version of Lemma 3.1 holds:

Lemma 3.6. *Let X be a Banach space, let (S, Σ, μ) be a σ -finite measure space and let $p \in (1, 2]$. The following assertions hold:*

- (1) *If X has type p , then there exists a constant C such that for all $(f_n)_{n=1}^N$ in $L^{p,\infty}(S)$ which are identically distributed and $(x_n)_{n=1}^N$ in X*

$$(3.3) \quad \|f_1\|_{L^{p,\infty}(S)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)} \leq C \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^p(S; L^2(\Omega; X))}.$$

- (2) *If X has type p , then for all $\tilde{p} \in [1, p)$ there exists a constant C such that for all $(f_n)_{n=1}^N$ in $L^{\tilde{p}}(S)$ and $(x_n)_{n=1}^N$ in X*

$$(3.4) \quad \inf_{1 \leq n \leq N} \|f_n\|_{L^{\tilde{p}}(S)} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)} \leq C \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^{\tilde{p}}(S; L^2(\Omega; X))}.$$

Moreover, if $p \in \{1, 2\}$, then (3.2) holds with $\tilde{p} = p$.

- (3) *Assume (S, Σ, μ) contains N disjoint sets of equal finite positive measure for every $N \in \mathbb{Z}_+$. If there exists a constant C such that (3.3) holds for all $(f_n)_{n=1}^N$ in $L^{p,1}(S)$ which are identically distributed and $(x_n)_{n=1}^N$ in X , then X has type p .*

A similar statement as in Remark 3.4 also holds.

Proof. (1) : Without loss of generality, $\|f_1\|_{L^{p,\infty}(S)} = \sup_{t>0} t^{1/p} f_1^*(t) = 1$. Choose t_0 so that $f_1^*(t_0) > (2t_0)^{-1/p}$, or equivalently $t_0 < \mu(|f_1| > (2t_0)^{-1/p})$. Let $A_n :=$

$\{|f_n| > (2t_0)^{-1/p}\}$, so by equidistribution, $\mu(A_n) = \mu(A_1) > t_0$. It follows that

$$\begin{aligned} \left| \sum_{n=1}^N \langle x_n, x_n^* \rangle \right| &= \frac{1}{\mu(A_1)} \left| \int_S \mathbb{E} \left\langle \sum_{n=1}^N r_n \mathbf{1}_{A_n} x_n, \sum_{m=1}^N r_m \mathbf{1}_{A_m} x_m^* \right\rangle d\mu(s) \right| \\ &\leq \frac{1}{\mu(A_1)} \left\| \sum_{n=1}^N r_n \mathbf{1}_{A_n} x_n \right\|_{L^p(S; L^2(\Omega; X))} \left\| \sum_{m=1}^N r_m \mathbf{1}_{A_m} x_m^* \right\|_{L^{p'}(S; L^2(\Omega; X^*))}. \end{aligned}$$

Now X^* has cotype p' , hence by Lemma 3.1(1) there holds

$$\left\| \sum_{m=1}^N r_m \mathbf{1}_{A_m} x_m^* \right\|_{L^{p'}(S; L^2(\Omega; X^*))} \leq C \mu(A_1)^{1/p'} \left\| \sum_{m=1}^N r_m x_m^* \right\|_{L^2(\Omega; X^*)}.$$

Since X has non-trivial type, taking the supremum over all $\sum_{n=1}^N r_n x_n^* \in \text{Rad}_N(X^*)$ with norm one, it follows that

$$\begin{aligned} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; X)} &\lesssim \left\| \sum_{n=1}^N r_n \frac{\mathbf{1}_{A_n}}{\mu(A_1)^{1/p}} x_n \right\|_{L^p(S; L^2(\Omega; X))} \\ &\lesssim \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^p(S; L^2(\Omega; X))}, \end{aligned}$$

where the last estimate used the contraction principle and the fact that $|f_n| > (2t_0)^{-1/p} \mathbf{1}_{A_n} \gtrsim \mu(A_1)^{-1/p} \mathbf{1}_{A_n}$ by the definition of A_n .

(2) : The case $p = \tilde{p} = 1$ follows from

$$\begin{aligned} LHS(3.4) &\leq \left\| \sum_{n=1}^N r_n \int_S |f_n(s)| d\mu(s) x_n \right\|_{L^2(\Omega; X)} \\ &\leq \int_S \left\| \sum_{n=1}^N r_n |f_n(s)| x_n \right\|_{L^2(\Omega; X)} d\mu(s) = RHS(3.4), \end{aligned}$$

where the first estimate was the contraction principle. For $p > 1$, we argue by duality in a similar spirit as in (1): assuming $\min_{1 \leq n \leq N} \|f_n\|_{L^{\tilde{p}}(S)} = 1$, choose $g_n \in L^{\tilde{p}'}(S)$ of at most unit norm so that $\int_S f_n \cdot g_n d\mu = 1$ and write

$$\sum_{n=1}^N \langle x_n, x_n^* \rangle = \mathbb{E} \int_S \left\langle \sum_{n=1}^N r_n f_n x_n, \sum_{m=1}^N r_m g_m x_m^* \right\rangle d\mu.$$

Then proceed as in (1), only using Lemma 3.1(2) instead of Lemma 3.1(1).

(3): This follows in a similar way as the corresponding claim in Lemma 3.1. \square

4. INTEGRAL OPERATORS

An operator-valued function $T : S \rightarrow \mathcal{B}(X, Y)$ will be called *X-strongly measurable* if for all $x \in X$, the Y -valued function $s \mapsto T(s)x$ is strongly measurable.

Let $r \in [1, \infty]$. For an X -strongly measurable mapping $T : S \rightarrow \mathcal{B}(X, Y)$ with $\|T(s)x\|_{L^r(S; Y)} \leq M\|x\|$ and $f \in L^r(S)$ we will define $T_f \in \mathcal{B}(X, Y)$ as

$$T_f x = \int_S T(s)x f(s) d\mu(s).$$

By Hölder's inequality, we have $\|T_f\|_{\mathcal{B}(X,Y)} \leq M\|f\|_{L^{r'}(S)}$. If $r = 1$, then by [21, Corollary 2.17]

$$(4.1) \quad R(\{T_f : \|f\|_{L^\infty(S)} \leq 1\}) \leq 2M.$$

In the next result we will obtain R -boundedness of $\{T_f : \|f\|_{L^{r'}(S)} \leq 1\}$ for different exponents r under assumptions on the cotype of X and type of Y .

Proposition 4.1. *Let X and Y be Banach spaces and let (S, Σ, μ) be a σ -finite measure space. Let $p_0 \in [1, 2]$ and $q_0 \in [2, \infty]$. Assume that X has cotype q_0 and Y has type p_0 . The following assertions hold:*

- (1) *If $r \in [1, \infty)$ is such that $\frac{1}{r} > \frac{1}{p_0} - \frac{1}{q_0}$. Then there exists a constant $C = C(r, p_0, q_0, X, Y)$ such that for all $T \in L^r(S; \mathcal{B}(X, Y))$,*

$$(4.2) \quad R(\{T_f \in \mathcal{B}(X, Y) : \|f\|_{L^{r'}(S)} \leq 1\}) \leq C\|T\|_{L^r(S; \mathcal{B}(X, Y))}.$$

- (2) *Assume the pair (p_0, q_0) is an element in $\{(1, \infty), (2, \infty), (2, 2), (1, 2)\}$. If $r \in [1, \infty)$ is such that $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{q_0}$, then there exists a constant $C = C(X, Y)$ such that for all $T \in L^r(S; \mathcal{B}(X, Y))$ (4.2) holds.*

Remark 4.2. Since $\mathcal{B}(X, Y)$ is usually non-separable, it could happen that $T : S \rightarrow \mathcal{B}(X, Y)$ is not strongly measurable and therefore not in $L^r(S; \mathcal{B}(X, Y))$. However, one can replace the assumption that $T \in L^r(S; \mathcal{B}(X, Y))$ by the condition that T is X -strongly measurable and $s \mapsto \|T(s)\|$ is in $L^r(S)$ or is dominated by a function in $L^r(S)$. This does not affect the assertion in Proposition 4.1 and the proof is the same.

The following will be clear from the proof of Proposition 4.1 and Remark 3.4.

Remark 4.3. Let $(A_i, \mathcal{A}_i, \nu_i)$, $i = 1, 2$ be a σ -finite measure space. Let $1 < p_0 < p_1 \leq 2$ and $2 \leq q_1 < q_0 < \infty$. Let \tilde{X} be a Banach space with cotype q_1 and \tilde{Y} be a Banach space with type p_1 . If $X = L^{q_0}(A_1; \tilde{X})$ and $Y = L^{p_0}(A_2; \tilde{Y})$, then (4.2) of Proposition 4.1 (1) holds with $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{q_0}$.

Proof of Proposition 4.1. (1): Let $(f_n)_{n=1}^N$ in $L^{r'}(S)$ be such that $\sup_n \|f_n\|_{L^{r'}(S)} \leq 1$ and $x_1, \dots, x_N \in X$.

