

DIAGONALIZATION AND REPRESENTATION RESULTS FOR NONPOSITIVE SESQUILINEAR FORM MEASURES

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ABSTRACT. We study decompositions of operator measures and more general sesquilinear form measures E into linear combinations of positive parts, and their diagonal vector expansions. The underlying philosophy is to represent E as a trace class valued measure of bounded variation on a new Hilbert space related to E . The choice of the auxiliary Hilbert space fixes a unique decomposition with certain properties, but this choice itself is not canonical. We present relations to Naimark type dilations and direct integrals.

1. INTRODUCTION

The idea of rigged Hilbert spaces arises in attempts to develop mathematically rigorous interpretations of the intuitively appealing Dirac formalism of Quantum Mechanics. With the help of generalized eigenvectors lying outside the Hilbertian state space, one is able to write eigenvalue expansions, with formal similarity to the finite-dimensional case, even for self-adjoint operators with a continuous spectrum. By the Spectral Theorem, self-adjoint operators may be identified with spectral measures on the real line, and they are the mathematical representatives of physical observables in the traditional von Neumann approach to Quantum Mechanics. It is, however, well known that this point of view becomes too restrictive already when considering such basic physical examples as phase-like quantities (see e.g. [6]), but they can still be incorporated into the mathematical formalism by allowing more general positive operator measures in place of spectral measures. Also for them, and for the yet larger class of positive sesquilinear form measures, generalized eigenvalue expansions have been obtained in the literature. See e.g. [3] and the references therein.

The present note is concerned with similar results for sesquilinear form measures without any positivity conditions. Besides purely mathematical interest, motivation comes from important physical questions. Let us consider an example.

Example 1.1. Let H be a complex Hilbert space with an orthonormal basis $(e_n)_{n=0}^\infty$. Let $z \in \mathbb{C} \setminus \{0\}$ and define a *coherent state*

$$\psi_z := e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e_n.$$

It describes quasimonochromatic laser light (in a single-mode quantum optical system), where $|z|$ is the energy parameter and $z/|z| \in \mathbb{T}$ is the phase parameter of the laser light (see e.g. [8]). The vector e_n , the so-called *number state* or *Fock state*, describes an optical field which contains n photons of the same frequency.

2000 *Mathematics Subject Classification.* 47A70 (Primary); 47A07 (Secondary).

Key words and phrases. Sesquilinear form, operator measure, bounded variation, diagonalization, Naimark dilation, direct integral.

A measurement of the phase parameter can be described by using a phase shift covariant semispectral measure [6, p. 23]

$$E(X) := \sum_{m,n=0}^{\infty} c_{mn} \int_X w^{n-m} d\mu(w) |e_m\rangle \langle e_n|$$

where X is a Borel set of \mathbb{T} , μ is the normalized Haar measure of \mathbb{T} and (c_{mn}) is a positive semidefinite complex matrix with the unit diagonal; the probability of getting a value w from a set X when the system is prepared in a state ψ_z is thus $\langle \psi_z | E(X) \psi_z \rangle$.

In realistic physical situations we cannot produce arbitrarily high photon numbers, that is, we cannot prepare number states e_n for an arbitrarily large n . In fact, as of 2004, a method described in [11] “still remains the only experiment in principle capable of providing an arbitrary Fock state (at least up to $n = 4$) on demand”. But still we need the whole Hilbert space H to define coherent states. Hence, we can relax the definition of (c_{mn}) : we need only assume that the first, say, 10×10 block of (c_{mn}) is positive semidefinite (so that we get probability distributions also for superpositions of number states e_n , $n \leq 10$). Moreover, we assume that for any coherent state ψ_z with sufficiently low energy $|z| \leq r \in \mathbb{R}$ we can define a probability measure $X \mapsto e^{-|z|^2} \sum_{m,n=0}^{\infty} c_{mn} \int_X w^{n-m} d\mu(w) \frac{z^{n-m}}{\sqrt{n!m!}}$. Further restrictions can be imposed, if we assume that some superpositions of coherent states can be measured. If (c_{mn}) is not assumed to be positive semidefinite, then $E(X)$ may be a nonpositive operator, or even a sesquilinear form on $V = \text{lin}\{e_n \mid n = 0, 1, \dots\}$, for some X . Then the mapping $X \mapsto E(X)$ can be understood as a (nonpositive) *sesquilinear form measure*. It can be shown that some (phase shift covariant) sesquilinear form measures give more accurate phase distributions in coherent states than any (covariant) positive semispectral measures [7].

The sesquilinear form measures we study here generalize operator measures which have already received a fair amount of attention in the mathematical literature. For example, we may quote a well-known decomposition result from [5], pp. 104–105: A regular Borel operator measure on a compact Hausdorff space (with values in the space of bounded operators on a Hilbert space) is, as a consequence of Wittstock’s decomposition theorem, completely bounded if, and only if, it can be expressed as a linear combination of positive operator valued measures. In this paper an analogous decomposition problem in the setting of sesquilinear form measures is in a central role. We consider a σ -algebra Σ , a vector space V with a countable Hamel basis, and measures $E : \Sigma \rightarrow S(V)$ where $S(V)$ is the space of sesquilinear forms on V . This generalizes the more standard setting of operator measures in the context of a separable Hilbert space, and it turns out that our more flexible framework yields new information even there: An operator measure may be decomposed into a linear combination of positive parts without the condition of complete boundedness. Of course there is a price to pay: these positive parts are not necessarily operator valued but only sesquilinear form valued. While this on the one hand may be seen as a drawback, on the other hand it highlights the usefulness of general sesquilinear form measures.

