

VECTOR-VALUED MULTIPARAMETER SINGULAR INTEGRALS AND PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We consider multiparameter singular integrals and pseudodifferential operators acting on mixed-norm Bochner spaces $L_{p_1, \dots, p_N}(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}; X)$ where X is a UMD Banach space satisfying Pisier's property (α) . These geometric conditions are shown to be necessary. We obtain a vector-valued version of a result by R. Fefferman and Stein, also providing a new, inductive proof of the original scalar-valued theorem. Then we extend a result of Bourgain on singular integrals in UMD spaces with an unconditional basis to a multiparameter situation. Finally we carry over a result of Yamazaki on pseudodifferential operators to the Bochner space setting, improving the known vector-valued results even in the one-parameter case. 2000 *Mathematics Subject Classification*. 42B20, 47G30, 46B09, 46E40.

1. INTRODUCTION

In the last twenty-five years or so, much of the classical Calderón–Zygmund theory of singular integrals has been extended to the vector-valued situation, by which we understand results concerning functions f , typically of n real variables, which take their values in a Banach space X , usually of infinite dimensions. It is well-known that the class of so-called UMD spaces provides the most general setting in which the typical classical results of this theory – those dealing with classes of singular kernels which are invariant under the natural one-parameter dilations $x \mapsto \delta x$, $\delta > 0$ – remain valid.

In the scalar-valued case, there is also a well-developed multiparameter (or product) theory where, given positive integers N, n_1, \dots, n_N and $n := n_1 + \dots + n_N$, one thinks of \mathbb{R}^n as

$$(1) \quad \mathbb{R}^n = \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N}.$$

Then one considers classes of singular kernels which are invariant under the N -parameter dilations $x = (x_1, \dots, x_N) \mapsto (\delta_1 x_1, \dots, \delta_N x_N)$ of \mathbb{R}^n , where $\delta_1, \dots, \delta_N > 0$.

There are two interesting aspects to this multiparameter theory when considered in the vector-valued situation (which, so far, has been mostly done in

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the Fourier multiplier representation of these operators). First, it turns out that the UMD condition alone is not sufficient anymore for the underlying Banach space X , but one needs to strengthen this by the so-called property (α) , introduced by G. Pisier, in order to extend the scalar-valued theorems; cf. [13, 21]. Second, when the property (α) is assumed, a nice inductive approach becomes available which effectively reduces the N -parameter results to their one-parameter versions, the point being that there is a natural identification of the vector-valued L_p spaces

$$L_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}; X) = L_p(\mathbb{R}^{n_N}; L_p(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_{N-1}}; X)).$$

This was realized in [8] to give new proofs of the vector-valued Littlewood–Paley and Mihlin–Lizorkin multiplier theorems for UMD spaces with (α) .

In the present paper, we develop these ideas further to cover boundedness results for wider classes of operators. We start, in Sec. 2, by recalling the basic definitions needed in the rest of the paper. Then, in Sec. 3, we consider multiparameter singular integrals of convolution type, which were treated in the scalar case by R. Fefferman and E. M. Stein [7]. Besides extending their results to the vector-valued function spaces, we also obtain a new approach to (some of) the original results from [7], where the vector-valued arguments replace the use of various maximal function techniques employed by Fefferman and Stein. We also demonstrate the necessity of property (α) for our results. In the same spirit we reprove and extend in Sec. 4 a result of J. Bourgain [2] about singular integrals on Banach spaces with an unconditional basis. In Sec. 5 we collect some results concerning vector-valued Littlewood–Paley decompositions, which are then applied in Sec. 6 to carry out some work of M. Yamazaki [20] on pseudodifferential operators of rather general kind in the vector-valued setting. This improves the earlier vector-valued results from [15] and [18] even in the one-parameter case. Given the importance of pseudodifferential operators with limited smoothness in PDE and the fact that vector-valued results have already appeared to be useful in applications (see for instance [1] and [6]) it is our hope that this last result in particular as well as the other results from this paper will find genuine applications.

2. BASIC DEFINITIONS

For the convenience of the reader we briefly recall the notions from Banach space theory which are used in this paper. First, we define the fundamental property which we always assume for our spaces:

Definition 1. *A Banach space X is UMD if the Hilbert transform*

$$Hf(x) := \text{pv} \int_{-\infty}^{\infty} \frac{1}{y} f(x-y) dy$$

defines a bounded operator on $L_2(\mathbb{R}; X)$.

The name UMD (unconditional martingale differences) comes from an equivalent probabilistic definition, which we shall not need in this paper. Classical examples of UMD spaces are the (possibly non-commutative) L_p spaces in the range $1 < p < \infty$. A survey of UMD spaces is found in [16].

Another unconditionality property which turns out to be needed when moving from the one parameter to the multiparameter situation is Pisier's property (α) (see [14]). In the definition below and always thereafter we denote by ε_i independent Rademacher functions, i.e., random variables with distribution $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 2^{-1}$. Often two or more sequences of such variables are needed, which we then denote by $\varepsilon_i, \varepsilon'_j, \varepsilon_k^{(1)}, \dots$ and again all these are assumed independent. By \mathbb{E} we designate the mathematical expectation on the probability space supporting these random variables.

Definition 2. *A Banach space X has property (α) if there exists $C > 0$ such that for all $N \in \mathbb{N}$, all $(\alpha_{i,j})_{1 \leq i,j \leq N}$ in the complex unit disc and all $(x_{i,j})_{1 \leq i,j \leq N} \subset X$*

$$\mathbb{E} \left\| \sum_{1 \leq i,j \leq N} \varepsilon_i \varepsilon'_j \alpha_{i,j} x_{i,j} \right\| \leq C \mathbb{E} \left\| \sum_{1 \leq i,j \leq N} \varepsilon_i \varepsilon'_j x_{i,j} \right\|,$$

where $(\varepsilon_k)_{k \in \mathbb{N}}$ and $(\varepsilon'_k)_{k \in \mathbb{N}}$ are sequences of independent Rademacher variables.

This property differs from UMD and is enjoyed in particular by the (commutative) L_p spaces for $1 \leq p < \infty$ but not by the Schatten ideal C_p if $p \neq 2$.

Finally the boundedness assumptions from the scalar-valued case usually needs to be strengthened when one deals with operator-valued kernels or symbols. It was first realized by Weis in [19] that the following randomized boundedness was needed.

Definition 3. *Let X be a Banach space. $\Psi \subset B(X)$ is called R-bounded if $\exists C > 0, \forall N \in \mathbb{N}, \forall T_1, \dots, T_N \in \Psi, \forall x_1, \dots, x_N \in X$*

$$\mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j T_j x_j \right\| \leq C \mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|.$$

It should be pointed out that this corresponds to square functions estimates if X is for instance an L_p space and that Hilbert spaces are the only Banach spaces in which every uniformly bounded family is in fact R-bounded. In recent years this circle of ideas has appeared to be crucial in vector-valued harmonic analysis as well as in the H^∞ -functional calculus theory and, ultimately, in the applications of those theories to PDE's. The literature is now fairly extensive and we just refer to [12] for further information and references.