First assume $p_0 > 1$ and $q_0 < \infty$. Let $p \in (1, p_0)$ and $q \in (q_0, \infty)$ be such that $\frac{1}{r} := \frac{1}{p} - \frac{1}{q}$, and hence $\frac{1}{r'} = \frac{1}{p'} + \frac{1}{q}$. Let $g_n = |f_n|^{r'/q}$ and $h_n = \text{sign}(f_n)|f_n|^{r'/p'}$ for $n = 1, \dots, N$. Then $\|g_n\|_{L^q(S)} \leq 1$, $\|h_n\|_{L^{p'}(S)} \leq 1$ and $f_n = g_n h_n$ for all n . Let $(y_n^*)_{n=1}^N$ in Y^* be such that $\left\| \sum_{n=1}^N r_n y_n^* \right\|_{L^{p'}(\Omega; Y^*)} \leq 1$. Then it follows from

Hölder's inequality and $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ that

$$\begin{aligned}
\mathbb{E} \left\langle \sum_{n=1}^N r_n T_{f_n} x_n, \sum_{n=1}^N r_n y_n^* \right\rangle &= \sum_{n=1}^N \langle T_{f_n} x_n, y_n^* \rangle \\
&= \int_S \sum_{n=1}^N \langle g_n(s) T(s) x_n, h_n(s) y_n^* \rangle d\mu(s) \\
&= \int_S \mathbb{E} \left\langle T(s) \sum_{n=1}^N r_n g_n(s) x_n, \sum_{n=1}^N r_n h_n(s) y_n^* \right\rangle d\mu(s) \\
&\leq \left\| T \sum_{n=1}^N r_n g_n x_n \right\|_{L^p(S \times \Omega; Y)} \left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{p'}(S \times \Omega; Y^*)} \\
&\leq \|T\|_{L^r(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n g_n x_n \right\|_{L^q(S \times \Omega; X)} \left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{p'}(S \times \Omega; Y^*)}.
\end{aligned}$$

Since X has cotype $q_0 < q$ it follows from Lemma 3.1(2) that

$$\left\| \sum_{n=1}^N r_n g_n x_n \right\|_{L^q(S \times \Omega; X)} \leq C_1 \left\| \sum_{n=1}^N r_n x_n \right\|_{L^q(\Omega; X)}$$

Since Y has type p_0 it follows that Y^* has cotype $p'_0 < p'$ (cf. [10, Proposition 11.10]) and therefore it follows from Lemma 3.1(2) that

$$\left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{p'}(S \times \Omega; Y^*)} \leq C_2 \left\| \sum_{n=1}^N r_n y_n^* \right\|_{L^{p'}(\Omega; Y^*)} \leq C_2.$$

We may conclude that

$$\mathbb{E} \left\langle \sum_{n=1}^N r_n T_{f_n} x_n, \sum_{n=1}^N r_n y_n^* \right\rangle \leq C_1 C_2 \|T\|_{L^r(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^{q_0}(\Omega; X)}.$$

By assumption Y has non-trivial type, hence $\text{Rad}_N(Y)^* = \text{Rad}_N(Y^*)$ isomorphically (see (2.2)). Taking the supremum over all $y_1^*, \dots, y_N^* \in Y^*$ as above, we obtain that

$$\left\| \sum_{n=1}^N r_n T_{f_n} x_n \right\|_{L^p(\Omega; Y)} \lesssim \|T\|_{L^r(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^q(\Omega; X)}.$$

The result now follows from (2.1).

If $p_0 > 1$ and $q_0 = \infty$ one can easily adjust the above argument to obtain the result. In particular, $g_n = 1$ for $n = 1, \dots, N$ in this case.

If $p_0 = 1$ and $q_0 < \infty$, then the duality argument does not hold since Y only has the trivial type 1. However, one can argue more directly in this case. Now $r' > q_0$.

By the triangle inequality, Hölder's inequality and Lemma 3.1, we obtain that

$$\begin{aligned} \left\| \sum_{n=1}^N r_n T_{f_n} x_n \right\|_{L^2(\Omega; Y)} &\leq \int_S \left\| T \sum_{n=1}^N r_n f_n x_n \right\|_{L^2(\Omega; Y)} d\mu(s) \\ &\leq \|T\|_{L^r(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^{r'}(S; L^2(\Omega; Y))} \\ &\leq C \|T\|_{L^r(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(\Omega; Y)}. \end{aligned}$$

(2): The case $p_0 = 1$ and $q_0 = \infty$ follows from (4.1). The cases ($p_0 = 2$ and $q_0 = \infty$) and ($p_0 = q_0 = 2$) follow from Lemma 3.1 in the same way as in as (1).

If $p_0 = 1$ and $q_0 = 2$ then $r = 2$ and by the Cauchy-Schwartz inequality and Lemma 3.1 we obtain that

$$\begin{aligned} \left\| \sum_{n=1}^N r_n T_{f_n} x_n \right\|_{L^2(\Omega; Y)} &\leq \int_S \left\| T(s) \sum_{n=1}^N r_n f_n(s) x_n \right\|_{L^2(\Omega; X)} d\mu(s) \\ &\leq \|T\|_{L^2(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n f_n x_n \right\|_{L^2(S \times \Omega; X)} \\ &\leq C \|T\|_{L^2(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n x_n \right\|_{L^2(S \times \Omega; X)}. \end{aligned}$$

□

With a certain price to pay, it is possible to relax the norm integrability condition of Proposition 4.1 to the uniform L^r -integrability of the orbits $s \mapsto T(s)x$. This is reached at the expense of not being able to exploit the information about the cotype of X but only the type of Y , as shown in the following remark. In the example further below, it is shown that in general the L^r -integrability of the orbits is not sufficient for the full conclusion of Proposition 4.1.

Remark 4.4. Let X and Y be Banach spaces and let (S, Σ, μ) be a σ -finite measure space. Let $p_0 \in [1, 2]$. Assume that Y has type p_0 . The following assertions hold:

(1) Assume $p_0 \in (1, 2)$. If $r \in (1, p_0)$ and $T : S \rightarrow \mathcal{B}(X, Y)$ is such that

$$(4.3) \quad \|Tx\|_{L^r(S; Y)} \leq C_T \|x\|, \quad x \in X.$$

Then there exists a constant $C = C(r, p_0, Y)$ such that

$$(4.4) \quad R(\{T_f \in \mathcal{B}(X, Y) : \|f\|_{L^{r'}(S)} \leq 1\}) \leq CC_T.$$

(2) Assume that $p_0 = 1$ or $p_0 = 2$ and that (4.3) holds for $r = p_0$. Then there exists a constant $C = C(Y)$ such that (4.4) holds.

If Y is as in Remark 4.3, then (4.4) holds for $r = p_0$.

Proof. (1): One can argue as in the proof of Proposition 4.1 with $p = r$, $g_n = 1$ and $h_n = f_n$. Indeed, we have

$$\mathbb{E} \left\langle \sum_{n=1}^N r_n T_{f_n} x_n, \sum_{n=1}^N r_n y_n^* \right\rangle \leq \left\| T \sum_{n=1}^N r_n x_n \right\|_{L^r(S \times \Omega; Y)} \left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{r'}(S \times \Omega; Y^*)}.$$

By the assumption 4.3 one can estimate

$$\left\| T \sum_{n=1}^N r_n x_n \right\|_{L^p(S \times \Omega; Y)} \leq C_T \left\| \sum_{n=1}^N r_n x_n \right\|_{L^p(S \times \Omega; X)}.$$

The term $\left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{r'}(S \times \Omega; Y^*)}$ can be treated in the same way as in the proof of Proposition 4.1.

(2): The case $p_0 = 1$ follows from (4.1). The case $p_0 = 2$ can be proved as above. \square

In the next example, we will show that even if X has cotype q for some $q \in (2, \infty)$ the result in Remark 4.4 cannot be improved.

Example 4.5. Consider the spaces $X = \ell^q$, $q \in (2, \infty)$, and $Y = \mathbb{R}$, so that X has cotype q and Y has type 2. Let $S = \mathbb{Z}_+$ with the counting measure, and define the $\mathcal{B}(X, Y)$ -valued function T on S by $T(s)x := t(s)x(s)$ for some $t : S \rightarrow \mathbb{R}$. Then we can make the following observations:

$T \in L^r(S; \mathcal{B}(X, Y))$ if and only if $t \in L^r(S) = \ell^r$. The condition (4.3) means $\|tx\|_r \lesssim \|x\|_q$ (where tx is the pointwise product), which holds if and only if $t \in \ell^u$, $1/u = (1/r - 1/q) \vee 0$. Under this condition, the operators $x \mapsto \int_S T(s)f(s)x ds$ are well-defined and bounded from X to Y for $f \in L^{r'}(S) = \ell^{r'}$.

