

ON OPERATOR-MULTIPLIERS FOR MIXED-NORM $L^{\bar{p}}$ SPACES

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ABSTRACT. We use a bootstrapping property of R -boundedness to show that some (known) n -dimensional Fourier multiplier theorems on UMD spaces with property α are immediate corollaries of their one-dimensional versions. The method also gives easy extensions of these results to mixed-norm $L^{\bar{p}}$ spaces.

The purpose of this note is to consider operator-valued Fourier multipliers on the class of UMD Banach spaces having so-called property α of Pisier (call them α -UMD spaces for short), with two principal goals in mind: First, to show that the (known) Mihlin–Lizorkin and Littlewood–Paley multiplier theorems on $L^p(\mathbf{R}^n, X)$, for such spaces X and any $n \in \mathbf{Z}_+$, are in fact almost immediate consequences of the $n = 1$ case. Second, to extend these theorems to the mixed-norm spaces $L^{\bar{p}}(\mathbf{R}^n, X)$, where $\bar{p} = (p_1, \dots, p_n)$. We start by recalling relevant definitions and earlier results, and then proceed to the above-mentioned contributions.

1. NOTATION AND KNOWN RESULTS

Let us write $\bar{p} < \bar{q}$ if $p_j < q_j$ for all j , and moreover $\bar{1} := (1, 1, \dots, 1)$ and $\bar{\infty} := (\infty, \infty, \dots, \infty)$. The mixed-norm spaces (mainly of our interest for $\bar{1} < \bar{p} < \bar{\infty}$) can be defined inductively as follows: For $n = 1$, of course $L^{(p_1)}(\mathbf{R}, X) := L^{p_1}(\mathbf{R}, X)$. Then $L^{(\bar{p}, q)}(\mathbf{R}^{n+1}, X) := L^q(\mathbf{R}, L^{\bar{p}}(\mathbf{R}^n, X))$, and actually this definition already contains the simple idea of our approach, for if X is an α -UMD space, so is $L^{\bar{p}}(\mathbf{R}^n, X)$ for $\bar{1} < \bar{p} < \bar{\infty}$ (use induction on n). Note that for $\bar{p} = p \cdot \bar{1}$, the spaces $L^{\bar{p}}(\mathbf{R}^n, X)$ and $L^p(\mathbf{R}^n, X)$ are canonically isometric by Fubini’s theorem.

Denote $\mathbf{R}_* := \mathbf{R} \setminus \{0\}$, and then $\mathbf{R}_*^n := (\mathbf{R}_*)^n$ is the Euclidean n -space minus the coordinate hyperplanes. We say that m is a Y -valued *Mihlin–Lizorkin-multiplier* on \mathbf{R}^n if m is a Y -valued distribution on \mathbf{R}_*^n , and all the distributional derivatives $D^\alpha m$, $\alpha \in \{0, 1\}^n$, coincide with locally integrable functions on \mathbf{R}_*^n .

Let \mathcal{I} denote the collection of all real intervals of either of the forms $[2^j, 2^{j+1}[$ or $] -2^{j+1}, -2^j]$ for $j \in \mathbf{Z}$. \mathcal{I}^n is the set of all n -fold products of such intervals. We say that m is a Y -valued *Littlewood–Paley-multiplier* on \mathbf{R}^n if m is a Y -valued function on \mathbf{R}_*^n which takes a constant value $m(I)$ on every $I \in \mathcal{I}^n$.

If m is an $\mathcal{L}(X)$ -valued multiplier of either type, we denote by T_m the operator $f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ acting on $X \otimes \mathcal{F}C_c^\infty(\mathbf{R}_*^n)$. Of course $\mathcal{F}f \equiv \hat{f}$ is the Fourier transform of f . The density of $\mathcal{F}C_c^\infty(\mathbf{R}_*)$ in $L^p(\mathbf{R})$ for $1 < p < \infty$ is easily seen. Taking into account the density of $X \otimes L^p(\mathbf{R})$ in $L^p(\mathbf{R}, X)$, the density of $X \otimes \mathcal{F}C_c^\infty(\mathbf{R}_*^n)$ in $L^{\bar{p}}(\mathbf{R}^n, X)$ is readily established by induction.