The paper is organized as follows. After the setting is explained in Section 2, the next section establishes a connection with operator measures taking their values in the trace class $\mathcal{L}^1(H)$ of a separable Hilbert space. Since $\mathcal{L}^1(H)$ has the Radon-Nikodým property, a sesquilinear form measure can be expressed in terms of integrating an $\mathcal{L}^1(H)$ -valued density function with respect to a basic positive scalar measure. In Section 4 the desired decomposition is effected by utilizing the operator density found in Section 3. While the basic idea is straightforward enough, one must take care of rather delicate measurability issues. To this end, a classical result of

Kuratowski and Ryll-Nardzewski on measurable selectors is used. The final Section 5 deals with an analogue of the Naimark dilation theorem: The decomposition of a sesquilinear form measure into positive parts also yields a spectral dilation in a generalized sense involving a unitary operator W on the dilation space where the spectral measure acts. The characteristic feature of W is that $W^4 = I$. The paper concludes with a remark on formulating the dilation result in terms of a direct integral representation.

2. BASICS

We write $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$, $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$ and $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{N}$. For $p > 0$ and \mathcal{I} an index set, $\ell^p(\mathcal{I})$ is the space of the complex families $c = (c_n)_{n \in \mathcal{I}}$ such that $\sum_{n \in \mathcal{I}} |c_n|^p < \infty$.

Let V be a vector space. The scalar field is always \mathbb{C} . A mapping $\Phi : V \times V \rightarrow \mathbb{C}$ is called a *sesquilinear form* (SF), if it is antilinear (i.e., conjugate linear) in the first and linear in the second variable. It is *symmetric* if $\Phi(\phi, \psi) = \overline{\Phi(\psi, \phi)} =: \Phi^*(\phi, \psi)$ and *positive* if $\Phi(\phi, \phi) \geq 0$ for all $\phi, \psi \in V$. Any positive SF is symmetric, and any SF Φ is a linear combination of two symmetric SFs:

$$(2.1) \quad \Phi = \frac{1}{2}(\Phi + \Phi^*) + \frac{i}{2}(i\Phi^* - i\Phi).$$

We let $S(V)$ (resp. $PS(V)$) denote the set of sesquilinear forms (resp. positive sesquilinear forms) on $V \times V$.

Our basic reference on measure and (vector) integration is [2]. Measurability means μ -measurability where μ is a fixed positive measure. Let (Ω, Σ) be a measurable space, i.e., Σ is a σ -algebra of subsets of Ω .

Definition 2.2. Let $E : \Sigma \rightarrow S(V)$ be a mapping and denote $E(X) = E_X$ for $X \in \Sigma$. We call E a *sesquilinear form measure* (SFM) if the mapping $X \mapsto E_X(\phi, \psi)$ is σ -additive, i.e. a complex measure, for all $\phi, \psi \in V$. If in addition $E(X)$ is symmetric (resp. positive) for all $X \in \Sigma$, E is called a *symmetric* (resp. *positive*) *sesquilinear form measure*.

The inner product of any Hilbert space H is linear in the second variable and denoted by $\langle \cdot | \cdot \rangle$. We let $\mathcal{L}(H)$ stand for the bounded linear operators on H , $\mathcal{L}_s(H) \subset \mathcal{L}(H)$ for the self-adjoint operators, and $\mathcal{L}_+(H) \subset \mathcal{L}_s(H)$ for the positive ones. The trace class is denoted by $\mathcal{L}^1(H)$, and $\mathcal{L}_s^1(H) := \mathcal{L}^1(H) \cap \mathcal{L}_s(H)$, $\mathcal{L}_+^1(H) := \mathcal{L}^1(H) \cap \mathcal{L}_+(H)$.

Definition 2.3. Let H be a Hilbert space and $E_0 : \Sigma \rightarrow \mathcal{L}(H)$ a mapping. We call E_0 an *operator measure* (OM) if it is weakly σ -additive, i.e. the mapping $X \mapsto \langle \phi | E_0(X) \psi \rangle$ is σ -additive for all $\phi, \psi \in H$. If in addition $E_0(\Sigma) \subset \mathcal{L}_s(H)$ (resp. $E_0(\Sigma) \subset \mathcal{L}_+(H)$) we say that E_0 is a *self-adjoint* (resp. *positive*) *operator measure*, and if $E_0(X)^2 = E_0(X) = E_0(X)^*$ for all $X \in \Sigma$, E_0 is called a *projection measure*. An OM $E_0 : \Sigma \rightarrow \mathcal{L}(H)$ is called *normalized* if $E_0(\Omega) = I$, the identity operator on H . A normalized positive OM is also called a *semispectral measure* and a normalized projection measure a *spectral measure*.

Every (self-adjoint or positive) OM E_0 can be identified with a (symmetric or positive) SFM E by setting $E_X(\phi, \psi) := \langle \phi | E_0(X) \psi \rangle$.