Let us then derive a necessary condition for the R -boundedness of the mentioned operators. Let $x_i \in X = \ell^q$ be $\alpha(i)e_i$, where e_i is the i th standard unit-vector and $\alpha(i) \in \mathbb{R}$. Let $f_i = e_i$. The defining inequality of R -boundedness for these functions reduces to

$$\|t\alpha\|_2 \approx \mathbb{E} \left\| \sum_{i=1}^N r_i t(i) \alpha(i) \right\| \lesssim \mathbb{E} \left\| \sum_{i=1}^N r_i \alpha(i) e_i \right\|_q \approx \|\alpha\|_q.$$

This holds if and only if $t \in \ell^v$, $1/v = 1/2 - 1/q$, which is stronger than (4.3) unless $r \leq 2$. Conversely, the condition (4.3) suffices for R -boundedness in this range, as shown in Remark 4.4.

As a final result in this section we will show how one can use Lemma 3.1 to obtain a version of Proposition 4.1 with sharp exponents. It may seem artificial at first sight, but it enables us to obtain a sharp version of Theorem 5.1 below. For $f \in L^1(S) + L^\infty(S)$ let

$$(4.5) \quad L_f(S) = \{g \in L^1(S) + L^\infty(S) : \mu(|g| > t) = \mu(|f| > t) \text{ for all } t > 0\}$$

Proposition 4.6. *Let X and Y be Banach spaces and let (S, Σ, μ) be a σ -finite measure space. Let $p \in [1, 2]$ and $q \in [2, \infty]$. Assume that X has cotype q and Y has type p . If $r \in [1, \infty]$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then there exists a constant $C = C(p, q, X, Y)$ such that for all $f_0 \in L^{r', 1}(S)$ and $T \in L^r(S; \mathcal{B}(X, Y))$,*

$$(4.6) \quad R(\{T_f \in \mathcal{B}(X, Y) : f \in L_{f_0}(S)\}) \leq C \|T\|_{L^r(S; \mathcal{B}(X, Y))} \|f_0\|_{L^{r', 1}(S)}.$$

Since each $f \in L_{f_0}(S)$ is also in $L^{r'}(S)$, $T_f \in \mathcal{B}(X, Y)$ is well-defined. Note that Remark 4.2 applies also here. In the limit cases $p \in \{1, 2\}$ and $q \in \{2, \infty\}$, Proposition 4.1 (2) yields a stronger result.

Proof. Without loss of generality we may assume that $\|f_0\|_{L^{r', 1}(S)} = 1$. Let $(f_n)_{n=1}^N$ in $L_{f_0}(S)$ and $x_1, \dots, x_N \in X$.

First assume $p > 1$ and $q < \infty$. Let $g_n = |f_n|^{r'/q}$ and $h_n = \text{sign}(f_n)|f_n|^{r'/p'}$ for $n = 1, \dots, N$. Then the $|g_n|$ have the same distribution as $|f_0|^{r'/q}$ and the $|h_n|$ have the same distribution as $|f_0|^{r'/p'}$, and $f_n = g_n h_n$ for all n . Let $(y_n^*)_{n=1}^N$ in Y^* be such that $\left\| \sum_{n=1}^N r_n y_n^* \right\|_{L^{p'}(\Omega; Y^*)} \leq 1$. Then it follows from Hölder's inequality and $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ that

$$\begin{aligned} \mathbb{E} \left\langle \sum_{n=1}^N r_n T_{f_n} x_n, \sum_{n=1}^N r_n y_n^* \right\rangle &= \sum_{n=1}^N \langle T_{f_n} x_n, y_n^* \rangle \\ &= \int_S \sum_{n=1}^N \langle g_n(s) T(s) x_n, h_n(s) y_n^* \rangle d\mu(s) \\ &= \int_S \mathbb{E} \left\langle T(s) \sum_{n=1}^N r_n g_n(s) x_n, \sum_{n=1}^N r_n h_n(s) y_n^* \right\rangle d\mu(s) \\ &\leq \left\| T \sum_{n=1}^N r_n g_n x_n \right\|_{L^p(S \times \Omega; Y)} \left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{p'}(S \times \Omega; Y^*)} \\ &\leq \|T\|_{L^r(S; \mathcal{B}(X, Y))} \left\| \sum_{n=1}^N r_n g_n x_n \right\|_{L^q(S \times \Omega; X)} \left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{p'}(S \times \Omega; Y^*)}. \end{aligned}$$

Since X has cotype q it follows from Lemma 3.1 (1) that

$$\left\| \sum_{n=1}^N r_n g_n x_n \right\|_{L^q(S \times \Omega; X)} \leq C_1 \int_0^\infty \mu(|f_0| > t^{q/r'})^{1/q} dt \left\| \sum_{n=1}^N r_n x_n \right\|_{L^q(\Omega; X)}.$$

It follows from (2.3) and $L^{r',1} \hookrightarrow L^{r', \frac{r'}{q}}$ that

$$\begin{aligned} \int_0^\infty \mu(|f_0| > t^{q/r'})^{1/q} dt &= \frac{r'}{q} \int_0^\infty \mu(|f_0| > t)^{1/q} t^{r'/q} \frac{dt}{t} \\ &= \frac{r'}{q} (r')^{-r'/q} \|f_0\|_{L^{r', \frac{r'}{q}}(S)}^{r'/q} \leq C_{q,r}. \end{aligned}$$

Since Y has type p it follows that Y^* has cotype p' (cf. [10, Proposition 11.10]) and therefore it follows from Lemma 3.1 (1) that

$$\left\| \sum_{n=1}^N r_n h_n y_n^* \right\|_{L^{p'}(S \times \Omega; Y^*)} \leq C_2 \int_0^\infty \mu(|f_0| > t^{p'/r'})^{1/q} dt \left\| \sum_{n=1}^N r_n y_n^* \right\|_{L^{p'}(\Omega; Y^*)}.$$

As before it holds that

$$\int_0^\infty \mu(|f_0| > t^{p'/r'})^{1/q} dt \leq C_{p,r}$$

The result now follows with the same duality argument for $\text{Rad}_N(Y)$ as in Proposition 4.1.

If $p > 1$ and $q = \infty$ one can easily adjust the above argument to obtain the result. In particular, $g_n = 1$ for $n = 1, \dots, N$ in this case. If $p = 1$ and $q < \infty$, then one can argue as in Proposition 4.1, but instead of Lemma 3.1 (2) one has to apply Lemma 3.1 (1). If $p = 1$ and $q = \infty$ the result follows from Proposition 4.1. \square

Similar as in Remark 4.4 the following strong version of Proposition 4.6 holds.

Remark 4.7. Let X and Y be Banach spaces and let (S, Σ, μ) be a σ -finite measure space. Let $p \in (1, 2]$. Assume that Y has type p . If $f_0 \in L^{p',1}(S)$ and $T : S \rightarrow \mathcal{B}(X, Y)$ is such that (4.3) holds, then there exists a constant $C = C(p, Y)$ such that

$$(4.7) \quad R(\{T_f \in \mathcal{B}(X, Y) : f \in L_{f_0}(S)\}) \leq CC_T \|f_0\|_{L^{p',1}(S)}.$$

Proof. Similar as in Remark 4.4 one can argue as in the proof of Proposition 4.6 with $p = r$, $g_n = 1$ and $h_n = f_n$. □

5. BESOV SPACES AND R -BOUNDEDNESS

Recall from [40] that for an interval $I = (a, b)$ and $T \in W^{1,1}(I; \mathcal{B}(X, Y))$,

$$(5.1) \quad R(T(t) \in \mathcal{B}(X, Y) : t \in (a, b)) \leq \|T(a)\| + \|T'\|_{L^1(I; \mathcal{B}(X, Y))}$$

In [13, Theorem 5.1] this result has been improved under the assumption that Y has Fourier type p . In the next result we obtain R -boundedness for the range of smooth operator-valued functions under (co)type assumptions on the Banach spaces X and Y . The result below improves [13, Theorem 5.1].

Theorem 5.1. *Let X and Y be Banach spaces. Let $p \in [1, 2]$ and $q \in [2, \infty]$. Assume that X has cotype q and Y has type p . If $r \in [1, \infty]$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, then there exists a constant $C = C(p, q, X, Y)$ such that for all $T \in B_{r,1}^{\frac{d}{r}}(\mathbb{R}^d; \mathcal{B}(X, Y))$,*

$$(5.2) \quad R(\{T(t) \in \mathcal{B}(X, Y) : t \in \mathbb{R}^d\}) \leq C \|T\|_{B_{r,1}^{\frac{d}{r}}(\mathbb{R}^d; \mathcal{B}(X, Y))}.$$

Note that $B_{r,1}^{\frac{d}{r}}(\mathbb{R}^d; \mathcal{B}(X, Y)) \hookrightarrow BUC(\mathbb{R}^d; \mathcal{B}(X, Y))$ (the space of bounded, uniformly continuous functions) for all $r \in [1, \infty]$ (cf. [38, Theorem 2.8.1(c)]).

If $p = q = 2$ in (2), then the uniform boundedness of $\{T(t) : t \in \mathbb{R}^d\} \subset \mathcal{B}(X, Y)$ already implies R -boundedness (see [3, Proposition 1.13]).