3. SINGULAR INTEGRALS

Recall that $n = n_1 + \dots + n_N$. In accordance with the product philosophy (1), we denote

$$\mathbb{R}_*^n := (\mathbb{R}^{n_1} \setminus \{0\}) \times \dots \times (\mathbb{R}^{n_N} \setminus \{0\}).$$

Let $1 < p_1, \dots, p_N < \infty$ and $\bar{p} := (p_1, \dots, p_N)$. For a Banach space X , we consider the mixed-norm Bochner spaces having the inductive definition

$$(2) \quad L_{\bar{p}}(\mathbb{R}^n; X) := L_{p_N}(\mathbb{R}^{n_N}; L_{(p_1, \dots, p_{N-1})}(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_{N-1}}; X)),$$

with $L_{p_1}(\mathbb{R}^{n_1}, X)$ the usual Bochner space.

We consider subsets $J \subseteq \{1, \dots, N\}$ and their complements $J^c = \{1, \dots, N\} \setminus J$ and use $|J|$ to denote the cardinality of J . With $t = (t_1, \dots, t_N)$ and $I = \{i_1, \dots, i_{|I|}\} \subseteq \{1, \dots, N\}$, we employ the following notations:

$$\left(\prod_{i \in I} \int_{A_i} dt_i \right) F(t) = \begin{cases} F(t) & \text{if } I = \emptyset, \\ \int_{A_{i_1} \times \dots \times A_{i_{|I|}}} F(t) dt_{i_1} \dots dt_{i_{|I|}} & \text{otherwise.} \end{cases}$$

$$\Delta_h^j F(t) = F(t_1, \dots, t_{j-1}, t_j - h, t_{j+1}, \dots, t_N) - F(t).$$

The main result of this section is the following:

Theorem 4. *Let X be a UMD Banach space with property (α) and $\eta \in (0, 1]$. Let $K \in C(\mathbb{R}_*^n; B(X))$ be a kernel such that the collection $\tau(K) \subset B(X)$ of all the operators*

$$(3) \quad \prod_{i \in I} |t_i|^{n_i} \prod_{j \in J} \left(\frac{|t_j|}{|h_j|} \right)^\eta \Delta_{h_j}^j \left(\prod_{\ell \in I^c} \int_{\alpha_\ell < |t_\ell| < \beta_\ell} dt_\ell \right) K(t),$$

where the variables range over all $t, h \in \mathbb{R}^n$ with $|t_j| > 2|h_j| > 0$, all $\alpha, \beta \in \mathbb{R}_+^N$ with $\beta_j > \alpha_j$, and all $J \subseteq I \subseteq \{1, \dots, N\}$, is an R -bounded set. Assume further that the following limit exists in the norm of X :

$$(4) \quad \lim_{\epsilon_i \downarrow 0 \forall i \in I} \left(\prod_{i \in I} \int_{\epsilon_i < |t_i| \leq 1} \right) K(t)x$$

for all $I \subseteq \{1, \dots, N\}$, all $x \in X$, and almost all $(t_i)_{i \in I^c}$.

Then the limit

$$(5) \quad Tf(t) := \text{pv} \int_{\mathbb{R}^n} K(u)f(t-u) du := \lim_{\epsilon_1, \dots, \epsilon_N \downarrow 0} \left(\prod_{i=1}^N \int_{|u_i| > \epsilon_i} du_i \right) K(u)f(t-u)$$

exists for all $f \in \mathcal{S}(\mathbb{R}^{n_1}) \otimes \dots \otimes \mathcal{S}(\mathbb{R}^{n_N}) \otimes X$, a dense subspace of $L_{\bar{p}}(\mathbb{R}^n; X)$, and satisfies a norm estimate which permits the extension of T to a bounded operator on $L_{\bar{p}}(\mathbb{R}^n; X)$.

More generally, if \mathcal{K} is a collection of kernels verifying the above properties and such that $\bigcup_{K \in \mathcal{K}} \tau(K)$ is R -bounded in $B(X)$, then the collection \mathcal{T} of the associated operators T is R -bounded in $L_{\bar{p}}(\mathbb{R}^n; X)$.

We prove this result using an induction argument based on the identification (2). This is a rather natural method but one should remark that it requires a vector-valued result for the one parameter case, even if $X = \mathbb{C}$.

Proof. The proof is divided in two steps. First we show that T is well defined on the mentioned dense subspace of $L_{\bar{p}}(\mathbb{R}^n; X)$. Then we obtain $L_{\bar{p}}$ estimates using the induction argument.

Step 1 (existence):

Let $x \in X$, $\phi_i \in \mathcal{S}(\mathbb{R}^{n_i}) \quad \forall i = 1, \dots, N$ and consider $\phi(t) = \prod_{i=1}^N \phi_i(t_i)x$.

$$\begin{aligned} & \int_{\forall i: |u_i| > \epsilon_i} K(u) \phi(t - u) \, du \\ &= \sum_{L \subseteq \{1, \dots, N\}} \left(\prod_{\ell \in L^c} \int_{|u_\ell| > 1} du_\ell \phi_\ell(t_\ell - u_\ell) \right) \times \\ & \quad \times \left(\prod_{k \in L} \int_{\epsilon_k < |u_k| \leq 1} du_k [\phi_k(t_k - u_k) - \phi_k(t_k) + \phi_k(t_k)] \right) K(u) x \\ &= \sum_L \sum_{I \subseteq L} \left[\prod_{\ell \in L^c} \int_{|u_\ell| > 1} du_\ell \frac{\phi_\ell(t_\ell - u_\ell)}{|u_\ell|^{n_\ell}} \right] \prod_{i \in I} \int_{\epsilon_i < |u_i| \leq 1} du_i \frac{\phi_i(t_i - u_i) - \phi_i(t_i)}{|u_i|^{n_i}} \times \\ & \quad \times \prod_{j \in I \cup L^c} |u_j|^{n_j} \left(\prod_{k \in L \setminus I} \phi_k(t_k) \int_{\epsilon_k < |u_k| \leq 1} du_k \right) K(u) x \end{aligned}$$

By our assumptions, the last line is bounded by $C\|x\|_X$, and converges to a limit (for a.e. $(u_j)_{j \in I \cup L^c}$) as $\epsilon_k \downarrow 0$ for $k \in L \setminus I$. The rest of the integrand, for fixed L and I , is dominated by the integrable function

$$(u_k)_{k \in I \cup L^c} \mapsto \prod_{\ell \in L^c} \frac{|\phi_\ell(t_\ell - u_\ell)|}{|u_\ell|^{n_\ell}} 1_{|u_\ell| > 1} \times \prod_{i \in I} \frac{\|\nabla \phi_i\|_{L_\infty(B(t_i, 1))}}{|u_i|^{n_i - 1}} 1_{0 < |u_i| \leq 1},$$

so the existence of the asserted limit follows from the Dominated Convergence Theorem.