3. REDUCTION TO TRACE-CLASS OPERATOR MEASURES

For the rest of the note, we assume that V has a countably infinite Hamel basis $(e_n)_{n=0}^\infty$, and H is the Hilbert space completion of V such that $(e_n)_{n=0}^\infty$ is an

orthonormal basis of H . For any SF Φ on V we write (formally)

$$\Phi = \sum_{m,n=0}^{\infty} \Phi_{mn} |e_m\rangle \langle e_n|$$

where $\Phi_{mn} := \Phi(e_m, e_n)$. If Φ is bounded with respect to the norm of H , it determines a unique bounded linear operator $\tilde{\Phi} \in \mathcal{L}(H)$ satisfying $\langle \phi | \tilde{\Phi} \psi \rangle = \Phi(\phi, \psi)$. Then the series above is not just formal; when $|e_m\rangle \langle e_n|$ denotes as usual the rank one operator $\phi \mapsto \langle e_n | \phi \rangle e_m$, the series converges with respect to the weak operator topology to $\tilde{\Phi}$. We may identify Φ and $\tilde{\Phi}$, and then $\Phi_{mn} = \langle e_m | \Phi e_n \rangle$.

Lemma 3.1. *Let $\Phi \in S(V)$ be represented by an infinite matrix $(\Phi_{mn})_{m,n=0}^{\infty} \in \ell^1(\mathbb{N} \times \mathbb{N})$. Then Φ has a unique extension $\Phi \in \mathcal{L}^1(H)$ and $\|\Phi\|_{\mathcal{L}^1(H)} \leq \sum_{m,n=0}^{\infty} |\Phi_{mn}|$.*

Proof. Since $\mathcal{L}^1(H)$ is the dual of the space of finite rank operators on H , the first claim is equivalent to requiring that $\sup |\text{tr}(\Phi \Lambda)| < \infty$ where Φ is interpreted as a matrix (Φ_{mn}) and Λ ranges over the matrices (Λ_{nm}) of finite rank operators of norm ≤ 1 . But

$$|\text{tr}(\Phi \Lambda)| = \left| \sum_{m,n=0}^{\infty} \Phi_{mn} \Lambda_{nm} \right| \leq \sum_{m,n=0}^{\infty} |\Phi_{mn}| \cdot |\Lambda_{nm}| \leq \sum_{m,n=0}^{\infty} |\Phi_{mn}|,$$

since $|\Lambda_{nm}| \leq \|\Lambda\|_{\mathcal{L}(H)} \leq 1$. \square

Theorem 3.2. *For any SFM $E : \Sigma \rightarrow S(V)$ there exist an $\mathcal{L}^1(H)$ -valued measure F of bounded variation, and an injective operator $D \in \mathcal{L}_+(H)$ such that $DV = V$ and*

$$E_X(D\phi, D\psi) = \langle \phi | F(X) \psi \rangle, \quad \phi, \psi \in V.$$

There further exist a finite positive measure $\mu : \Sigma \rightarrow [0, \infty)$ and a function

$$T \in L^1(\Omega, \Sigma, \mu; \mathcal{L}^1(H))$$

such that

$$E_X(D\phi, D\psi) = \int_X \langle \phi | T(\omega) \psi \rangle d\mu(\omega), \quad \phi, \psi \in V.$$

Defining $C_\omega(\phi, \psi) := \langle D^{-1}\phi | T(\omega) D^{-1}\psi \rangle$, we also obtain the integral representation

$$E_X(\phi, \psi) = \int_X C_\omega(\phi, \psi) d\mu(\omega), \quad \phi, \psi \in V.$$

Proof. We denote $E_{mn}(X) := E_X(e_m, e_n)$, and write $|E_{mn}|(X)$ for its total variation on X .

Choose any bounded positive sequence $(d_m)_{m=0}^{\infty}$ such that

$$\delta := \sum_{m,n=0}^{\infty} d_m d_n |E_{mn}|(\Omega) < \infty.$$

For example, we may take $d_m = \alpha_m / \max\{1, \sqrt{|E_{kl}|(\Omega)} \mid 0 \leq k, l \leq m\}$ where $(\alpha_m)_{m=0}^{\infty}$ is any summable positive sequence.

Let D be the diagonal operator

$$D\phi := \sum_{n=0}^{\infty} d_n |e_n\rangle \langle e_n | \phi \rangle.$$

Then for $\phi, \psi \in V$,

$$(3.3) \quad E_X(D\phi, D\psi) = \sum_{m,n=0}^{\infty} \langle \phi | e_m \rangle d_m d_n E_{mn}(X) \langle e_n | \psi \rangle =: \langle \phi | F(X) \psi \rangle,$$

and we have $F(X) \in \mathcal{L}^1(H)$ with $\|F(X)\|_{\mathcal{L}^1(H)} \leq \delta$ by Lemma 3.1. If $(X_k)_{k=0}^\infty$ is any countable partition of $X \subset \Omega$, then

$$(3.4) \quad \sum_{k=0}^{\infty} \|F(X_k)\|_{\mathcal{L}^1(H)} \leq \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} d_m d_n |E_{mn}(X_k)| \leq \sum_{m,n=0}^{\infty} d_m d_n |E_{mn}|(X) \leq \delta.$$

This justifies the computation

$$\sum_{k=0}^{\infty} F(X_k) = \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} d_m d_n E_{mn}(X_k) |e_m\rangle \langle e_n| = \sum_{m,n=0}^{\infty} d_m d_n E_{mn}(X) |e_m\rangle \langle e_n| = F(X),$$

which shows that F is σ -additive, and (3.4) with $X = \Omega$ also shows that F is of bounded variation.