Let ε_j be independent random variables with distribution $\mathbf{P}(\varepsilon_j = 1) = \mathbf{P}(\varepsilon_j = -1) = \frac{1}{2}$ (so called Rademacher variables), and \mathbf{E} be the related expectation. An operator collection $\mathcal{T} \subset \mathcal{L}(X)$ is called R -bounded if

$$\mathbf{E} \left| \sum_{j=1}^n \varepsilon_j T_j x_j \right|_X \leq C \mathbf{E} \left| \sum_{j=1}^n \varepsilon_j x_j \right|_X$$

for all $n \in \mathbf{Z}_+$ and $(x_j) \subset X$, $(T_j) \subset \mathcal{T}$. The smallest C is denoted by $\mathcal{R}[\mathcal{T} | \mathcal{L}(X)]$. We often abbreviate $\mathcal{R}[\{T : \mathcal{C}(T)\} | \mathcal{L}(X)] =: \mathcal{R}[T : \mathcal{C}(T) | \mathcal{L}(X)]$.

Recall Kahane's inequality: $\mathbf{E}|\sum \varepsilon_j x_j|_X \leq (\mathbf{E}|\sum \varepsilon_j x_j|_X^p)^{1/p} \leq K_p \mathbf{E}|\sum \varepsilon_j x_j|_X$, where $K_p < \infty$ for all $p \in [1, \infty[$.

Let $\text{Rad}(X)$ be the completion of all finitely non-zero sequences $(x_j)_{j=1}^\infty$ in the norm $\|(x_j)_{j=1}^\infty\|_{\text{Rad}(X)} := \mathbf{E}|\sum_{j=1}^\infty \varepsilon_j x_j|_X$. Since $(\mathbf{E}|\sum_{j=1}^\infty \varepsilon_j x_j|^p)^{1/p}$ is an equivalent norm for every $p \in [1, \infty[$, Fubini's theorem gives a canonical isomorphism of $L^p(S, \text{Rad}(X))$ and $\text{Rad}(L^p(S, X))$ for all these p .

For $\mathcal{T} \subset \mathcal{L}(X)$, denote by $\tilde{\mathcal{T}} \subset \mathcal{L}(\text{Rad}(X))$ the collection of all finitely non-zero $\tilde{T} = (T_j)_{j=1}^\infty$, where the action of such a \tilde{T} on $\tilde{x} = (x_j)_{j=1}^\infty \in \text{Rad}(X)$ is defined in the obvious way $\tilde{T}\tilde{x} := (T_j x_j)_{j=1}^\infty$. It is clear that $\mathcal{R}[\mathcal{T}|\mathcal{L}(X)] = \sup_{\tilde{T} \in \tilde{\mathcal{T}}} \|\tilde{T}|\mathcal{L}(\text{Rad}(X))\|$. If X has property α (i.e., if $\mathbf{E}\tilde{\mathbf{E}}|\sum_{i,j=1}^n \varepsilon_i \tilde{\varepsilon}_j \alpha_{ij} x_{ij}|_X \leq C\mathbf{E}\tilde{\mathbf{E}}|\sum_{i,j=1}^n \varepsilon_i \tilde{\varepsilon}_j x_{ij}|_X$ for all $n \in \mathbf{N}$, $x_{ij} \in X$ and $|\alpha_{ij}| \leq 1$, where ε_i and $\tilde{\varepsilon}_j$ are independent Rademacher variables), then more can be said (also cf. [2]):

1.1. Proposition (Clément *et al.* [1]). *If X has property α , then for all $\mathcal{T} \subset \mathcal{L}(X)$*

$$\mathcal{R}[\tilde{\mathcal{T}}|\mathcal{L}(\text{Rad}(X))] \leq C\mathcal{R}[\mathcal{T}|\mathcal{L}(X)].$$

Next, we recall the following operator-valued multiplier theorem:

1.2. Theorem (Weis [4]). *Let X be a UMD space, and $1 < p < \infty$. If m is a Littlewood–Paley-multiplier on \mathbf{R} , then*

$$\|T_m|\mathcal{L}(L^p(\mathbf{R}, X))\| \leq C\mathcal{R}[m(I) : I \in \mathcal{I}|\mathcal{L}(X)].$$

If m is a Mihlin–Lizorkin-multiplier on \mathbf{R} , then

$$\|T_m|\mathcal{L}(L^p(\mathbf{R}, X))\| \leq C\mathcal{R}[m(\xi), \xi m'(\xi) : \xi \in \mathbf{R}_*|\mathcal{L}(X)].$$

For α -UMD spaces, Prop. 1.1 can be used to bootstrap Theorem 1.2 as follows, and this observation was in fact the principal inspiration of the present note:

1.3. Theorem (Girardi, Weis [2]). *Let X be an α -UMD space, and $1 < p < \infty$. If $m \in \mathcal{M}$ are $\mathcal{L}(X)$ -valued Mihlin–Lizorkin-multipliers on \mathbf{R} , then*

$$\mathcal{R}[T_m : m \in \mathcal{M}|\mathcal{L}(L^p(\mathbf{R}, X))] \leq C\mathcal{R}[m(\xi), \xi m'(\xi) : \xi \in \mathbf{R}_*, m \in \mathcal{M}|\mathcal{L}(X)].$$

2. NEW RESULTS AND PROOFS

We first prove a result analogous to Theorem 1.3 for Littlewood–Paley-multipliers. The idea of the proof is the same, it is borrowed from Girardi and Weis [2].