Note that

$$\int_{|u_\ell| > 1} \frac{|\phi_\ell(t_\ell - u_\ell)|}{|u_\ell|^{n_\ell}} \, du_\ell \leq \|\phi_\ell\|_{L_1}$$

for all $t_\ell \in \mathbb{R}^{n_\ell}$, but also, if $|t_\ell| > 2$,

$$\left[\int_{1 < |u_\ell| \leq |t_\ell|/2} + \int_{|u_\ell| > |t_\ell|/2} \right] \frac{|\phi_\ell(t_\ell - u_\ell)|}{|u_\ell|^{n_\ell}} \, du_\ell \leq \|\phi_\ell\|_{L_1(B(t_\ell, |t_\ell|/2))} + \frac{2^{n_\ell}}{|t_\ell|^{n_\ell}} \|\phi_\ell\|_{L_1}.$$

In view of the rapid decay at infinity of the ϕ_i and their derivatives, we conclude that

$$\left\| \int_{\forall i: |u_i| > \epsilon_i} K(u) \phi(t-u) du \right\|_X \leq C(\phi) \prod_{i=1}^N (1 + |t_i|)^{-n_i}.$$

The function on the right is in $L_{\bar{p}}(\mathbb{R}^n)$ whenever $p_i > 1$ for all i , and hence the established pointwise convergence also implies, via dominated convergence, the existence of the asserted limit in $L_{\bar{p}}(\mathbb{R}^n; X)$.

Step 2 (boundedness):

For $N = 1$, the base of induction, the theorem asserts that the R -boundedness of the set

$$\begin{aligned} \tau(K) = & \left\{ \int_{\alpha < |t| < \beta} K(t) dt : 0 < \alpha < \beta \right\} \\ & \cup \left\{ |t|^{n+\eta} |h|^{-\eta} (K(t-h) - K(t)) : t, h \in \mathbb{R}^{n_1}, |t| > 2|h| > 0 \right\}, \end{aligned}$$

together with the existence of $\lim_{\epsilon \downarrow 0} \int_{\epsilon < |t| \leq 1} K(t)x dt$ for all $x \in X$, implies that $f \mapsto K * f$, initially defined on $\mathcal{S}(\mathbb{R}^{n_1}) \otimes X$, extends to a bounded linear operator on $L_{p_1}(\mathbb{R}^{n_1}; X)$. This is precisely Theorem 5.10 of [10], which actually uses only the UMD property of X . Moreover, the theorem claims that the R -boundedness of a union $\bigcup_{K \in \mathcal{K}} \tau(K)$ implies the R -boundedness of the associated operators on $L_{p_1}(\mathbb{R}^{n_1}; X)$. This result, where the need for (α) appears, is essentially Theorem 6.4 of [10], although it is stated there with the unnecessary restriction that the kernels be odd.

Now let us assume the validity of the theorem with $N - 1 \geq 1$ parameters and deduce it for N . Observe that for fixed $t_N, h_N \in \mathbb{R}^{n_N} \setminus \{0\}$ ($|t_N| > 2|h_N|$), $\beta_N > \alpha_N > 0$, the $(N - 1)$ -parameter kernels (considered as functions of $(t_i)_{1 \leq i < N-1} \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_{N-1}}$)

$$|t_N|^{n_N} K(t), \quad |t_N|^{n_N+\eta} |h_N|^{-\eta} \Delta_{h_N}^N K(t), \quad \int_{\alpha_N < |t_N| < \beta_N} K(t) dt_N$$

inherit the R -boundedness conditions on K , with $\tau(\tilde{K}) \subseteq \tau(K)$ when \tilde{K} is any of the above-defined new kernels. If \mathcal{K} is a set of kernels with $\tau(\mathcal{K}) := \bigcup_{K \in \mathcal{K}} \tau(K)$ R -bounded, the union of $\tau(\tilde{K})$ ranging over all the kernels \tilde{K} so derived remains R -bounded, and in fact coincides with $\tau(\mathcal{K})$.

For each $t_N \in \mathbb{R}^{n_N} \setminus \{0\}$, we define the $(N - 1)$ -parameter convolution operator

$$\begin{aligned} & (\tilde{K}(t_N)f)(u_1, \dots, u_{N-1}) \\ & := \text{pv} \int_{\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_{N-1}}} K(t) f(u_1 - t_1, \dots, u_{N-1} - t_{N-1}) dt_1 \cdots dt_{N-1}, \end{aligned}$$

initially on $\mathcal{S}(\mathbb{R}^{n_1}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R}^{n_{N-1}}) \otimes X$. By the induction assumption, this extends to a bounded linear operator on $L_{(p_1, \dots, p_{N-1})}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_{N-1}}; X)$, and more precisely we have the R -boundedness of

$$|t_N|^{n_N} \bar{K}(t_N), \quad |t_N|^{n_N+\eta} \frac{\bar{K}(t_N - h_N) - \bar{K}(t_N)}{|h_N|^\eta}, \quad \int_{\alpha_N < |t_N| < \beta_N} \bar{K}(t_N) dt_N$$

for t_N, h_N, α, β as before and $K \in \mathcal{K}$.

By the base of our induction, this implies that the operators $f \mapsto \bar{K} * f$ (convolution with respect to the t_N variable), initially defined on $\mathcal{S}(\mathbb{R}^{n_N}) \otimes L_{(p_1, \dots, p_{N-1})}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_{N-1}}; X)$, extend to bounded linear operators on $L_{\bar{p}}(\mathbb{R}^n, X)$, and that the set of all these operators is again R -bounded. But clearly $\bar{K} * f$ coincides with $K * f$ for $f \in \mathcal{S}(\mathbb{R}^{n_1}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R}^{n_N}) \otimes X$, so that we have proved the existence of the asserted R -bounded extensions. \square

Denoting by m the Fourier transform of the $B(X)$ -valued tempered distribution given by

$$\phi \in \mathcal{S}(\mathbb{R}^n) \mapsto [x \mapsto \text{pv} \int K(t)\phi(t)x dt] \in B(X),$$

the convolution operator $f \mapsto K * f$ also has the Fourier multiplier representation $\hat{f} \mapsto m\hat{f}$. If the $B(X)$ -valued distribution m coincides with a locally integrable function, a result of Clément and Prüss [4] says that the essential range of m is R -bounded, since the associated Fourier multiplier transformation is bounded according to Theorem 4.