For the measure μ one can take any finite positive measure with respect to which the vector measure F , or equivalently E , is absolutely continuous (i.e., whenever $\mu(X) = 0$, we have also $F(X) = 0$, or equivalently $E_X = 0$ as a sesquilinear form). To be specific, we take μ to be the total variation of F ,

$$|F|(X) := \sup \sum_{k=1}^N \|F(X_k)\|_{\mathcal{L}^1(H)}$$

where the supremum is over all finite Σ -partitions of X . As in the proof of Proposition 7.1 of [3], the existence of T then follows from the vector-valued Radon–Nikodým theorem, since $\mathcal{L}^1(H)$ (as a separable dual space) has the Radon–Nikodým property. \square

Remark 3.5. The above theorem shows that a sesquilinear form measure on V can always be viewed as an operator measure on a new Hilbert space. In fact, let us denote by H_D the range of $D \in \mathcal{L}(H)$ equipped with the inner product $\langle \eta | \theta \rangle_D := \langle D^{-1} \eta | D^{-1} \theta \rangle$ and the induced norm. Then $D : H \rightarrow H_D$ is an isometric Hilbert space isomorphism. Observe that in (3.3) the series in the middle is absolutely convergent, and the right-hand side makes sense, for all $\phi, \psi \in H$. Thus E_X extends continuously to a sesquilinear form on H_D , and for $\eta = D\phi, \theta = D\psi \in H_D$ we have

$$\begin{aligned} E_X(\eta, \theta) &= \langle D^{-1} \eta | F(X) D^{-1} \theta \rangle = \langle D^{-1} \eta | D^{-1} D F(X) D^{-1} \theta \rangle \\ &= \langle \eta | D F(X) D^{-1} \theta \rangle_D =: \langle \eta | \tilde{E}(X) \theta \rangle_D. \end{aligned}$$

Due to the operator-ideal property of the trace class, we find that $X \in \Sigma \mapsto \tilde{E}(X) = D F(X) D^{-1}$ is an $\mathcal{L}^1(H_D)$ -valued measure of bounded variation. By the Radon–Nikodým theorem, it can be written as

$$\tilde{E}(X) = \int_X S(\omega) d\mu(\omega), \quad S \in L^1(\Omega, \Sigma, \mu; \mathcal{L}^1(H_D)).$$

Remark 3.6. In the rest of the paper we take μ to be the measure constructed in the above proof. Assume now that $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$. If we let $\mu = \mu_1 + \mu_2$ be the decomposition of μ as the sum of a discrete measure μ_1 and a continuous measure μ_2 , the integral formula in the above theorem may be used to decompose E as $E = E_1 + E_2$ where E_1 is a discrete SFM, i.e. vanishes outside a countable set, and the SFM E_2 is continuous, i.e., vanishes at every singleton. Clearly such a decomposition is unique.

4. DIAGONALIZATION; POSITIVE AND NEGATIVE PARTS

By formula (2.1) we may decompose the measures E and F as well as the operator density T into linear combinations of two symmetric parts, and by linearity the representation formulae of Theorem 3.2 remain true for these parts. In this section

we obtain a further decomposition of these symmetric parts. We will need the following classical result on *measurable selectors* from [4]; it is also stated in [1], Lemma 1.9:

Lemma 4.1. *Let E be a compact metric space and let $\psi : E \times \Omega \rightarrow \mathbb{R}$ be a mapping such that $\psi(x, \cdot)$ is measurable for arbitrary $x \in E$ and $\psi(\cdot, \omega)$ is continuous for arbitrary $\omega \in \Omega$. Then there exists a measurable $\xi : \Omega \rightarrow E$ such that*

$$\psi(\xi(\omega), \omega) = \max_{x \in E} \psi(x, \omega), \quad \omega \in \Omega.$$

Corollary 4.2. *Let $T : \Omega \rightarrow \mathcal{L}^1(H)$ be a measurable function. Then there exists a measurable $\Phi : \Omega \rightarrow \bar{B}_H$, the closed unit ball of H , such that*

$$|\langle \Phi(\omega) | T(\omega) \Phi(\omega) \rangle| = \max_{\phi \in \bar{B}_H} |\langle \phi | T(\omega) \phi \rangle|, \quad \omega \in \Omega.$$

Proof. It is well known that the unit ball \bar{B}_H of a separable Hilbert space, when equipped with the weak topology, is a compact metrizable space. We consider the mapping

$$\psi : \bar{B}_H \times \Omega \rightarrow \mathbb{R}, (\phi, \omega) \mapsto \langle \phi | T(\omega) \phi \rangle,$$

and it suffices to check the conditions of Lemma 4.1.

That $\psi(\phi, \cdot)$ is measurable is clear from the assumptions. To see that $\psi(\cdot, \omega)$ is continuous, denote $\Lambda := T(\omega) \in \mathcal{L}^1(H)$. Assume first that $\Lambda = |\psi_1\rangle\langle\psi_2|$ has rank 1. The mappings $\phi \mapsto \langle\psi_i|\phi\rangle$ are obviously continuous in the topology in question, and so is their product. In general, we have

$$\Lambda = \sum_{k=1}^{\infty} |\psi_k\rangle\langle\rho_k|, \quad \sum_{k=1}^{\infty} \|\psi_k\| \cdot \|\rho_k\| < \infty.$$

Since uniformly convergent series of continuous functions are continuous, we have reached the conclusion. \square

We can now prove a measurable diagonalization of an $\mathcal{L}_s^1(H)$ -valued function. The proof follows closely the same pattern as the special case for $\mathcal{L}_+^1(H)$ -valued functions given in [1], Proposition 1.8, but we include the details for the reader's convenience.