2.1. Proposition. *Let X be an α -UMD space, and $1 < p < \infty$. If $m \in \mathcal{M}$ are $\mathcal{L}(X)$ -valued Littlewood–Paley multipliers on \mathbf{R} , then*

$$\mathcal{R}[T_m : m \in \mathcal{M}|\mathcal{L}(L^p(\mathbf{R}, X))] \leq C\mathcal{R}[m(I) : I \in \mathcal{I}, m \in \mathcal{M}|\mathcal{L}(X)].$$

Proof. If m_j , $j = 1, 2, \dots$, is a finitely non-zero sequence of $\mathcal{L}(X)$ -valued Littlewood–Paley-multipliers on \mathbf{R} , then $\tilde{m} := (m_j)_{j=1}^\infty$ is an $\mathcal{L}(\text{Rad}(X))$ -valued multiplier of the same type, and we have $(T_{m_j})_{j=1}^\infty = \mathcal{F}^{-1}(m_j)_{j=1}^\infty \mathcal{F} = T_{\tilde{m}}$. Thus

$$\begin{aligned} & \mathcal{R}[T_m : m \in \mathcal{M}|\mathcal{L}(L^p(\mathbf{R}, X))] \\ &= \sup \{ \|(T_{m_j})_{j=1}^\infty|\mathcal{L}(\text{Rad}(L^p(\mathbf{R}, X)))\| : (m_j)_{j=1}^\infty \in \mathcal{M}^\infty \text{ finitely non-zero} \} \\ &\leq C \sup \{ \|\tilde{T}_{\tilde{m}}|\mathcal{L}(L^p(\mathbf{R}, \text{Rad}(X)))\| : \tilde{m} \in \mathcal{M}^\infty \text{ finitely non-zero} \} \\ &\leq C' \sup \{ \mathcal{R}[\tilde{m}(I) : I \in \mathcal{I}|\mathcal{L}(\text{Rad}(X))] : \tilde{m} \in \mathcal{M}^\infty \text{ finitely non-zero} \} \\ &\leq C'' \sup \{ \mathcal{R}[m_j(I) : I \in \mathcal{I}, j \in \mathbf{Z}_+|\mathcal{L}(X)] : (m_j)_{j=1}^\infty \in \mathcal{M}^\infty \} \\ &\leq C'' \mathcal{R}[m(I) : I \in \mathcal{I}, m \in \mathcal{M}|\mathcal{L}(X)]. \end{aligned}$$

The first inequality used the isomorphism of $\text{Rad}(L^p(\mathbf{R}, X))$ and $L^p(\mathbf{R}, \text{Rad}(X))$, the second was an application of Theorem 1.2 (with the UMD-space $\text{Rad}(X)$ in place of X), the third follows from Prop. 1.1, and the fourth is obvious. \square

Now we come to the main theorem:

2.2. Theorem. *Let X be an α -UMD space, and $\bar{1} < \bar{p} < \infty$. If $m \in \mathcal{M}$ are $\mathcal{L}(X)$ -valued Littlewood–Paley-multipliers on \mathbf{R}^n , then*

$$\mathcal{R}[T_m : m \in \mathcal{M} | \mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \leq C\mathcal{R}[m(I) : I \in \mathcal{I}^n, m \in \mathcal{M} | \mathcal{L}(X)].$$

If $m \in \mathcal{M}$ are $\mathcal{L}(X)$ -valued Mihlin–Lizorkin-multipliers on \mathbf{R}^n , then

$$\begin{aligned} \mathcal{R}[T_m : m \in \mathcal{M} | \mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \\ \leq C\mathcal{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}_*^n, \alpha \in \{0, 1\}^n, m \in \mathcal{M} | \mathcal{L}(X)]. \end{aligned}$$

Proof. We use induction on n . The case $n = 1$ is just Theorem 1.3 for Mihlin–Lizorkin-multipliers and Prop. 2.1 for Littlewood–Paley-multipliers. Suppose that we have proved the theorem for some $n \geq 1$, and let us prove it for $n + 1$. We consider the case of Mihlin–Lizorkin-multipliers, the other one being similar.