Proof. (1): Let $Z = \mathcal{B}(X, Y)$. We may write $T = \sum_{k \geq 0} T_k = \sum_{k \geq 0} \sum_{n \geq 0} \varphi_n * T_k$, where $T_k = \varphi_k * T$ and the series converges in Z uniformly in \mathbb{R}^d . Let $\varphi_{-1} = 0$. By [40, Lemma 2.4] we obtain that

$$\begin{aligned} R(T(t) \in Z : t \in \mathbb{R}^d) &\leq \sum_{k \geq 0} \sum_{n \geq 0} R(\varphi_n * T_k(t) \in Z : t \in \mathbb{R}^d) \\ &= \sum_{k \geq 0} \sum_{n=k-1}^{k+1} R(\varphi_n * T_k(t) \in Z : t \in \mathbb{R}^d). \end{aligned}$$

Fix $n \in \{0, 1, 2, \dots\}$ and $t \in \mathbb{R}^d$ and define $\varphi_{n,t} \in \mathcal{S}(\mathbb{R}^d)$ by $\varphi_{n,t}(s) = \varphi_n(t - s)$. Then for all $t \in \mathbb{R}^d$, $\varphi_{n,t} \in L_{\varphi_n}(\mathbb{R}^d)$, where $L_{\varphi_n}(\mathbb{R}^d)$ is as in (4.5) and $\|\varphi_n\|_{L^{r',1}(\mathbb{R}^d)} \leq c 2^{\frac{dn}{r}}$. Indeed, it is elementary to check that $\varphi_n^*(t) = 2^{dn} \varphi^*(2^{nd}t)$. Therefore,

$$\|\varphi_n\|_{L^{r',1}(\mathbb{R}^d)} = 2^{dn} \int_0^\infty t^{\frac{1}{r'}} \varphi^*(2^{nd}t) \frac{dt}{t} = 2^{\frac{dn}{r}} \int_0^\infty t^{\frac{1}{r'}} \varphi^*(t) \frac{dt}{t} = 2^{\frac{dn}{r}} \|\varphi\|_{L^{r',1}(\mathbb{R}^d)}$$

Letting $T_{k,\varphi_{n,t}} \in Z$ be the integral operator from Propositions 4.1 and 4.6, it follows that for all $k \geq 0$,

$$R(\varphi_n * T_k(t) \in Z : t \in \mathbb{R}^d) = R(T_{k,\varphi_{n,t}} \in Z : t \in \mathbb{R}^d) \leq C_1 2^{\frac{dn}{r}} \|T_k\|_{L^r(\mathbb{R}^d; Z)}.$$

We may conclude that

$$\begin{aligned} R(T(t) \in Z : t \in \mathbb{R}^d) &\leq C_1 \sum_{k \geq 0} \sum_{n=k-1}^{k+1} 2^{\frac{dn}{r}} \|T_k\|_{L^r(\mathbb{R}^d; Z)} \\ &\leq C_2 \sum_{k \geq 0} 2^{\frac{dk}{r}} \|T_k\|_{L^r(\mathbb{R}^d; Z)} = C_2 \|T\|_{B_{r,1}^{\frac{d}{r}}(\mathbb{R}^d; Z)}. \end{aligned}$$

□

In [13, Remark 5.2] the following result is presented for operator families which are in a Besov space in the strong sense. If Y has Fourier type p and $T : \mathbb{R}^d \rightarrow \mathcal{B}(X, Y)$ is such that for all $x \in X$,

$$\|Tx\|_{B_{p,1}^{d/p}(\mathbb{R}^d; Y)} \leq C_T \|x\|,$$

then $\{T(t) : t \in \mathbb{R}^d\}$ is R -bounded. We will obtain the same conclusion assuming only type p . Notice that many Banach spaces have type 2, whereas all Banach spaces with Fourier 2 are isomorphic to a Hilbert space.

We first need an analogue of Proposition 4.1 involving the space $\gamma(H, Y)$ of γ -radonifying operators from H to Y . See [29] for information on this space. We note that a version of the following Lemma is also in [15, Proposition 3.19], where it is instead assumed that Y has so-called property (α) . Moreover, in [15, Remark 5.3 and Proof of Proposition 3.19] it is claimed that this assumption can be relaxed to non-trivial cotype, which is weaker than our assumption below. However, there seems to be a small confusion there: in [15, Remark 5.3] it is observed that non-trivial cotype suffices, thanks to a result in [17], if (a certain Hilbert space) $H_1 = \mathbb{C}$, whereas in [15, Proposition 3.19] and the following lemma one has the dual situation: $H_1 = H$ is a general Hilbert space and $H_2 = \mathbb{C}$. Indeed one could deduce the following lemma by a standard duality argument from the result in [17], but since a self-contained argument is only slightly longer, we provide it for completeness:

Lemma 5.2. *Let Y be a Banach space with non-trivial type and let H be a Hilbert space. Then there exists a constant C such that for all $\Psi_n \in \gamma(H, Y)$ and $f_n \in H$,*

$$\left\| \sum_{n=1}^N r_n \Psi_n f_n \right\|_{L^2(\Omega; Y)} \leq C \sup_{1 \leq n \leq N} \|f_n\|_H \left\| \sum_{n=1}^N r_n \Psi_n \right\|_{L^2(\Omega; \gamma(H, Y))}$$

Proof. Let $(h_k)_{k=1}^K$ be an orthonormal basis for the span of $(f_n)_{n=1}^N$ in H , so that $f_n = \sum_{k=1}^K (h_k, f_n) h_k$. Let further $(r_n)_{n=1}^N$ be a Rademacher sequence on a probability space Ω , and $(\gamma_k)_{k=1}^K$ a Gaussian sequence on Ω' . Then

$$\begin{aligned} & \left| \sum_{n=1}^N \langle \Psi_n f_n, y_n^* \rangle \right| = \left| \sum_{n=1}^N \sum_{k=1}^K \langle \Psi_n h_k, (h_k, f_n) y_n^* \rangle \right| \\ & = \left| \mathbb{E} \mathbb{E}' \left\langle \sum_{k=1}^K \gamma_k \left(\sum_{n=1}^N r_n \Psi_n \right) h_k, \sum_{m=1}^N r_m \sum_{\ell=1}^K \gamma_\ell (h_\ell, f_m) y_m^* \right\rangle \right| \\ & \leq \left\| \sum_{k=1}^K \gamma_k \left(\sum_{n=1}^N r_n \Psi_n \right) h_k \right\|_{L^2(\Omega; L^2(\Omega'; Y))} \\ & \quad \times \left\| \sum_{m=1}^N r_m \sum_{\ell=1}^K \gamma_\ell (h_\ell, f_m) y_m^* \right\|_{L^q(\Omega'; L^2(\Omega; Y^*))}, \quad q \in [2, \infty). \end{aligned}$$

The first factor is bounded by

$$\left\| \sum_{n=1}^N r_n \Psi_n \right\|_{L^2(\Omega; \gamma(H, Y))}$$

by the definition of the norm in $\gamma(H, Y)$. As for the second, the non-trivial type of Y implies some non-trivial cotype $q_0 \in [2, \infty)$ for Y^* , and then, taking $q \in (q_0, \infty)$ and applying Lemma 3.1(2),

$$\begin{aligned} & \left\| \sum_{m=1}^N r_m \sum_{\ell=1}^K \gamma_\ell (h_\ell, f_m) y_m^* \right\|_{L^q(\Omega'; L^2(\Omega; Y^*))} \\ & \leq C \sup_{1 \leq m \leq N} \left\| \sum_{\ell=1}^K \gamma_\ell (h_\ell, f_m) \right\|_{L^q(\Omega')} \left\| \sum_{m=1}^N r_m y_m^* \right\|_{L^2(\Omega; Y^*)}. \end{aligned}$$

Here $\sum_{\ell=1}^K \gamma_\ell (h_\ell, f_m)$ is a centered Gaussian variable with variance $\sum_{\ell=1}^K (h_\ell, f_m)^2 = \|f_m\|_H^2$, hence its L^q norm is $c_q \|f_m\|_H$ for a constant c_q .

The assertion follows by taking the supremum over all $\sum_{m=1}^N r_m y_m^* \in \text{Rad}_N(Y^*)$ of norm 1, using the non-trivial type of Y . \square

Proposition 5.3. *Let X and Y be Banach spaces. Let $p \in [1, 2]$, and assume that Y has type p . If $T : \mathbb{R}^d \rightarrow \mathcal{B}(X, Y)$ satisfies*

$$(5.3) \quad \|Tx\|_{B_{p,1}^{d/p}(\mathbb{R}^d; Y)} \leq C_T \|x\|, \quad x \in X,$$

then there exists a constant $C = C(p, Y)$ such that

$$(5.4) \quad R(\{T(t) \in \mathcal{B}(X, Y) : t \in \mathbb{R}^d\}) \leq CC_T.$$

Proof. If $p = 1$, the result follows from [13, Remark 5.2]. Let then $p \in (1, 2]$.