Theorem 4.3. *Given a measurable function $T : \Omega \rightarrow \mathcal{L}_s^1(H)$, there exist measurable functions $\phi_k : \Omega \rightarrow H$ and $\lambda_k : \Omega \rightarrow \mathbb{R}$, $k \in \mathbb{Z}_+$, such that for any fixed $\omega \in \Omega$ there holds*

$$\begin{aligned} \langle \phi_k(\omega) | \phi_\ell(\omega) \rangle &= \delta_{k\ell}, & |\lambda_k(\omega)| &\geq |\lambda_\ell(\omega)| \quad \text{if } k \leq \ell, \\ T(\omega) &= \sum_{k=1}^{\infty} \lambda_k(\omega) |\phi_k(\omega)\rangle\langle\phi_k(\omega)|, & \|T(\omega)\|_{\mathcal{L}^1(H)} &= \sum_{k=1}^{\infty} |\lambda_k(\omega)|. \end{aligned}$$

Proof. This representation of $T(\omega)$ for each fixed $\omega \in \Omega$ is just the usual spectral representation, but the point is to obtain this with a measurable dependence on ω . To see this, we recall an algorithm for computing the spectral representation. An eigenvalue λ of $\Lambda \in \mathcal{L}_s^1(H)$ of largest modulus satisfies

$$|\lambda| = \max_{\phi \in \bar{B}_H} |\langle \phi | \Lambda \phi \rangle|,$$

and any $\phi \in \bar{B}_H$, which gives the maximum, is an eigenvector related to $\pm\lambda$. By Corollary 4.2, there is a measurable function $\phi_1 : \Omega \rightarrow \bar{B}_H$ such that

$$\lambda_1(\omega) := \langle \phi_1(\omega) | T(\omega) \phi_1(\omega) \rangle,$$

which is also a measurable function of ω by the above formula, is an eigenvalue of $T(\omega)$ of maximal modulus, with the eigenvector $\phi_1(\omega)$.

We then repeat the same procedure with $T_1(\omega) := T(\omega) - \lambda_1(\omega) |\phi_1(\omega)\rangle \langle \phi_1(\omega)|$ in place of $T(\omega)$, obtaining new measurable functions $\lambda_2(\omega)$ and $\phi_2(\omega)$. Proceeding inductively, we obtain sequences of measurable functions $(\lambda_k(\omega))_{k=1}^{\infty}$ and $(\phi_k(\omega))_{k=1}^{\infty}$. At each fixed $\omega \in \Omega$, these give the spectral decomposition of $T(\omega)$ by standard results about compact selfadjoint operators. \square

It is now also easy to separate the positive and negative parts of the operator density in a measurable way:

Corollary 4.4. *Given a measurable function $T : \Omega \rightarrow \mathcal{L}_s^1(H)$, there exist measurable functions $g_k : \Omega \rightarrow H$, $k \in \mathbb{Z} \setminus \{0\}$, such that for any fixed $\omega \in \Omega$ there holds*

$$\begin{aligned} \langle g_k(\omega) | g_\ell(\omega) \rangle &= \delta_{k\ell} \|g_k(\omega)\|^2, \\ \|g_k(\omega)\| &\geq \|g_\ell(\omega)\| \quad \text{if } 0 < k < \ell \quad \text{or} \quad 0 > k > \ell, \\ T(\omega) &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \operatorname{sgn}(k) |g_k(\omega)\rangle \langle g_k(\omega)|, \quad \|T(\omega)\|_{\mathcal{L}^1(H)} = \sum_{k \in \mathbb{Z} \setminus \{0\}} \|g_k(\omega)\|^2. \end{aligned}$$

Proof. With the notation of Theorem 4.3, we define the measurable functions

$$\begin{aligned} n_0(\omega) &:= 0, \quad g_0(\omega) := 0, \\ n_{\pm k}(\omega) &:= \inf\{n \in \mathbb{Z}_+ | n > n_{\pm(k-1)}(\omega), \pm\lambda_n(\omega) > 0\}, \quad k \in \mathbb{Z}_+, \\ g_{\pm k}(\omega) &:= |\lambda_{n_{\pm k}(\omega)}(\omega)|^{1/2} \phi_{n_{\pm k}(\omega)}(\omega), \quad k \in \mathbb{Z}_+, \end{aligned}$$

where it is understood that $\inf \emptyset := \infty$ and $\lambda_\infty(\omega) := 0 =: \phi_\infty(\omega)$. \square

Corollary 4.5. *Given a measurable function $T : \Omega \rightarrow \mathcal{L}_s^1(H)$, there exists a pair of measurable functions $T^\pm : \Omega \rightarrow \mathcal{L}_\pm^1(H)$, such that for any fixed $\omega \in \Omega$ we have*

- (i) $T(\omega) = T^+(\omega) - T^-(\omega)$,
- (ii) $T^+(\omega)T^-(\omega) = 0$, and
- (iii) $\|T(\omega)\|_{\mathcal{L}^1} = \|T^+(\omega)\|_{\mathcal{L}^1} + \|T^-(\omega)\|_{\mathcal{L}^1}$.

Moreover, if (i) and (ii), or alternatively (i) and (iii), hold for all $\omega \in \Omega$, the functions T^+ and T^- are uniquely determined.