Conversely, the existence of an R -bounded Fourier transform of K can replace the existence of the limit (4) in the assumptions of Theorem 4. More precisely:

Corollary 5. *Let X be a UMD space with property (α) . Let $K \in C(\mathbb{R}_*^n; B(X))$ satisfy the same R -boundedness conditions as in Theorem 4, and let there be $m : \mathbb{R}^n \rightarrow B(X)$ such that $[\xi \mapsto m(\xi)x] \in L^\infty(\mathbb{R}^n, X)$ and its inverse Fourier transform, in the sense of distributions, coincides with $t \mapsto K(t)x$ on \mathbb{R}_*^n for every $x \in X$. Denote $\tilde{\tau}(K) := \tau(K) \cup \text{range}(m)$.*

If $\tilde{\tau}(K)$ is R -bounded, then the operator defined by

$$(6) \quad Tf(x) := \int_{\mathbb{R}^n} e^{i2\pi x \cdot \xi} m(\xi) \hat{f}(\xi) d\xi,$$

initially for $f \in \mathcal{S}(\mathbb{R}^{n_1}) \otimes \cdots \otimes \mathcal{S}(\mathbb{R}^{n_N}) \otimes X$, extends to a bounded linear operator on $L_{\bar{p}}(\mathbb{R}^n; X)$. If \mathcal{K} is a collection of such kernels with $\bigcup_{K \in \mathcal{K}} \tau(K)$ R -bounded in $B(X)$, then the set of associated operators T is R -bounded in $B(L_{\bar{p}}(\mathbb{R}^n; X))$.

Proof. We indicate the required modifications in the proof of Theorem 4. This time, the existence of the operator on the indicated test function class presents

no problem, as the defining integral in (6) converges absolutely and gives $\|Tf\|_{C_0(\mathbb{R}^n, X)} \leq \|m\|_\infty \|\hat{f}\|_{L_1(\mathbb{R}^n, X)}$.

Concerning the boundedness, the base of induction with a single kernel K is now contained in Theorem 4.1 of [10], and the general case with a family of kernels \mathcal{K} is obtained from this by the methods in Sec. 6 of [10].

Finally, the induction step is proved almost the same way as in the proof of Theorem 4, with obvious modifications. \square

It should be noticed that requiring Pisier's property (α) for the Banach space X in the above statements is in fact necessary. This is shown in the following Proposition, which was inspired by an analogous result for Fourier multipliers from [13].

Proposition 6. *Let X be a Banach space and assume that each kernel satisfying the assumptions of Theorem 4 gives rise to a bounded convolution operator on $L_p(\mathbb{R}^n; X)$. Then X is a UMD space with property (α) .*

Proof. The UMD property is necessary to obtain the boundedness of the Hilbert transform given by

$$n = N = 1, \quad K(x) = \frac{1}{x} \quad \forall x \in \mathbb{R}^*.$$

For the (α) property we consider

$$N = 2 \quad K(t, s) = \sum_{k=1}^M \sum_{l=1}^M \alpha_{k,l} 2^{k+l} \psi(2^k t) \psi(2^l s) \quad \forall (t, s) \in \mathbb{R}^* \times \mathbb{R}^*$$

where ψ is an odd Schwartz function with $\hat{\psi}(1) = 1$, the support of $\hat{\psi}$ is contained in $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and $(\alpha_{k,l})_{k,l=1,\dots,M}$ is a sequence of elements of the complex disc.

The convolution operator with kernel K can be viewed as a Fourier multiplier operator with symbol

$$\hat{K}(\xi, \eta) = \sum_{k=1}^M \sum_{l=1}^M \alpha_{k,l} \hat{\psi}(2^{-k}\xi, 2^{-l}\eta).$$

As this is continuous at the lattice points, the Fourier multiplier operator T on $L_p(\mathbb{T} \times \mathbb{T}; X)$ associated to the restriction of the symbol \hat{K} to \mathbb{Z}^2 is bounded if the original operator $K*$ is bounded on $L_p(\mathbb{R} \times \mathbb{R}; X)$, and there is a contractive estimate of the operator norms. This is a classical result for $X = \mathbb{C}$, and one can check as an exercise that the proof given in [17], Theorem VII.3.8, extends verbatim to the vector-valued situation.

An application of the boundedness of T to the periodic function $f(t, s) = \sum_{k,\ell=1}^M e^{i2\pi(2^k t + 2^\ell s)} x_{k\ell}$ and the equivalence of vector-valued Rademacher and

Steinhaus random series gives

$$\begin{aligned} \mathbb{E}_\varepsilon \mathbb{E}_{\varepsilon'} \left\| \sum_{k,\ell=1}^M \alpha_{k\ell} \varepsilon_k \varepsilon'_\ell x_{k\ell} \right\|_X^p &\simeq \iint_{[0,1]^2} \left\| \sum_{k,\ell=1}^M \alpha_{k\ell} x_{k\ell} e^{i2\pi(2^k t + 2^\ell s)} \right\|_X^p dt ds \\ &\leq \|T\|^p \iint_{[0,1]^2} \left\| \sum_{k,\ell=1}^M x_{k\ell} e^{i2\pi(2^k t + 2^\ell s)} \right\|_X^p dt ds \simeq \mathbb{E}_\varepsilon \mathbb{E}_{\varepsilon'} \|T\|^p \left\| \sum_{k,\ell=1}^M \varepsilon_k \varepsilon'_\ell x_{k\ell} \right\|_X^p. \end{aligned}$$

This gives property (α) provided that we can bound the operator norm of T with an estimate independent of M and the complex numbers $|\alpha_{k\ell}| \leq 1$. By what was said above, this follows if we show that the kernels K satisfy the assumptions of Theorem 4 uniformly with respect to these quantities.

So let us first remark that

$$\int_{\alpha_1 < |s| < \beta_1} K(t, s) ds = 0 = \int_{\alpha_2 < |t| < \beta_2} K(t, s) dt \quad \forall \alpha_1, \alpha_2, \beta_1, \beta_2 > 0.$$

Moreover, for all $(t, s) \in \mathbb{R}^* \times \mathbb{R}^*$

$$\begin{aligned} |t| \cdot |s| \cdot |K(t, s)| &\leq \sum_{k=1}^M \sum_{l=1}^M |\alpha_{k,l}| 2^k |t| \cdot |\psi(2^k t)| 2^l |s| |\psi(2^l s)| \\ &\preceq \sup_{t>0} \left(\sum_{k=1}^M \min(2^k t, (2^k t)^{-1}) \right)^2 \preceq 1, \end{aligned}$$

and

$$\begin{aligned} &|t| \cdot |s| \cdot \left| \frac{t}{h} \right| \cdot |\Delta_h^1 K(t, s)| \\ &\leq \sum_{k=1}^M \sum_{l=1}^M |\alpha_{k,l}| (2^k |t|)^2 \|\nabla \psi(2^k t + 2^k \lambda h)\|_{L^\infty([-1,1], d\lambda)} 2^l |s| |\psi(2^l s)| \\ &\preceq \sup_{t>0} \left(\sum_{k=1}^M \min(2^k t, (2^k t)^{-1}) \right)^2 \preceq 1. \end{aligned}$$

$\Delta_h^2 K(t, s)$ and $\Delta_{h_1, h_2}^{1,2} K(t, s)$ can then be handled in the same way. \square

4. EXTENSION OF A RESULT OF J. BOURGAIN

With the methods of this paper, we can easily reprove (in a manner completely different from the original) and extend a result of Bourgain [2] concerning singular integrals defined on a UMD space with an unconditional basis $(e_j)_{j=1}^\infty$. To us it is important to know that these spaces enjoy property (α) : A space with an unconditional basis has *a fortiori* local unconditional structure

(see [14], Def. 1.1) and a UMD space does not contain the ℓ_∞^n 's uniformly, so that property (α) follows from [14], Proposition 2.1.