Fix for the moment $x \in X$ and $k \geq 0$. Let $f_k : \mathbb{R}^d \rightarrow Y$ be defined as $f_k(t) = T_k(t)x = \varphi_k * T(t)x$. Then by [19, Theorem 1.1],

$$(5.5) \quad \|f_k\|_{\gamma(L^2(\mathbb{R}^d), Y)} \leq C \|f_k\|_{B_{p,p}^{d(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^d; Y)} \leq C 2^{dk(\frac{1}{p}-\frac{1}{2})} \|f_k\|_{L^p(\mathbb{R}^d; Y)}.$$

Choose $(t_m)_{m=1}^M$ in \mathbb{R}^d and $(x_m)_{m=1}^M$ in X arbitrarily. Since Y has type $p > 1$ it follows from Lemma 5.2 that

$$\begin{aligned}
& \left\| \sum_{m=1}^M r_m T(t_m) x_m \right\|_{L^2(\Omega; Y)} \leq \sum_{k \geq 0} \sum_{n=k-1}^{k+1} \left\| \sum_{m=1}^M r_m \varphi_n * T_k(t_m) x_m \right\|_{L^2(\Omega; Y)} \\
& = \sum_{k \geq 0} \sum_{n=k-1}^{k+1} \left\| \sum_{m=1}^M r_m \int_{\mathbb{R}^d} T_k(u) x_m \varphi_n(t_m - u) du \right\|_{L^2(\Omega; Y)} \\
& \lesssim \sum_{k \geq 0} \sum_{n=k-1}^{k+1} \sup_{1 \leq m \leq M} \|\varphi_n(t_m - \cdot)\|_{L^2(\mathbb{R}^d)} \left\| \sum_{m=1}^M r_m T_k x_m \right\|_{L^2(\Omega; \gamma(L^2(\mathbb{R}^d), Y))} \\
& \approx \sum_{k \geq 0} 2^{kd/2} \left\| T_k \left(\sum_{m=1}^M r_m x_m \right) \right\|_{L^2(\Omega; \gamma(L^2(\mathbb{R}^d), Y))}.
\end{aligned}$$

Applying (5.5) pointwise in Ω yields that

$$(5.6) \quad \left\| T_k \left(\sum_{m=1}^M r_m x_m \right) \right\|_{\gamma(L^2(\mathbb{R}^d), Y)} \lesssim 2^{kd(\frac{1}{p} - \frac{1}{2})} \left\| T_k \left(\sum_{m=1}^M r_m x_m \right) \right\|_{L^p(\mathbb{R}^d; Y)}.$$

Therefore, we obtain from (5.6) and (2.1) that

$$\begin{aligned}
& \sum_{k \geq 0} 2^{kd/2} \left\| T_k \left(\sum_{m=1}^M r_m x_m \right) \right\|_{L^2(\Omega; \gamma(L^2(\mathbb{R}^d), Y))} \\
& \lesssim \sum_{k \geq 0} 2^{kd/2} 2^{kd(\frac{1}{p} - \frac{1}{2})} \left\| T_k \left(\sum_{m=1}^M r_m x_m \right) \right\|_{L^2(\Omega; L^p(\mathbb{R}^d; Y))} \\
& \approx \int_{\Omega} \sum_{k \geq 0} 2^{kd/p} \left\| T_k \left(\sum_{m=1}^M r_m x_m \right) \right\|_{L^p(\mathbb{R}^d; Y)} d\mathbb{P} \\
& = \int_{\Omega} \left\| T \left(\sum_{m=1}^M r_m x_m \right) \right\|_{B_{p,1}^{d/p}(\mathbb{R}^d; Y)} d\mathbb{P} \\
& \leq \int_{\Omega} C_T \left\| \sum_{m=1}^M r_m x_m \right\|_X d\mathbb{P} \leq C_T \left\| \sum_{m=1}^M r_m x_m \right\|_{L^2(\Omega; X)}.
\end{aligned}$$

Putting things together yields the required R -boundedness estimate. \square

As a consequence of Theorem 5.1 we have the following two results. One can similarly derive strong type results from Proposition 5.3.

Corollary 5.4. *Let X and Y be Banach spaces. Let $p \in [1, 2]$ and $q \in [2, \infty]$. Assume that X has cotype q and Y has type p . Let $r \in [1, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. If there exists an M such that*

$$(5.7) \quad \left(\int_{\mathbb{R}^d} \|D^\alpha T\|_{\mathcal{B}(X, Y)}^r \right)^{\frac{1}{r}} \leq M$$

for every $\alpha \in \{0, 1, \dots, d\}^d$ with $|\alpha| \leq \lfloor \frac{d}{r} \rfloor + 1$, then $\{T(t) \in \mathcal{B}(X, Y) : t \in \mathbb{R}^d\}$ is R -bounded.

Corollary 5.5. *Let X and Y be Banach spaces. Let $p \in [1, 2]$ and $q \in [2, \infty]$. Assume that X has cotype q and let Y have type p . Let $I = (a, b)$ with $-\infty \leq a < b \leq \infty$. Let $r \in (1, \infty]$ be such that $\frac{1}{r} \geq \frac{1}{p} - \frac{1}{q}$. Let $\alpha \in (\frac{1}{r}, 1)$. If $T \in L^r(\mathbb{R}; \mathcal{B}(X, Y))$ and there exists an A such that*

$$(5.8) \quad \|T(s+h) - T(s)\| \leq A|h|^\alpha(1+|s|)^{-\alpha}, \quad s, s+h \in I, \quad h \in I,$$

then $\{T(t) \in \mathcal{B}(X, Y) : t \in I\}$ is R -bounded by a constant times A .

Note that in the case that I is bounded, the factor $(1+|s|)^{-\alpha}$ can be omitted.

Proof. By taking a worse p or q it suffices to consider the case that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. First consider the case that $I = \mathbb{R}$. As in [13, Corollary 5.4] one may check that $T \in \Lambda_{r,1}^{\frac{1}{r}}(\mathbb{R}; \mathcal{B}(X, Y))$, where the latter is defined in Section 2.5, and therefore the result follows from Theorem 5.1.

If $I \neq \mathbb{R}$, then one can reduce to the above case by (2.5). \square

6. APPLICATIONS

6.1. R -boundedness of semigroups. In the next result we will give a sufficient condition for R -boundedness of strongly continuous semigroups restricted to fractional domain spaces.

Theorem 6.1. *Let $(T(t))_{t \in \mathbb{R}_+}$ be a strongly continuous semigroup on a Banach space X with $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$. Assume X has type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $\alpha > \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and let $i_\alpha : D((-A)^\alpha) \rightarrow X$ be the inclusion mapping. Then*

$$\{T(t)i_\alpha : t \in \mathbb{R}_+\} \subset \mathcal{B}(D((-A)^\alpha), X)$$

is R -bounded.

Proof. For $\theta \in (0, 1)$ let $X_\theta = (X, D(A))_{\theta, \infty}$. Then $x \in X_\theta$ if and only if

$$\|x\|_{X_\theta} := \|x\| + \sup_{t \in \mathbb{R}_+} t^{-\theta} \|T(t)x - x\|$$

is finite, and this expression defines an equivalent norm on X_θ (cf. [25, Proposition 3.2.1]). If we fix $\theta \in (\frac{1}{r}, \alpha)$, then we obtain that

$$\begin{aligned} \sup_{t \in \mathbb{R}_+} t^{-\alpha} \|T(t)i_\alpha - i_\alpha\|_{\mathcal{B}(D((-A)^\alpha), X)} &= \sup_{\|x\|_{D((-A)^\alpha)} \leq 1} \sup_{t \in \mathbb{R}_+} t^{-\alpha} \|T(t)x - x\|_X \\ &\leq \sup_{\|x\|_{D((-A)^\alpha)} \leq 1} \|x\|_{X_\theta} \\ &\lesssim \sup_{\|x\|_{D((-A)^\alpha)} \leq 1} \|x\|_{D((-A)^\alpha)} = 1. \end{aligned}$$

Therefore,

$$\|T(s+h)i_\alpha - T(s)i_\alpha\|_{\mathcal{B}(D((-A)^\alpha), X)} \lesssim Me^{-\omega s} h^\alpha$$

and the result follows from Corollary 5.5. \square

The result in Theorem 6.1 is quite sharp as follows from the next example. An application of Theorem 6.1 will be given in Theorem 6.3.

For $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$, let $H^{\alpha,p}(\mathbb{R})$ be the Bessel-potential spaces (cf. [38, 2.3.3]).

Example 6.2. Let $p \in [1, \infty)$. Let $(T(t))_{t \in \mathbb{R}}$ be the left-translation group on $X = L^p(\mathbb{R})$ with generator $A = \frac{d}{dx}$. Then for all $\alpha \in (|\frac{1}{p} - \frac{1}{2}|, 1)$ and $M \in \mathbb{R}_+$,

$$(6.1) \quad \{T(t)i_\alpha : t \in [-M, M]\} \subset \mathcal{B}(H^{\alpha,p}(\mathbb{R}), L^p(\mathbb{R})),$$

is R -bounded, where $i_\alpha : H^{\alpha,p}(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ denotes the embedding.