Proof. For existence, it suffices to set

$$T^\pm(\omega) := \sum_{k \in \mathbb{Z}_\pm} |g_k(\omega)\rangle \langle g_k(\omega)|.$$

The uniqueness statement assuming (i) and (ii) follows e.g. from Corollary 2.10 in [9]. Assuming (i) and (iii), the uniqueness claim is a consequence of Theorem 4.2 in [10], since $\mathcal{L}^1(H)$ with its norm and order may be identified with the predual of $\mathcal{L}(H)$. \square

In the case of a *symmetric* SFM E , its trace-class density T is self-adjoint operator valued and, using the above corollaries, we get

$$\begin{aligned} (4.6) \quad E_X(\phi, \psi) &= \int_X \langle D^{-1}\phi | [T^+(\omega) - T^-(\omega)] D^{-1}\psi \rangle d\mu(\omega) \\ &= \int_X \sum_{k \in \mathbb{Z} \setminus \{0\}} \operatorname{sgn}(k) \langle D^{-1}\phi | g_k(\omega) \rangle \langle g_k(\omega) | D^{-1}\psi \rangle d\mu(\omega) \\ &= \int_X \sum_{k \in \mathbb{Z} \setminus \{0\}} \operatorname{sgn}(k) \langle \phi | d_k(\omega) \rangle \langle d_k(\omega) | \psi \rangle d\mu(\omega), \end{aligned}$$

where we have defined

$$d_k(\omega) := D^{-1}g_k(\omega) \in H_{D^{-1}},$$

and $H_{D^{-1}}$ is the Hilbert space consisting of all the formal sums $\sum_{n=0}^{\infty} c_n e_n$ such that $\sum_{n=0}^{\infty} d_n^2 |c_n|^2 < \infty$. Note that we have a Hilbert space triplet $H_D \subset H \subset H_{D^{-1}}$, where $H_{D^{-1}}$ is the topological antidual of H_D . Note that the conclusion of (4.6) could also have been reached by applying Corollary 4.4 to (the symmetric parts of) the function $S : \Omega \rightarrow L^1(\Omega, \Sigma, \mu; \mathcal{L}^1(H_D))$ from Remark 3.5.

Denoting

$$E_X^{\pm}(\phi, \psi) := \int_X \sum_{k \in \mathbb{Z}_{\pm}} \langle \phi | d_k(\omega) \rangle \langle d_k(\omega) | \psi \rangle d\mu(\omega)$$

we obtain a splitting

$$(4.7) \quad E_X = E_X^+ - E_X^-$$

of an arbitrary symmetric sesquilinear form measure into a difference of two positive sesquilinear form measures. Despite the above notation, this splitting is not canonical, and a different choice of the operator D typically yields a different decomposition. (The choice of μ is less important: it only affects the normalization of the vectors $d_k(\omega)$.) However, by Corollary 4.5, *given* the choice of D , there is a unique splitting with the stated properties. In particular, the $\mathcal{L}^1(H_D)$ -valued extension \tilde{E} (cf. Remark 3.5) has a canonical splitting into $\mathcal{L}_+^1(H_D)$ -valued operator measures. Also, if E is already positive in the beginning, then the process used in the proof of the decomposition only gives $T^+ = T$ and $E^+ = E$.

Let then $E : \Sigma \rightarrow S(V)$ be an arbitrary SFM.

Definition 4.8. The family $(E^{(k)})_{k=0}^3$ of positive SFMs $E^{(k)} : \Sigma \rightarrow PS(V)$ is a *decomposition of E (into positive parts)* if

$$E = \sum_{k=0}^3 i^k E^{(k)}.$$

From eqs. (2.1) and (4.7) one sees easily that for any SFM E there exists a decomposition of E into positive parts.

5. DILATIONS

Definition 5.1. Let K be a Hilbert space, $F : \Sigma \rightarrow \mathcal{L}(K)$ a spectral measure, and $W \in \mathcal{L}(K)$ a unitary operator whose spectrum $\sigma(W)$ is contained in $\{1, -1, i, -i\}$. Let $J : V \rightarrow K$ be a linear map. We say that the quadruple (K, F, W, J) is a (*spectral W -*)*dilation* of a SFM $E : \Sigma \rightarrow S(V)$ if the following conditions hold:

- (1) $\langle J\phi | F(X)WJ\psi \rangle = E_X(\phi, \psi)$ for all $X \in \Sigma$ and $\phi, \psi \in V$,
- (2) $WF(X) = F(X)W$ for all $X \in \Sigma$.
- (3) the linear span of the set $\{W^k F(X)J\phi \mid k = 0, 1, 2, 3, X \in \Sigma, \phi \in V\}$ is dense in K .

For $k \in \{0, 1, 2, 3\}$, let K_k be the eigenspace of W corresponding to i^k (define $K_k = \{0\}$ if $i^k \notin \sigma(W)$), I_k the identity of K_k , P_k the projection of K onto K_k , $J_k := P_k \circ J$, $F_k : \Sigma \rightarrow \mathcal{L}(K_k)$ the restriction $F_k(X) := F(X)|_{K_k}$, and $E^{(k)} : \Sigma \rightarrow S(V)$ the positive SFM defined by

$$(5.2) \quad E_X^{(k)}(\phi, \psi) := \langle J_k \phi | F_k(X) J_k \psi \rangle.$$

Theorem 5.3. *Let E be a SFM. Any dilation (K, F, W, J) of E defines by (5.2) a decomposition $(E^{(k)})_{k=0}^3$ of E into positive parts. Conversely, for any decomposition $(E^{(k)})_{k=0}^3$, there exists a dilation (K, F, W, J) such that (5.2) holds.*

In particular, any SFM has a spectral W -dilation. In the situation of Theorem 5.3, we say that $(E^{(k)})_{k=0}^3$ is the *decomposition of E associated to the dilation (K, F, W, J)* .