Let us denote diagonal operators with respect to the basis $(e_j)_{j=1}^\infty$ by $u = (u_j)_{j=1}^\infty$, so that $u(\sum_{j=1}^\infty x_j e_j) = \sum_{j=1}^\infty u_j x_j e_j$. The following lemma will be useful:

Lemma 7. *Let X be a UMD space with an unconditional basis $(e_j)_{j=1}^\infty$. All the diagonal operators $u = (u_j)_{j=1}^\infty$ with $|u_j| \leq 1$ for all $j = 1, 2, \dots$ form an R -bounded subset of $B(X)$.*

Proof. Let $u^{(k)} = (u_j^{(k)})_{j=1}^\infty$ be such operators and $x^{(k)} = \sum_{j=1}^\infty x_j^{(k)} e_j \in X$ for $k = 1, 2, \dots, m$. Then

$$\begin{aligned} \mathbb{E}_\varepsilon \left\| \sum_{k=1}^m \varepsilon_k u^{(k)} x^{(k)} \right\|_X &= \mathbb{E}_\varepsilon \left\| \sum_{j=1}^\infty \left(\sum_{k=1}^m \varepsilon_k u_j^{(k)} x_j^{(k)} \right) e_j \right\|_X \\ &\preceq \mathbb{E}_\varepsilon \mathbb{E}_{\varepsilon'} \left\| \sum_{j=1}^\infty \varepsilon'_j \left(\sum_{k=1}^m \varepsilon_k u_j^{(k)} x_j^{(k)} \right) e_j \right\|_X \preceq \mathbb{E}_\varepsilon \left\| \sum_{j=1}^\infty \sum_{k=1}^m \varepsilon_j^{(k)} u_j^{(k)} x_j^{(k)} e_j \right\|_X \\ &\preceq \mathbb{E}_\varepsilon \left\| \sum_{j=1}^\infty \sum_{k=1}^m \varepsilon_j^{(k)} x_j^{(k)} e_j \right\|_X \preceq \mathbb{E}_\varepsilon \mathbb{E}_{\varepsilon'} \left\| \sum_{j=1}^\infty \varepsilon'_j \sum_{k=1}^m \varepsilon_k x_j^{(k)} e_j \right\|_X \\ &\preceq \mathbb{E}_\varepsilon \left\| \sum_{j=1}^\infty \sum_{k=1}^m \varepsilon_k x_j^{(k)} e_j \right\|_X = \mathbb{E}_\varepsilon \left\| \sum_{k=1}^m \varepsilon_k x^{(k)} \right\|_X, \end{aligned}$$

where the first and last inequalities were applications of the unconditionality of the basis $(e_j)_{j=1}^\infty$, the second and second to last used property (α) , and the middle step exploited the contraction principle. \square

Theorem 8. *Let X be a UMD space with an unconditional basis $(e_j)_{j=1}^\infty$. Let $K_j \in C(\mathbb{R}_*^n)$, $j = 1, 2, \dots$, be kernels which satisfy uniformly the assumptions of either Theorem 4 or Corollary 5, and let $T_j \in B(L_{\bar{p}}(\mathbb{R}^n, X))$ be the operator associated with K_j . If we define a new operator T by*

$$(7) \quad T\left(\sum_{j=1}^\infty f_j e_j\right) := \sum_{j=1}^\infty T_j f_j e_j,$$

initially on $L_{\bar{p}}(\mathbb{R}^n) \otimes \text{sp}(e_j)_{j=1}^\infty$, then this extends boundedly to all $L_{\bar{p}}(\mathbb{R}^n; X)$. Moreover, any family of such operators satisfying the assumptions uniformly is R -bounded on $L_{\bar{p}}(\mathbb{R}^n; X)$.

Bourgain's result corresponds to the assumptions of Corollary 5 in the one-parameter case $N = 1$.

Proof. It is readily seen that the operator T from (7) has the associated diagonal-operator-valued kernel $K(t) = (K_j(t))_{j=1}^\infty$. By the assumptions, all

the quantities (3) with K_j in place of K , are uniformly bounded; thus by Lemma 7, the quantities (3) as they stand are R -bounded.

Under the assumptions of Theorem 4 for the kernels K_j , the existence of the limits (4) follows at once for $x \in \text{sp}(e_j)_{j=1}^\infty$, and then for all $x \in X$ by the density of this span and the uniform boundedness of the operators involved.

In the situation of Corollary 5, on the other hand, we have a diagonal operator-valued multiplier $m(\xi) = (m_j(\xi))_{j=1}^\infty$. The measurability of $m(\xi)x$ follows from its being the pointwise limit of the obviously measurable finite-dimensional functions $\sum_{j=1}^k m_j(\xi)x_j e_j$, and the boundedness of each m_j together with Lemma 7 gives the R -boundedness of the range of m .

It follows that either Theorem 4 or Corollary 5 gives the asserted conclusion. \square

5. LITTLEWOOD-PALEY DECOMPOSITION

Let us recall that the classical Littlewood-Paley decomposition is obtained by considering dyadic partitions of the form

$$\begin{cases} \phi_0(\xi) = \psi(\xi), \\ \phi_k(\xi) = \psi(2^{-k}\xi) - \psi(2^{1-k}\xi) \quad \forall k \geq 1, \end{cases}$$

for some function $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ such that $\psi(\xi) = 1$ if $|\xi| \leq 1$, $\psi(\xi) = 0$ if $|\xi| > 2$. The decomposition is then given by a sequence $(D_{\phi_k})_{k \in \mathbb{N}} \in B(L_p(\mathbb{R}^n; X))$ defined by

$$(8) \quad \begin{array}{ccc} D_{\phi_k} : L_p(\mathbb{R}^n; X) & \rightarrow & L_p(\mathbb{R}^n; X), \\ f & \mapsto & f * \check{\phi}_k. \end{array}$$

The result in UMD spaces, which is due to Bourgain [3], is the following.

Theorem 9. (Bourgain 1986)

Let X be UMD and $1 < p < \infty$.

Then there exists $C > 0$ such that for all $f \in L_p(\mathbb{R}^n; X)$

$$\frac{1}{C} \|f\|_{L_p(\mathbb{R}^n; X)} \leq \mathbb{E} \left\| \sum_{k \in \mathbb{N}} \varepsilon_k D_{\phi_k} f \right\|_{L_p(\mathbb{R}^n; X)} \leq C \|f\|_{L_p(\mathbb{R}^n; X)}.$$

In the product setting, we choose for each of the N components \mathbb{R}^{n_j} such a dyadic partition with the functions $\phi_k^{(j)}$, $k = 0, 1, \dots$; $j = 1, \dots, N$. Given $J \subseteq \{1, \dots, N\}$ and $K = (k(j_1), \dots, k(j_{|J|})) \in \mathbb{N}^J$ we consider the product Littlewood-Paley operators defined by

$$\mathcal{D}_{K,J} = \prod_{j \in J} D_{\phi_{k(j)}^{(j)}}.$$

We also use the notation $\mathcal{D}_K := \mathcal{D}_{K, \{1, \dots, N\}}$.