On the other hand, for $\alpha \in (0, |\frac{1}{p} - \frac{1}{2}|)$ and $M = 1$, the family (6.1) is not R -bounded.

Proof. Note that $L^p(\mathbb{R})$ and $H^{\alpha,p}$ have type $p \wedge 2$ and cotype $p \vee 2$. Therefore, for $\alpha > |\frac{1}{p} - \frac{1}{2}|$ the R -boundedness of

$$\{e^{-t}T(t)i_\alpha : t \in \mathbb{R}_+\} \subset \mathcal{B}(H^{\alpha,p}(\mathbb{R}), L^p(\mathbb{R}))$$

follows from Theorem 6.1. Therefore, we obtain from the Kahane-contraction principle that

$$\{T(t)i_\alpha : t \in [0, M]\} \subset \mathcal{B}(H^{\alpha,p}(\mathbb{R}), L^p(\mathbb{R}))$$

is R -bounded. Since a similar argument works for $T(-t)$, the R -boundedness of (6.1) follows from the fact that the union of two R -bounded sets is again R -bounded.

For the converse, let $\psi \in C^\infty(\mathbb{R}) \setminus \{0\}$ be such that $\text{supp}(\psi) \subset (0, 1)$. For $c \in (0, \infty)$ let $\psi_c(t) = \psi(ct)$. Then $(-A)^\alpha \psi_c = c^\alpha [(-A)^\alpha \psi]_c$. Fix an integer N and let $f_n = f_0 := \psi_N$ for all n . Then f_0 has support in $(0, 1/N)$ and $\|f_0\|_{L^p(\mathbb{R})}^p = N^{-1} \|\psi\|_{L^p(\mathbb{R})}^p$.

There holds, on the one hand,

$$\begin{aligned} \left\| \sum_{n=1}^N r_n T(n/N) i_\alpha f_n \right\|_{L^2(\Omega; X)}^p &= \left\| \sum_{n=1}^N r_n f_0(\cdot + n/N) \right\|_{L^2(\Omega; X)}^p \\ &= \sum_{n=1}^N \|f_0(\cdot + n/N)\|_{L^p(\mathbb{R})}^p = N \|f_0\|_{L^p(\mathbb{R})}^p = \|\psi\|_{L^p(\mathbb{R})}^p, \end{aligned}$$

and on the other hand,

$$\begin{aligned} \left\| \sum_{n=1}^N r_n f_n \right\|_{L^2(\Omega; D((-A)^\alpha))}^p &= N^{\frac{p}{2}} \|f_0\|_{D((-A)^\alpha)}^p = N^{\frac{p}{2}} (\|f_0\|_{L^p(\mathbb{R})}^p + \|(-A)^\alpha f_0\|_{L^p(\mathbb{R})}^p) \\ &= N^{\frac{p}{2}} (\|\psi_N\|_{L^p(\mathbb{R})}^p + \|N^\alpha [(-A)^\alpha \psi]_N\|_{L^p(\mathbb{R})}^p) \\ &= N^{-1 + \frac{p}{2}} (\|\psi\|_{L^p(\mathbb{R})}^p + N^{\alpha p} \|(-A)^\alpha \psi\|_{L^p(\mathbb{R})}^p). \end{aligned}$$

Therefore, if $\tau := \{T(t)i_\alpha : t \in [-1, 1]\}$ is R -bounded, then it follows that there exists a constant C such that

$$1 \leq CN^{-\frac{1}{p} + \frac{1}{2} + \alpha}.$$

Letting N tend to infinity, this implies that $\alpha \geq \frac{1}{p} - \frac{1}{2}$, i.e., τ can only be R -bounded in this range.

We still have to prove that the R -boundedness also implies $\alpha \geq \frac{1}{2} - \frac{1}{p}$. This can be proved by duality. If $\{T(t) \in \mathcal{B}(H^{\alpha,p}(\mathbb{R}), L^p(\mathbb{R})) : t \in [-1, 1]\}$ is R -bounded, then $\{T^*(t) \in \mathcal{B}(L^{p'}(\mathbb{R}), H^{-\alpha,p'}(\mathbb{R})) : t \in [-1, 1]\}$ is R -bounded as well. It follows that $\{(1-A)^{-\alpha} T^*(t) \in \mathcal{B}(L^{p'}(\mathbb{R}), L^{p'}(\mathbb{R})) : t \in [-1, 1]\}$ is R -bounded. This implies that $\{T^*(t) \in \mathcal{B}(H^{\alpha,p'}(\mathbb{R}), L^{p'}(\mathbb{R})) : t \in [-1, 1]\}$ is R -bounded. According to the first part of the proof this implies that $\alpha \geq \frac{1}{p'} - \frac{1}{2} = \frac{1}{2} - \frac{1}{p}$. \square

6.2. Stochastic Cauchy problems. We apply Theorem 6.1 to stochastic equations with additive Brownian noise. We refer the reader to [29] for details on stochastic Cauchy problems, stochastic integration and γ -radonifying operators. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let H be a separable Hilbert space and let W_H be a cylindrical Wiener process. Recall from [29] that for an operator-valued function $\Phi : [0, t] \rightarrow \mathcal{B}(H, E)$ which belongs to $\gamma(L^2(0, t; H), X)$ (the space of γ -radonifying operators from $L^2(0, t; H)$ to X) we have

$$\left\| \int_0^t \Phi(s) dW_H(s) \right\|_{L^2(\Omega; X)} = \|\Phi\|_{\gamma(L^2(0, t; H), X)}.$$

On a real Banach space X we consider the following equation.

$$(SE) \quad \begin{cases} dU(t) = AU(t) dt + B(t)dW_H(t), & t \in \mathbb{R}_+, \\ U(0) = x, \end{cases}$$

Here A is the generator of a strongly continuous semigroup $(T(t))_{t \in \mathbb{R}_+}$, $B : \mathbb{R}_+ \rightarrow \mathcal{B}(H, E)$ and $x \in X$. We say that a strongly measurable process $U : \mathbb{R}_+ \times \Omega \rightarrow X$ is a mild solution of (SE) if for all $t \in \mathbb{R}_+$, almost surely we have

$$U(t) = T(t)x + \int_0^t T(t-s)B(s) dW_H(s).$$

In general (SE) does not have a solution (cf. [29, Example 7.3]). In the case when $B(t) = B \in \gamma(H, X)$ is constant, there are some sufficient conditions for existence. Indeed, if X has type 2 or $(T(t))_{t \in \mathbb{R}_+}$ is an analytic semigroup, then (SE) always has a unique mild solution and it has a version with continuous paths (see [39, Corollary 3.4] and [9] respectively). In the next result we prove such an existence and regularity result under assumptions on the noise in terms of the type and cotype of X .

Theorem 6.3. *Assume X has type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $w \in \mathbb{R}$ be such that $\lim_{t \rightarrow \infty} e^{wt}T(t) = 0$. Let $\alpha > \frac{1}{p} - \frac{1}{q}$ and $B \in \gamma(L^2(\mathbb{R}_+; H), D((w-A)^\alpha))$. Then (SE) has a unique mild solution U . Moreover, if there exists an $\varepsilon > 0$ such that for all $M \in \mathbb{R}_+$,*

$$(6.2) \quad \sup_{t \in [0, M]} \|s \mapsto (t-s)^{-\varepsilon} B(s)\|_{\gamma(L^2(0, t; H), D((w-A)^\alpha))} < \infty$$

then U has a version with continuous paths.

In particular we note that if $B(t) = B \in \gamma(H, D((w-A)^\alpha))$ is constant then for all $\varepsilon \in (0, \frac{1}{2})$

$$\|s \mapsto (t-s)^{-\varepsilon} B(s)\|_{\gamma(L^2(0, t; H), D((w-A)^\alpha))} = (1-2\varepsilon)^{-1} t^{\frac{1}{2}-\varepsilon} \|B\|_{\gamma(H, D((w-A)^\alpha))}.$$

Remark 6.4. Here is a sufficient condition for (6.2): there is an $s \in (2, \infty)$ such that for all $M \in \mathbb{R}_+$,

$$B \in B_{s,p}^{\frac{1}{s}-\frac{1}{2}}(0, M; D((w-A)^\alpha)).$$

Indeed, it follows from [28, Lemma 3.3] that (6.2) holds for all $\varepsilon \in (0, \frac{1}{2} - \frac{1}{s})$.

Proof. Assume that (6.2) holds for some $\varepsilon \in [0, \frac{1}{2})$. In the case $\varepsilon = 0$ we will show existence of a solution, and in the other case we show that the solution has a version with continuous paths.