The defining equality $L^{(\bar{p}, q)}(\mathbf{R}^{n+1}, X) = L^q(\mathbf{R}, Y)$, where $Y := L^{\bar{p}}(\mathbf{R}^n, X)$, gives rise to the identification of an $\mathcal{L}(X)$ -valued Mihlin–Lizorkin-multiplier m on \mathbf{R}^{n+1} with an $\mathcal{L}(Y)$ -valued Mihlin–Lizorkin-multiplier \tilde{m} on \mathbf{R} such that $\tilde{m}(\eta) = T_{m(\cdot, \eta)}$. Then

$$\begin{aligned} \mathcal{R}[T_m : m \in \mathcal{M} | \mathcal{L}(L^{(\bar{p}, q)}(\mathbf{R}^{n+1}, X))] &= \mathcal{R}[T_{\tilde{m}} : m \in \mathcal{M} | \mathcal{L}(L^q(\mathbf{R}, L^{\bar{p}}(\mathbf{R}^n, X)))] \\ &\leq C\mathcal{R}[\tilde{m}(\eta), \eta \cdot \tilde{m}'(\eta) : \eta \in \mathbf{R}_*, m \in \mathcal{M} | \mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \\ &= C\mathcal{R}[T_{m(\cdot, \eta)}, T_{\eta D_\eta m(\cdot, \eta)} : \eta \in \mathbf{R}_*, m \in \mathcal{M} | \mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))] \\ &\leq \tilde{C}\mathcal{R}[\zeta^\alpha D_\zeta^\alpha m(\zeta, \eta), \zeta^\alpha \eta D_\zeta^\alpha D_\eta m(\zeta, \eta) : (\zeta, \eta) \in \mathbf{R}_*^{n+1}, \alpha \in \{0, 1\}^n, m \in \mathcal{M} | \mathcal{L}(X)] \\ &= \tilde{C}\mathcal{R}[\xi^\beta D^\beta m(\xi) : \xi \in \mathbf{R}_*^{n+1}, \beta \in \{0, 1\}^{n+1}, m \in \mathcal{M} | \mathcal{L}(X)]. \end{aligned}$$

The first estimate was an application of Theorem 1.3, while the second was the induction assumption. \square

Immediate corollaries of Theorem 2.2 for a single multiplier are the following: Let X be an α -UMD space and $\bar{1} < \bar{p} < \infty$. If m is an $\mathcal{L}(X)$ -valued Littlewood–Paley-multiplier on \mathbf{R}^n , then

$$\|T_m | \mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))\| \leq C\mathcal{R}[m(I) : I \in \mathcal{I}^n | \mathcal{L}(X)].$$

If m is an $\mathcal{L}(X)$ -valued Mihlin–Lizorkin-multiplier on \mathbf{R}^n , then

$$\|T_m | \mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))\| \leq C\mathcal{R}[\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbf{R}_*^n, \alpha \in \{0, 1\}^n | \mathcal{L}(X)].$$

For $\bar{p} = p \cdot \bar{1}$, these are due to Ž. Štrkalj and L. Weis [3], and the bootstrapped Mihlin–Lizorkin part of Theorem 2.2 is due to Girardi and Weis [2]. The case when m is scalar-valued was already proved by F. Zimmermann [5]. We feel that our approach to these theorems is simpler, even when specialized to the situation of [5]. Observe, however, that it is essential for our induction that we have an operator-valued multiplier theorem in one dimension as a starting point: Even if m were a \mathbf{C} -valued multiplier on \mathbf{R}^{n+1} , the related multiplier \tilde{m} appearing in the proof of Theorem 2.2 would be $\mathcal{L}(L^{\bar{p}}(\mathbf{R}^n, X))$ -valued.

The n -dimensional Littlewood–Paley and Mihlin–Lizorkin multiplier theorems, as formulated above, are not valid in general UMD spaces for $n > 1$, as shown in [5]. There are, however, slightly weaker substitute results. Their proof does not seem to allow such a trivial reduction to the one-dimensional case as in the present situation, but requires a more genuine n -dimensional theory.

Finally, we note that all our results remain valid for $\mathcal{L}(X, Y)$ -valued multipliers acting from $L^{\bar{p}}(\mathbf{R}^n, X)$ to $L^{\bar{p}}(\mathbf{R}^n, Y)$, where X and Y are two different α -UMD spaces. In fact, this situation can be easily reduced to the one already investigated by considering the new α -UMD space $X \oplus Y$.

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