Lemma 10. *Let X be a UMD space with property (α) , $J \subseteq \{1, \dots, N\}$ and f belong to $L_{\bar{p}}(\mathbb{R}^n; X)$. Then*

$$\mathbb{E} \left\| \sum_{K \in \mathbb{N}^J} \varepsilon_K \mathcal{D}_{K,J} f \right\|_{L_{\bar{p}}(\mathbb{R}^n; X)} \simeq \|f\|_{L_{\bar{p}}(\mathbb{R}^n; X)}.$$

Proof. This is a direct consequence of property (α) (the first norm equivalence below) and a $|J|$ -fold application of Bourgain's parameter result, Theorem 9 (the second norm equivalence):

$$\mathbb{E} \left\| \sum_{K \in \mathbb{N}^J} \varepsilon_K \mathcal{D}_{K,J} f \right\|_{\bar{p}} \simeq \mathbb{E} \left\| \sum_{K \in \mathbb{N}^J} \prod_{j \in J} (\varepsilon_{k^{(j)}}^{(j)} D_{\phi_{k^{(j)}}}^{(j)}) f \right\|_{\bar{p}} \simeq \|f\|_{\bar{p}}.$$

□

Using property (α) and Theorem 3.3 in [11] we also have the following.

Lemma 11. *Let X be a UMD space with property (α) . Then the set $\{\mathcal{D}_K; K \in \mathbb{N}^N\}$ defined above is R -bounded.*

We shall further need a multiparameter version of the following result of Bourgain [3]:

Proposition 12. (Bourgain 1986)

Let X be UMD and $(f_j)_{j \in \mathbb{Z}} \subset L_p(\mathbb{R}^n; X)$ be a finitely non-zero sequence such that $\text{supp } \widehat{f}_j \subseteq \bar{B}(0, 2^{-j})$. Let $(h_j)_{j \in \mathbb{Z}} \subset \mathbb{R}^n$ be a sequence, lying on a line through the origin and such that $|h_j| < K2^j$ for some constant $K > 0$. Then there exists $C > 0$ such that

$$\mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j f_j(\cdot - h_j) \right\|_{L_p(\mathbb{R}^n; X)} \leq C \log(2 + K) \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j f_j \right\|_{L_p(\mathbb{R}^n; X)}.$$

We also make use of the functions $\widetilde{\phi}_{k^{(j)}}^{(j)} := \psi^{(j)}(2^{-k^{(j)}} \cdot)$, which is supported in a ball of radius $2^{k^{(j)}}$, and denote by $D_{\widetilde{\phi}_{k^{(j)}}}^{(j)}$ the corresponding Fourier multiplier.

This is the case in the next Lemma where, given a function $u \in \mathcal{S}(\mathbb{R}^n; X)$, a vector $y = (y^{(1)}, \dots, y^{(N)}) \in \mathbb{R}^n$ and a set $J \subseteq \{1, \dots, N\}$ we consider

$$u_{y,K,J} = \prod_{j \in J} D_{\widetilde{\phi}_{k^{(j)}}}^{(j)} (\Delta_{2^{-k^{(j)}} y^{(j)}}^{(j)} + I) u.$$

Lemma 13. *Let X be a UMD space with property (α) . Then there exists $C > 0$ such that for all $J \subseteq \{1, \dots, N\}$ and all $y \in \mathbb{R}^n$ we have*

$$\mathbb{E} \left\| \sum_{K \in \mathbb{N}^J} \varepsilon_K u_{y,K,J} \right\|_{\bar{p}} \leq C \prod_{j \in J} \log(2 + |y^{(j)}|) \times \|u\|_{\bar{p}}.$$

Proof. Without loss of generality we assume that $1 \in J$. Using property (α) one obtains

$$\mathbb{E} \left\| \sum \varepsilon_K u_{y,K,J} \right\|_{\bar{p}} \preceq \mathbb{E} \left\| \sum_{k(1) \in \mathbb{N}} \varepsilon_{k(1)}^{(1)} (\Delta_{2^{-k(1)}y^{(1)}} + I) D_{\tilde{\phi}_{k(1)}^{(1)}} \sum_{K' \in \mathbb{N}^{J \setminus \{1\}}} \varepsilon'_{K'} u_{y,K',J \setminus \{1\}} \right\|_{\bar{p}}.$$

By Bourgain's Proposition 12, this is estimated by

$$\preceq \log(2 + |y^{(1)}|) \mathbb{E} \left\| \sum_{k(1) \in \mathbb{N}} \varepsilon_{k(1)}^{(1)} D_{\tilde{\phi}_{k(1)}^{(1)}} \sum_{K' \in \mathbb{N}^{J \setminus \{1\}}} \varepsilon'_{K'} u_{y,K',J \setminus \{1\}} \right\|_{\bar{p}},$$

and by Bourgain's Theorem 9 we further have

$$\preceq \log(2 + |y^{(1)}|) \mathbb{E} \left\| \sum_{K' \in \mathbb{N}^{J \setminus \{1\}}} \varepsilon'_{K'} u_{y,K',J \setminus \{1\}} \right\|_{\bar{p}}.$$

Iterating the argument gives the result. \square

Let us finally record a lemma from [9]. For its application, we recall that either one of the UMD and (α) properties of a space X implies that it cannot contain the ℓ_∞^n 's uniformly. Concerning (α) , this is stated in [14], Remark 2.2.

Lemma 14. *Let X be a Banach space which does not contain the ℓ_∞^n 's uniformly. Let K and J_k , for all $k \in K$, be disjoint index sets, and let $J := \bigcup_{k \in K} J_k$. For all $j \in J$, let $x_j \in X$ and $\lambda_j \in \mathbb{C}$. If the scalars satisfy*

$$\max_{k \in K} \sum_{j \in J_k} |\lambda_j|^2 \leq M^2,$$

then the following estimate holds with some finite C depending only on the space X :

$$\mathbb{E} \left\| \sum_{k \in K} \varepsilon_k \sum_{j \in J_k} \lambda_j x_j \right\|_X \leq CM \mathbb{E} \left\| \sum_{j \in J} \varepsilon_j x_j \right\|_X.$$

6. PSEUDODIFFERENTIAL OPERATORS

In this section we establish the boundedness of an operator-valued version of a class of pseudodifferential operators introduced by M. Yamazaki [20]. Despite the new vector-valued situation, the structure of the argument still reflects the original one from [20] to a considerable extent, and thus we have kept the details fairly brief, concentrating on the places where results from Banach space theory and vector-valued harmonic analysis play a decisive rôle. The reader may consult [20] for more details on those parts of the proof which are essentially similar for scalar and vector functions.