By Theorem 6.1, $\{e^{wt}T(t)i_\alpha \in \mathcal{B}(D((w-A)^\alpha), X) : t \geq 0\}$ is R -bounded. It follows that for fixed $M > 0$,

$$\{T(t)i_\alpha \in \mathcal{B}(D((w-A)^\alpha), X) : t \in [0, M]\}$$

is R -bounded by some constant C . Therefore, by [20] (see also [27, Theorem 9.14]), the function $s \mapsto T(s)i_\alpha$ acts as a multiplier between the spaces $\gamma(L^2(0, t; H), D((w-A)^\alpha))$ and $\gamma(L^2(0, t; H), X)$, and we conclude that

$$\begin{aligned} & \sup_{t \in [0, M]} \|s \mapsto (t-s)^{-\varepsilon} T(t-s)B(s)\|_{\gamma(L^2(0, t; H), X)} \\ & \leq C \sup_{t \in [0, M]} \|s \mapsto (t-s)^{-\varepsilon} B(s)\|_{\gamma(L^2(0, t; H), D((w-A)^\alpha))} < \infty. \end{aligned}$$

Now the result follows from [39, Proposition 3.1 and Theorem 3.3]. \square

6.3. R -boundedness of evolution families. In the next application we obtain R -boundedness of an evolution family generated by a family $(A(t))_{t \in [0, T]}$ of unbounded operators which satisfy the conditions (AT) of Acquistapace and Terreni (see [1]). For $\phi \in (0, \pi]$, we define the sector

$$\Sigma(\phi) := \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \phi\}.$$

The condition (AT) is said to be satisfied if the following two requirements hold:

(AT1) The $A(t)$ are linear operators on a Banach space E and there are constants $K \geq 0$, and $\phi \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(\phi) \subset \varrho(A(t))$ and for all $\lambda \in \Sigma(\phi)$ and $t \in [0, T]$,

$$\|R(\lambda, A(t))\| \leq \frac{K}{1 + |\lambda|}.$$

(AT2) There are constants $L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that for all $\lambda \in \Sigma(\phi, 0)$ and $s, t \in [0, T]$,

$$\|A(t)R(\lambda, A(t))(A(t)^{-1} - A(s)^{-1})\| \leq L|t-s|(|\lambda| + 1)^{-\nu}.$$

Under these assumptions there exists a unique strongly continuous evolution family $(P(t, s))_{0 \leq s \leq t \leq T}$ in $\mathcal{B}(X)$ such that $\frac{\partial P(t, s)}{\partial t} = A(t)P(t, s)$ for $0 \leq s < t \leq T$. Moreover, $\|A(t)P(t, s)\| \leq C(t-s)^{-1}$.

For analytic semigroup generators one has that for all $\varepsilon > 0$ and $T \in [0, \infty)$, $\{t^\varepsilon S(t) \in \mathcal{B}(X) : t \in [0, T]\}$ is R -bounded. This easily follows from (5.1). This may be generalized to evolution families $(P(t, s))_{0 \leq s \leq t \leq T}$, where $(A(t))_{t \in [0, T]}$ satisfies the (AT) conditions. Indeed, then by the same reasoning we obtain that for all $\alpha > 0$,

$$\sup_{s \in [0, T]} R(\{(t-s)^\alpha P(t, s) \in \mathcal{B}(X) : t \in [s, T]\}) < \infty.$$

This argument does not hold if one considers the R -bound with respect to $s \in [0, t]$ instead of $t \in [s, T]$. This is due to the fact that

$$(6.3) \quad \left\| \frac{\partial P(t, s)}{\partial s} \right\| \leq C(t-s)^{-1}$$

might not be true. The R -boundedness with respect to $s \in [0, t]$ has applications for instance in the study of non-autonomous stochastic Cauchy problems (see [39]). We also note that (6.3) does hold if $(A(t)^*)_{t \in [0, T]}$ satisfies the (AT)-conditions (see [2]).

Recall from [42, Theorem 2.3] that for all $\theta \in (0, \mu)$,

$$(6.4) \quad \|P(t, s)(-A(s))^\theta\| \leq C(t-s)^{-\theta}, \quad 0 \leq s < t \leq T.$$

Due to this inequality one might expect that under assumptions on μ , one can still obtain a fractional version of (6.3). This is indeed the case and in the next theorem we will give conditions under which the R -boundedness with respect to $s \in [0, t]$ holds.

The authors are grateful to Roland Schnaubelt for showing them the following result.

Proposition 6.5. *Assume (AT). Then for all $\theta \in (0, \mu)$ there exists a constant C such that for all $0 \leq s \leq t \leq T$,*

$$(6.5) \quad \|(-A(t))^{-\theta}(P(t, s) - I)\| \leq C(t - s)^\theta.$$

Proof. First let $\theta \in (1 - \nu, \mu)$. By [26, equation (A.5)] we can write

$$(-A(t))^{-\theta}(P(t, s) - I) = g(t, s) + \int_s^t (-A(t))^{-\theta} P(t, \tau) (-A(\tau))^\theta h(\tau, s) d\tau,$$

where

$$g(t, s) = (-A(t))^{-\theta}(e^{(t-s)A(s)} - I),$$

$$h(t, s) = (-A(t))^{1-\theta} [(-A(s))^{-1} - (-A(t))^{-1}] A(s) e^{(t-s)A(s)}.$$

We may write

$$\|g(t, s)\| \leq \|((-A(t))^{-\theta} - (-A(s))^{-\theta})(e^{(t-s)A(s)} - I)\|$$

$$+ \|(-A(s))^{-\theta}(e^{(t-s)A(s)} - I)\|.$$

By [37, equation (2.10)]

$$\|((-A(t))^{-\theta} - (-A(s))^{-\theta})(e^{(t-s)A(s)} - I)\| \lesssim |t - s|^\mu \lesssim (t - s)^\theta.$$

For the other term it is clear that

$$\|(-A(s))^{-\theta}(e^{(t-s)A(s)} - I)\| \leq \int_0^{t-s} \|(-A(s))^{1-\theta} e^{\tau A(s)}\| d\tau \lesssim (t - s)^\theta.$$

This shows that $\|g(t, s)\| \lesssim (t - s)^\theta$. By [41, equation (2.2)] we obtain that

$$\|h(t, s)\| \lesssim (t - s)^{\mu-1}.$$

Since by [26, Lemma A.1] $V(t, s) = (-A(t))^{-\theta} P(t, \tau) (-A(\tau))^\theta$ is uniformly bounded, it follows that

$$\int_s^t \|V(t, \tau) h(\tau, s)\| d\tau \lesssim (t - s)^\mu \lesssim (t - s)^\theta.$$

We may conclude (6.5) for the special choice of θ .

For general $\theta \in (0, \mu)$ choose $\varepsilon > 0$ so small that $\mu - \varepsilon > 1 - \nu$. Then by interpolation with $\theta/(\mu - \varepsilon)$ it follows that

$$\|(-A(t))^{-\theta}(P(t, s) - I)\|$$

$$\lesssim \|(-A(t))^{-(\mu-\varepsilon)}(P(t, s) - I)\|^{\theta/(\mu-\varepsilon)} \|P(t, s) - I\|^{1-\theta/(\mu-\varepsilon)} \lesssim (t - s)^\theta.$$

□

Theorem 6.6. *Let X be a Banach space with type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Assume (AT) with*

$$\mu > \frac{1}{p} - \frac{1}{q}.$$

Then for all $\varepsilon > 0$,

$$\sup_{t \in [0, T]} R(\{(t-s)^\varepsilon P(t, s) \in \mathcal{B}(X) : s \in [0, t]\}) < \infty.$$

As a consequence one obtains a version of [39, Corollary 4.5] without assuming $\|\frac{\partial P(t, s)}{\partial s}\| \leq C(t-s)^{-1}$.

Proof. Choose $\theta \in (\frac{1}{p} - \frac{1}{q}, \mu)$. Let $r \in (1, \infty)$ be such that $\theta > \frac{1}{r} \geq \frac{1}{p} - \frac{1}{q}$ and $\varepsilon > \theta - \frac{1}{r}$. Fix $t \in [0, T]$. We will apply Theorem 5.1, with the equivalent norm explained in Section 2.5, to the function $f : [0, t] \rightarrow \mathcal{B}(X)$ defined by $f(s) = (t-s)^\varepsilon P(t, s)$.