Proof. Given a dilation (K, F, W, J) , it follows from 5.1(2) that each K_k is invariant under $F(X)$, and (K_k, F_k, I_k, J_k) is a spectral dilation of $E^{(k)}$. Then 5.1(1) implies that $(E^{(k)})_{k=0}^3$ is a decomposition of E .

Conversely, let $(E^{(k)})_{k=0}^3$ be a decomposition of E . Then each $E^{(k)}$ is a positive SFM, for which there exists a spectral dilation of the form (K_k, F_k, I_k, J_k) by Theorem 3.6 of [3]. Define $K := K_0 \oplus K_1 \oplus K_2 \oplus K_3$, $F(X) := F_0(X) \oplus F_1(X) \oplus F_2(X) \oplus F_3(X)$, $W := I_0 \oplus (iI_1) \oplus (-I_2) \oplus (-iI_3)$, and $J := J_0 \oplus J_1 \oplus J_2 \oplus J_3$. To check that (K, F, W, J) is a dilation of E , conditions 5.1(1) and 5.1(2) are clear and 5.1(3) follows from $\text{lin}\{W^l \mid l = 0, \dots, 3\} = \text{lin}\{P_k \mid k = 0, \dots, 3\}$. It is also clear that (5.2) holds. \square

Let $\mathcal{M} = (K, F, W, J)$ and $\mathcal{M}' = (K', F', W', J')$ be two dilations of E . The quantities K'_k, F'_k, P'_k, I'_k and J'_k related to \mathcal{M}' are defined as before in the obvious way.

Definition 5.4. The dilations \mathcal{M} and \mathcal{M}' of E are *unitarily equivalent* if there exists a unitary map $U : K \rightarrow K'$ such that $UF(X)J\phi = F'(X)J'\phi$ for all $X \in \Sigma$, $\phi \in V$ and $UW = W'U$; in particular, $UJ\phi = J'\phi$ for all $\phi \in V$.

Theorem 5.5. *Two dilations \mathcal{M} and \mathcal{M}' of E are unitarily equivalent if and only if the decompositions of E associated to \mathcal{M} and \mathcal{M}' are the same, that is,*

$$E_X^{(k)}(\phi, \psi) = \langle J_k \phi | F(X) J_k \psi \rangle = \langle J'_k \phi | F'(X) J'_k \psi \rangle$$

for all $k \in \{0, 1, 2, 3\}$, $X \in \Sigma$ and $\phi, \psi \in V$. Moreover, then the relevant U is unique and $UF(X) = F'(X)U$ for all $X \in \Sigma$.

Proof. Assume first that the decompositions of E associated to \mathcal{M} and \mathcal{M}' are the same. Since both (K_k, F_k, I_k, J_k) and (K'_k, F'_k, I'_k, J'_k) are dilations of the positive SFM $E^{(k)}$, it follows from Theorem 3.6 of [3] that there is a unique unitary map $U_k : K_k \rightarrow K'_k$ such that $U_k F_k(X) J_k \phi = F'_k(X) J'_k \phi$ for all $X \in \Sigma$, $\phi \in V$, and $U_k F_k(X) = F'_k(X) U_k$ for all $X \in \Sigma$. Then $U := \sum_{k=0}^3 U_k P_k$ has the desired properties.

Suppose conversely that \mathcal{M} and \mathcal{M}' are unitarily equivalent. Since $UF(X)W^k F(Y)J\phi = UW^k F(X \cap Y)J\phi = W'^k UF(X \cap Y)J\phi = W'^k F'(X \cap Y)J'\phi = F'(X)W'^k F'(Y)UJ\phi = F'(X)UW^k F(Y)J\phi$, it follows from 5.1(3) that $UF(X) = F'(X)U$. As $UW = W'U$ implying $UP_k = P'_k U$, one sees that

$$\begin{aligned} \langle P'_k J' \phi | F'(X) P'_k J' \psi \rangle &= \langle P'_k U J \phi | F'(X) P'_k U J \psi \rangle = \langle UP_k J \phi | F'(X) UP_k J \psi \rangle \\ &= \langle P_k J \phi | F(X) P_k J \psi \rangle, \end{aligned}$$

i.e., the associated decompositions coincide. Since $UW^k F(X)J\phi = F'(X)W'^k J'\phi$, the uniqueness of U is clear. \square

Remark 5.6. Since $\|J\phi\|_K^2 = \sum_{k=0}^3 \|J_k \phi\|_{K_k}^2 = \sum_{k=0}^3 E_\Omega^{(k)}(\phi, \phi)$, we see that $J : V \rightarrow K$ is injective if and only if

$$(5.7) \quad \sum_{k=0}^3 E_\Omega^{(k)}(\phi, \phi) > 0 \quad \text{for all } \phi \in V \setminus \{0\}.$$

This situation can always be achieved by writing $E = (E + \epsilon E_0) - \epsilon E_0$, where $\epsilon > 0$ and $E_0 : \Sigma \rightarrow \mathcal{L}(H)$ is a semispectral measure, which automatically satisfies (5.7).