Given a set $J = \{j_1, \dots, j_{|J|}\} \subseteq \{1, \dots, N\}$ and a vector $y \in \mathbb{R}^n$ we denote by y_J the vector $(y_{j_1}, \dots, y_{j_{|J|}})$.

Definition 15. A set of functions $\{\omega_J \in C(\mathbb{R}_+^{|J|}; \mathbb{R}_+) ; J \subseteq \{1, \dots, N\}\}$ is called a modulus of continuity if

- (1) For each $J \subseteq \{1, \dots, N\}$, ω_J is increasing and concave in each variable.
- (2) For each $J \subseteq \{1, \dots, N\}$, ω_J is invariant under any permutation of the variables.
- (3) For each $J_1 \subset J_2 \subseteq \{1, \dots, N\}$, $2^{|J_1|}\omega_{J_1}(t) \leq 2^{|J_2|}\omega_{J_2}((t, t'))$ for each $t \in \mathbb{R}_+^{|J_1|}$ and $t' \in \mathbb{R}_+^{|J_2|-|J_1|}$.

Definition 16. A function $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow B(X)$ is called an R -Yamazaki symbol if there exists a modulus of continuity $(\omega_J)_{J \subseteq \{1, \dots, N\}}$ such that for some $C < \infty$

$$(i) \forall J \subseteq \{1, \dots, N\}, \int_{[0,1]^{|J|}} \frac{\omega_J(t)^2}{t_1 \dots t_{|J|}} dt_1 \dots dt_{|J|} \leq C.$$

$$(ii) \forall J \subseteq \{1, \dots, N\} \quad \forall l \in \{1, \dots, N\} \quad \forall m_l \in \{1, \dots, n_l\} \\ \forall k \in \{0, \dots, n+1\} \quad \forall x \in \mathbb{R}^n \quad \forall y \in \mathbb{R}^n$$

$$\mathcal{R}(\{\omega_J(|y_J|)^{-1}(1 + |\xi^{(l)}|)^k \prod_{j \in J} \Delta_{y^{(j)}}^{(j)} \partial_{\xi_{m_l}^{(l)}}^k a(x, \xi) ; \xi \in \mathbb{R}^n\}) \leq C.$$

The main Theorem of this section is then the following.

Theorem 17. Let X be a UMD space with property (α) and a be an R -Yamazaki symbol. Then the pseudodifferential operator T_a ,

$$T_a u(x) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi,$$

is bounded on $L_{\bar{p}}(\mathbb{R}^n; X)$.

We will prove this Theorem as a consequence of a decomposition of R -Yamazaki symbols into elementary ones in the spirit of Coifman-Meyer [5] and of the following boundedness result for elementary symbols.

Proposition 18. Let $b_{K,h}(\xi) = \exp(i\pi \sum_{j=1}^N 2^{-k(j)-2} h^{(j)}) \prod_{j=1}^N \phi_{k^{(j)}}^{(j)}(\xi^{(j)})$ and a be a symbol of the form

$$a(x, \xi) = \sum_{K \in \mathbb{N}^N} a_{K,h}(x) b_{K,h}(\xi).$$

Assume that for some $J \subseteq \{1, \dots, N\}$ we have the following:

- (i) If $\xi \in \text{supp } \hat{a}_{K,h} + \text{supp } b_{K,h}(\xi)$ then

$$\xi_J \in \prod_{j \in J} (\text{supp } \phi_{k^{(j)}}^{(j)} \cup \text{supp } \phi_{k^{(j)+1}}^{(j)} \cup \text{supp } \phi_{k^{(j)-1}}^{(j)}).$$

- (ii) $\mathcal{R}(\{a_{K,h}(x) ; K_J \in \mathbb{N}^J\}) \leq C(K_{J^c})$ for all $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$.

Then, uniformly in h we have

$$\|T_a\|_{B(L_{\bar{p}}(\mathbb{R}^n; X))} \preceq \left(\sum_{K_{J^c} \in \mathbb{N}^{J^c}} C(K_{J^c})^2 \right)^{1/2} \prod_{j=1}^N \log(2 + |h^{(j)}|).$$

Proof. Let $A_{K,h}$ denote the multiplication operator by $a_{K,h}$ and $M_{K,h}$ the Fourier multiplier with symbol $b_{K,h}$ and let $u \in L_{\bar{p}}(\mathbb{R}^n; X)$. Using (i) and Lemmas 10, 11 and 14 we then have

$$\begin{aligned} & \left\| \sum_{K \in \mathbb{N}^N} A_{K,h} M_{K,h} u \right\|_{\bar{p}} \preceq \mathbb{E} \left\| \sum_{L \in \mathbb{N}^J} \varepsilon_L \mathcal{D}_{L,J} \left(\sum_{K \in \mathbb{N}^N} A_{K,h} M_{K,h} u \right) \right\| \\ & \preceq \mathbb{E} \left\| \sum_{K_J \in \mathbb{N}^J} \varepsilon_{K_J} \left(\sum_{K_{J^c} \in \mathbb{N}^{J^c}} A_{K,h} M_{K,h} u \right) \right\| \\ & \preceq \left(\sum_{K_{J^c} \in \mathbb{N}^{J^c}} C(K_{J^c})^2 \right)^{1/2} \mathbb{E} \left\| \sum_{K \in \mathbb{N}^N} \varepsilon_K C(K_{J^c})^{-1} A_{K,h} M_{K,h} u \right\|. \end{aligned}$$

Now using (ii), property (α) and Lemma 13 we get

$$\begin{aligned} & \mathbb{E} \left\| \sum_{K \in \mathbb{N}^N} \varepsilon_K C(K_{J^c})^{-1} A_{K,h} M_{K,h} u \right\| \\ & \preceq \mathbb{E} \left\| \sum_{K \in \mathbb{N}^N} \varepsilon_K M_{K,h} u \right\| \preceq \prod_{j=1}^N \log(2 + |h^{(j)}|) \|u\|_{\bar{p}}, \end{aligned}$$

completing the proof. \square

We now turn to the decomposition into elementary symbols. Given sets $K \in \mathbb{N}^N$ and $J = \{j_1, \dots, j_{|J|}\} \subseteq \{1, \dots, N\}$ we denote by 2^{-K_J} the vector $(2^{-k(j_1)}, \dots, 2^{-k(j_{|J|})})$.