Let $h \in (0, t)$. By the triangle inequality we can write

$$\begin{aligned} & \| (t-s-h)^\varepsilon P(t, s+h) - (t-s)^\varepsilon P(t, s) \| \\ & \leq \| (t-s-h)^\varepsilon (P(t, s+h) - P(t, s)) \| + | (t-s-h)^\varepsilon - (t-s)^\varepsilon | \| P(t, s) \|. \end{aligned}$$

Since the main point is dealing with small ε , we may assume that $(\varepsilon-1)r < -1$. Then

$$\begin{aligned} & \int_0^{t-h} |(t-s)^\varepsilon - (t-s-h)^\varepsilon|^r ds = \left(\int_h^{2h} + \int_{2h}^t \right) |u^\varepsilon - (u-h)^\varepsilon|^r du \\ & \leq \int_h^{2h} u^{\varepsilon r} du + \int_{2h}^t |h\varepsilon(u-h)^{\varepsilon-1}|^r du \leq h(2h)^{\varepsilon r} + (h\varepsilon)^r \int_h^\infty u^{(\varepsilon-1)r} du \\ & \lesssim_{\varepsilon, r} h^{1+\varepsilon r} + h^r h^{1+(\varepsilon-1)r} \approx h^{1+\varepsilon r}. \end{aligned}$$

For the other part it follows from (6.4) and Proposition 6.5 that for all $\theta < \mu$

$$\begin{aligned} \|P(t, s+h) - P(t, s)\| &= \|P(t, s+h)(-A(s+h))^\theta (-A(s+h))^{-\theta} (I - P(s+h, s))\| \\ &\leq C(t-s-h)^{-\theta} \|(-A(s+h))^{-\theta} (P(s+h, s) - I)\| \\ &\leq C(t-s-h)^{-\theta} h^\theta. \end{aligned}$$

We conclude that

$$\begin{aligned} & \left(\int_0^{t-h} \|(t-s-h)^\varepsilon P(t, s+h) - (t-s)^\varepsilon P(t, s)\|^r ds \right)^{\frac{1}{r}} \\ & \lesssim h^{\varepsilon + \frac{1}{r}} + h^\theta \left(\int_0^{t-h} (t-s-h)^{(\varepsilon-\theta)r} ds \right)^{\frac{1}{r}} \\ & \approx h^{\varepsilon + \frac{1}{r}} + h^\theta \lesssim h^\theta \end{aligned}$$

where we used $\varepsilon > \theta - \frac{1}{r}$. Similar results hold for $h < 0$.

It follows that $\varrho_r(f, \tau) \lesssim \tau^\theta$, for $\tau \in (0, 1)$, where ϱ_r is defined as in Section 2.5. Since $\theta > \frac{1}{r}$ we can conclude that $f \in \Lambda_{r,1}^{\frac{1}{r}}(0, t; \mathcal{B}(X))$. Now the result follows from (2.5), the norm equivalence of $\Lambda_{r,1}^{\frac{1}{r}}(\mathbb{R}; \mathcal{B}(X))$ and $B_{r,1}^{\frac{1}{r}}(\mathbb{R}; \mathcal{B}(X))$, and Theorem 5.1. \square

Acknowledgment – The authors thank S. Kwapien for the helpful comments on Lemma 3.1 which eventually led us to parts (1) and (3) of that Lemma. The authors thank R. Schnaubelt for showing them Proposition 6.5.

REFERENCES

- [1] P. Acquistapace and B. Terreni. A unified approach to abstract linear nonautonomous parabolic equations. *Rend. Sem. Mat. Univ. Padova*, 78:47–107, 1987.
- [2] P. Acquistapace and B. Terreni. Regularity properties of the evolution operator for abstract linear parabolic equations. *Differential Integral Equations*, 5(5):1151–1184, 1992.
- [3] W. Arendt and S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.*, 240(2):311–343, 2002.
- [4] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.
- [5] E. Berkson and T. A. Gillespie. Spectral decompositions and harmonic analysis on UMD spaces. *Studia Math.*, 112(1):13–49, 1994.
- [6] J. Bourgain. Vector-valued singular integrals and the H^1 -BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986.
- [7] Ph. Clément, B. de Pagter, F. A. Sukochev, and H. Witvliet. Schauder decomposition and multiplier theorems. *Studia Math.*, 138(2):135–163, 2000.
- [8] R. Denk, M. Hieber, and J. Prüss. R -boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.
- [9] J. Dettweiler, J. M. A. M. van Neerven, and L. W. Weis. Space-time regularity of solutions of parabolic stochastic evolution equations. *Stoch. Anal. Appl.*, 24:843–869, 2006.
- [10] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
- [11] T. Figiel. On equivalence of some bases to the Haar system in spaces of vector-valued functions. *Bull. Polish Acad. Sci. Math.*, 36(3-4):119–131 (1989), 1988.
- [12] O. van Gaans. On R -boundedness of unions of sets of operators. In *Partial differential equations and functional analysis*, volume 168 of *Oper. Theory Adv. Appl.*, pages 97–111. Birkhäuser, Basel, 2006.
- [13] M. Girardi and L. W. Weis. Criteria for R -boundedness of operator families. In *Evolution equations*, volume 234 of *Lecture Notes in Pure and Appl. Math.*, pages 203–221. Dekker, New York, 2003.
- [14] L. Grafakos. *Classical and Modern Fourier Analysis*. Pearson/Prentice Hall, New Jersey, 2004.
- [15] B. H. Haak and P. C. Kunstmann. Admissibility of unbounded operators and wellposedness of linear systems in Banach spaces. *Integral Equations Oper. Theory*, 55(4):497–533, 2006.
- [16] T. Hytönen. Fourier embeddings and Mihlin-type multiplier theorems. *Math. Nachr.*, 274/275:74–103, 2004.
- [17] C. Kaiser and L. W. Weis. Wavelet transform for functions with values in UMD spaces. *Studia Math.*, 186(2):101–126, 2008.
- [18] N. J. Kalton and S. J. Montgomery-Smith. Set-functions and factorization. *Arch. Math. (Basel)*, 61(2):183–200, 1993.
- [19] N. J. Kalton, J. M. A. M. van Neerven, M. C. Veraar, and L. W. Weis. Embedding vector-valued Besov spaces into spaces of γ -radonifying operators. *Math. Nachr.*, 281(2):238–252, 2008.
- [20] N. J. Kalton and L. W. Weis. The H^∞ -calculus and square function estimates. Preprint, 2004.
- [21] P. C. Kunstmann and L. W. Weis. Maximal L_p -regularity for parabolic equations, Fourier multiplier theorems and H^∞ -functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.
- [22] S. Kwapien. Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Math.*, 44:583–595, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, VI.
- [23] S. Kwapien and W. A. Woyczyński. *Random series and stochastic integrals: single and multiple*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1992.
- [24] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. Isoperimetry and processes.
- [25] A. Lunardi. *Interpolation Theory*. Appunti. Scuola Normale Superiore Pisa, 1999.

- [26] L. Maniar and R. Schnaubelt. The Fredholm alternative for parabolic evolution equations with inhomogeneous boundary conditions. *J. Differential Equations*, 235(1):308–339, 2007.
- [27] J. M. A. M. van Neerven. Internet Seminar 2007/2008 Stochastic Evolution Equations. <http://fa.its.tudelft.nl/ise/wiki>.
- [28] J. M. A. M. van Neerven, M. C. Veraar, and L. W. Weis. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal.*, 255(4):940–993, 2008.
- [29] J. M. A. M. van Neerven and L. W. Weis. Stochastic integration of functions with values in a Banach space. *Studia Math.*, 166(2):131–170, 2005.
- [30] J. Peetre. Sur la transformation de Fourier des fonctions à valeurs vectorielles. *Rend. Sem. Mat. Univ. Padova*, 42:15–26, 1969.
- [31] A. Pelczyński and M. Wojciechowski. Molecular decompositions and embedding theorems for vector-valued Sobolev spaces with gradient norm. *Studia Math.*, 107(1):61–100, 1993.
- [32] G. Pisier. Factorization of operators through $L_{p\infty}$ or L_{p1} and noncommutative generalizations. *Math. Ann.*, 276(1):105–136, 1986.
- [33] G. Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 1986.
- [34] G. Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
- [35] E. Sawyer. Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator. *Trans. Amer. Math. Soc.*, 281(1):329–337, 1984.
- [36] H.-J. Schmeisser. Vector-valued Sobolev and Besov spaces. In *Seminar analysis of the Karl-Weierstraß-Institute of Mathematics 1985/86 (Berlin, 1985/86)*, volume 96 of *Teubner-Texte Math.*, pages 4–44. Teubner, Leipzig, 1987.
- [37] R. Schnaubelt. Asymptotic behaviour of parabolic nonautonomous evolution equations. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 401–472. Springer, Berlin, 2004.
- [38] H. Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.
- [39] M. C. Veraar and J. Zimmerschied. Non-autonomous stochastic Cauchy problems in Banach spaces. *Stud. Math.*, 185(1):1–34, 2008.
- [40] L. W. Weis. Operator-valued Fourier multiplier theorems and maximal L_p -regularity. *Math. Ann.*, 319(4):735–758, 2001.
- [41] A. Yagi. Parabolic evolution equations in which the coefficients are the generators of infinitely differentiable semigroups. II. *Funkcial. Ekvac.*, 33(1):139–150, 1990.
- [42] A. Yagi. Abstract quasilinear evolution equations of parabolic type in Banach spaces. *Boll. Un. Mat. Ital. B (7)*, 5(2):341–368, 1991.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI, GUSTAF HÄLLSTRÖMIN KATU 2B, FI-00014 HELSINKI, FINLAND

E-mail address: `tuomas.hytonen@helsinki.fi`

DELFT INSTITUTE OF APPLIED MATHEMATICS, DELFT UNIVERSITY OF TECHNOLOGY, P.O. BOX 5031, 2600 GA DELFT, THE NETHERLANDS

E-mail address: `m.c.veraar@tudelft.nl`, `mark@profsonline.nl`