Remark 5.8. In analogy with the case of positive SFMs treated in [3], it is possible to describe a concrete representation of the dilation (K, F, W, J) associated with any decomposition $(E^{(k)})_{k=0}^3$ of a SFM E into positive parts.

Let $L^2(\Omega, \mu; \ell^2(\mathbb{Z}_+^4)) \approx L^2(\Omega, \mu; \ell^2(\mathbb{Z}_+)^4)$ be the usual Bochner space of $\ell^2(\mathbb{Z}_+^4)$ -valued functions $f = (f^{(0)}, \dots, f^{(3)})$, where $f^{(k)} = (f_j^{(k)})_{j=1}^\infty \in L^2(\Omega, \mu; \ell^2(\mathbb{Z}_+))$.

Given a measurable $\mathbf{n}(\cdot) = (n_0(\cdot), \dots, n_3(\cdot)) : \Omega \rightarrow (\mathbb{N} \cup \{\infty\})^4$, we denote by $L_{\mathbf{n}(\cdot)}^2(\Omega, \mu; \ell^2(\mathbb{Z}_+^4))$ the closed subspace consisting of the functions f such that for a.e. $\omega \in \Omega$, all j and k , there holds $f_j^{(k)}(\omega) = 0$ if $j > n_k(\omega)$. This is analogous to the “direct integral” Hilbert space of a measurable family of ℓ^2 spaces of variable dimension considered in Section 5 of [3]; extending the notation used there we could write

$$L_{\mathbf{n}(\cdot)}^2(\Omega, \mu; \ell^2(\mathbb{Z}_+^4)) = \int_{\Omega}^{\oplus} (\ell^2)_{\mathbf{n}(\omega)}^4 d\mu(\omega).$$

Let then $E = \sum_{k=0}^3 i^k E^{(k)}$ be a SFM. By the construction of Section 4 (or Theorem 4.5 of [3]), the positive SFMs $E^{(k)}$ have representations

$$E_X^{(k)}(\phi, \psi) = \int_X \sum_{j \in \mathbb{Z}_+} \langle \phi | d_j^{(k)}(\omega) \rangle \langle d_j^{(k)}(\omega) | \psi \rangle d\mu(\omega),$$

where $d_j^{(k)}(\omega) = D^{-1} g_j^{(k)}(\omega)$, and the $g_j^{(k)}(\omega)$ are as the $g_j(\omega)$ in Corollary 4.4. We now fix a specific $\mathbf{n}(\cdot)$ by setting $n_k(\omega) := \sup\{j \in \mathbb{Z}_+ : d_j^{(k)}(\omega) \neq 0\}$ (with $\sup \emptyset := 0$), and define

$$K := L_{\mathbf{n}(\cdot)}^2(\Omega, \mu; \ell^2(\mathbb{Z}_+^4)), \quad F(X)f := 1_X f, \quad Wf := (f^{(0)}, if^{(1)}, -f^{(2)}, -if^{(3)}), \\ (J\phi)(\omega) := (\langle d_j^{(0)}(\omega) | \phi \rangle, \dots, \langle d_j^{(3)}(\omega) | \phi \rangle)_{j=1}^{\infty}.$$

The conditions 5.1(1) and 5.1(2) of a dilation follow from simple algebra. The density requirement 5.1(3) is a consequence of the fact that the component dilations (K_k, F_k, I_k, J_k) , $k = 0, \dots, 3$, are dilations of the positive parts $E^{(k)}$ of E by Theorem 5.1 of [3].

REFERENCES

- [1] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.
- [2] N. Dunford, J. T. Schwartz, *Linear Operators. I. General Theory*. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, 7. Interscience Publishers, New York–London, 1958.
- [3] T. Hytönen, J.-P. Pellonpää, K. Ylinen, Positive sesquilinear form measures and generalized eigenvalue expansions. *J. Math. Anal. Appl.*, in press; arXiv:math.FA/0703589v1
- [4] K. Kuratowski, C. Ryll-Nardzewski, A general theorem on selectors. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965), 397–403.
- [5] V. Paulsen, *Completely bounded maps and operator algebras*. Cambridge Studies in Advanced Mathematics 78. Cambridge University Press, Cambridge, U.K., 2002.
- [6] J.-P. Pellonpää, *Covariant Phase Observables in Quantum Mechanics*. Annales Universitatis Turkuensis A I, number 288, University of Turku, Finland, 2002. (<http://www.physics.utu.fi/theory/Opinnaytteita/jp.pdf>)
- [7] J.-P. Pellonpää, Phase observables, phase operators and operator orderings. *J. Phys. A* **34** (2001), 7901–7916.
- [8] J. Peřina, *Coherence of Light*. D. Reidel Publishing Company, Dordrecht, 1985.
- [9] S. Stratila, L. Zsido, *Lectures on von Neumann algebras*. Editura Academiei, Bucuresti, Romania; Abacus Press, Turnbridge Wells, Kent, England, 1979.
- [10] M. Takesaki, *Theory of operator algebras I*. Springer-Verlag, New York Heidelberg Berlin, 1979.
- [11] B. T. H. Varcoe, S. Brattke, H. Walther, The creation and detection of arbitrary photon number states using cavity QED. *New J. Phys.* **6** (2004) 97.

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