Proposition 19. *Let a be an R -Yamazaki symbol. Then*

$$a(x, \xi) = \sum_{h \in \mathbb{Z}^n} \sum_{J \subseteq \{1, \dots, N\}} \sum_{K \in \mathbb{N}^N} a_{K,h,J}(x) b_{K,h}(\xi)$$

where $b_{K,h}$ has the same meaning as in Proposition 18 and

(i) the set $\left\{ \frac{1 + |h_1|^{n+1} + \dots + |h_n|^{n+1}}{\omega_{J^c}(2^{-K_{J^c}})} a_{K,h,J}(x) ; K_J \in \mathbb{N}^{|J|} \right\}$ is R -bounded

uniformly in x, h and J ; and

(ii) for all K, h, J , if $\xi \in \text{supp } \widehat{a}_{K,h,J} + \text{supp } b_{K,h}$, then $\xi_J \in \prod_{j \in J} (\text{supp } \phi_{k^{(j)}}^{(j)} \cup$

$\text{supp } \phi_{k^{(j)+1}}^{(j)} \cup \text{supp } \phi_{k^{(j)-1}}^{(j)})$.

Proof. We first introduce another Littlewood-Paley-like decomposition given by functions $\tilde{\phi}_k^{(j)} \in \mathcal{S}(\mathbb{R}^{n_j}; \mathbb{R})$ of the form

$$\begin{cases} \tilde{\phi}_0^{(j)}(\xi) = \psi^{(j)}(2^{-1}\xi), \\ \tilde{\phi}_k^{(j)}(\xi) = \psi^{(j)}(2^{-k-1}\xi) - \psi(2^{-k+2}\xi) \quad \forall k \geq 1, \end{cases}$$

where $\psi^{(j)}$ is defined as in the beginning of Section 5. Remark that these functions have the following properties :

$$\begin{cases} \tilde{\phi}_k^{(j)}(\xi) = 1 & \text{if } \xi \in \text{supp } \phi_k^{(j)}, \\ \tilde{\phi}_k^{(j)}(\xi) = \tilde{\phi}_1^{(j)}(2^{1-k}\xi) & \forall k \geq 1, \\ \xi \in \text{supp } \tilde{\phi}_0^{(j)} \implies |\xi| \leq 4, \\ \xi \in \text{supp } \tilde{\phi}_k^{(j)} \implies 2^{k-2} \leq |\xi| \leq 2^{k+2}. \end{cases}$$

Now we define

$$a_K(x, \xi^{(1)}, \dots, \xi^{(N)}) = a(x, 2^{k(1)+2}\xi^{(1)}, \dots, 2^{k(N)+2}\xi^{(N)}) \prod_{j=1}^N \tilde{\phi}_{k(j)}^{(j)}(2^{k(j)+2}\xi^{(j)})$$

and

$$a_{K,h}(x) = 2^{-n} \int_{[-1,1]^n} \exp(-i\pi h \cdot \xi) a_K(x) d\xi.$$

Since a is a R-Yamazaki symbol, integration by parts (cf. [20], p. 219) shows that for some $C < \infty$ we have, for all $y, h \in \mathbb{R}^n$ and $J \subseteq \{1, \dots, N\}$, that

$$(9) \quad \mathcal{R}\left(\left\{\frac{1 + |h_1|^{n+1} + \dots + |h_n|^{n+1}}{\omega_{J^c}(2^{-K_{J^c}})} \prod_{j \notin J} \Delta_{y^{(j)}}^{(j)} a_{K,h}(x) ; K_J \in \mathbb{N}^{|J|}\right\}\right) \leq C.$$

Now consider the functions $\bar{\psi}_{k(j)}^{(j)}(\xi) := \psi^{(j)}(2^{2-k(j)}\xi)$. Then, given $K \in \mathbb{N}^N$ and $J \subseteq \{1, \dots, N\}$, let us define

$$\psi_{K,J}(\xi) = \prod_{j \in J} \bar{\psi}_{k(j)}^{(j)} \times \prod_{j \notin J} (1 - \bar{\psi}_{k(j)}^{(j)})$$

and let $M_{K,J}$ be the corresponding Fourier multiplier and $a_{K,h,J} = M_{K,J} a_{K,h}$. We then have $a_{K,h} = \sum_{J \subseteq \{1, \dots, N\}} a_{K,h,J}$ and (i) follows from (9) using the following equality (cf. [20], p. 219):

$$a_{K,h,J}(x) = \int_{\mathbb{R}^n} \prod_{j=1}^N \mathcal{F}^{-1} \phi_{k(j)}^{(j)}(y^{(j)}) (-1)^{N-|J|} \left(\prod_{j \in J} \Delta_{y^{(j)}}^{(j)} \prod_{j \notin J} (\Delta_{y^{(j)}}^{(j)} + I) \right) a_{K,h}(x) dy.$$

Now we remark that

$$\begin{aligned} a(x, \xi) &= \sum_{K \in \mathbb{N}^N} a_K(x, 2^{-k(1)-2}\xi^{(1)}, \dots, 2^{-k(N)-2}\xi^{(N)}) \\ &= \sum_{K \in \mathbb{N}^N} \sum_{h \in \mathbb{Z}^n} a_{K,h}(x) b_{K,h}(\xi) \\ &= \sum_{h \in \mathbb{Z}^n} \sum_{J \subseteq \{1, \dots, N\}} \sum_{K \in \mathbb{N}^N} a_{K,h,J}(x) b_{K,h}(\xi). \end{aligned}$$

Finally consider $\xi \in \text{supp } \widehat{a}_{K,h,J} + \text{supp } b_{K,h}$ and $j \in J$. Because of the form of the supports of $\widehat{a}_{K,h,J}$ and $b_{K,h}$ we have that $\xi^{(j)} \in \text{supp } \phi_{k(j)}^{(j)} \cup \text{supp } \phi_{k(j)+1}^{(j)} \cup \text{supp } \phi_{k(j)-1}^{(j)}$ which concludes the proof. \square

Theorem 17 now follows from Proposition 19, Proposition 18 and the following facts (the first one for all $J \subseteq \{1, \dots, N\}$):

$$\begin{aligned} (1) \quad & \sum_{K_{J^c} \in \mathbb{N}^{N-|J|}} \omega_{J^c}(2^{-K_{J^c}})^2 \leq \int_{[0,1]^{N-|J|}} \frac{\omega_{J^c}(t_1, \dots, t_{N-|J|})^2}{t_1 \dots t_{N-|J|}} dt_1 \dots dt_{N-|J|} \\ (2) \quad & \sum_{h \in \mathbb{Z}^n} \frac{\prod_{j=1}^N \log(2 + |h^{(j)}|)}{1 + |h^{(1)}|^{n+1} + \dots + |h^{(N)}|^{n+1}} < \infty. \end{aligned}$$

Remark 20. *The one-parameter case $N = 1$ of Theorem 17 is valid for all UMD spaces, without the assumption of property (α) .*

In fact, all the auxiliary results from Sec. 5 that we used also hold for arbitrary UMD spaces in the one-parameter situation. This result allows symbols with less regularity than the one-parameter results for operator-valued pseudodifferential operators considered earlier in [15] and [18] where moduli of continuity of the form $\omega(t) = t^r$ for some $r > 0$ were considered. Moreover, the admissible (lack of) regularity reached here can be considered optimal in view of Theorem 2 in [20].

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