ANISOTROPIC FOURIER MULTIPLIERS AND SINGULAR INTEGRALS FOR VECTOR-VALUED FUNCTIONS

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Abstract. We show that an anisotropic Mihlin-type condition on the symbol guarantees the boundedness of the associated Fourier multiplier operator on $L^p(\mathbf{R}^n, X)$ for $1 < p < \infty$ and an arbitrary UMD space X. In many cases, this result can be used as a substitute for the Marcinkiewicz–Lizorkin multiplier theorem, which in invalid in general UMD spaces. An application to anisotropic singular integrals is given.

1. Introduction

Denote by $\mathscr{F}$ the Fourier transform on $\mathbf{R}^n$, and by $T_m := \mathscr{F}^{-1}m\mathscr{F}$ the Fourier multiplier operator associated to the *multiplier* or *symbol* (a bounded measurable function) m, which above is identified with a multiplication operator in the obvious way. Several well-known *multiplier theorems* provide sufficient conditions on m to guarantee the boundedness of T_m as an operator on $L^p(\mathbf{R}^n)$, $1 < p < \infty$; see e.g. [22]. The multiplier problem has also attracted interest on the Bôchner spaces $L^p(\mathbf{R}^n, X)$ of functions taking values in a Banach space X, and the validity of the formal extension of some of the classical theorems is now known to be equivalent to certain other, probabilistic, properties of the Banach space X, which we recall. The first definition was introduced by D. L. Burkholder [4].

1.1. **Definition.** A Banach space X is a UMD space if the X-valued martingale difference sequences on any probability space $(\Omega, \mathscr{A}, \mathsf{P})$ are unconditional on $L^p(\Omega, X)$ for some (equivalently, all) $p \in]1, \infty[$. That is, X is UMD if there is a constant C such that

$$\Big(\mathsf{E}\Big\|\sum_{k=1}^{N}\epsilon_k d_k\Big\|_X^p\Big)^{1/p} \leq C\Big(\mathsf{E}\Big\|\sum_{k=1}^{N} d_k\Big\|_X^p\Big)^{1/p},$$

for all $N \in \mathbf{Z}_+$, all fixed signs $\epsilon_k \in \{-1, 1\}$, every increasing sequence $(\mathscr{A}_k)_{k=1}^N$ of sub-σ-algebras of $\mathscr{A}$, and for every adapted sequence $d_k \in L^p(\mathscr{A}_k, X)$ $(1 \leq k \leq N)$ with the martingale difference property $\mathsf{E}[d_k|\mathscr{A}_{k-1}] = 0$ $(1 < k \leq N)$. (Here E and $\mathsf{E}[\cdot|\mathscr{A}_{k-1}]$ are the expectation and conditional expectation operators, respectively.)

The second notion is due to G. Pisier [20]; it first appeared in a context rather different from the present one:

1.2. **Definition.** On a probability space $(\Omega, \mathscr{A}, \mathsf{P})$, we denote by ε_k, $k \in \mathbf{Z}$, independent random variables with distribution $\mathsf{P}(\varepsilon_k = +1) = \mathsf{P}(\varepsilon_k = -1) = 1/2$, and

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ε'_ℓ, $\ell \in \mathbf{Z}$, is an independent identical sequence. A Banach space satisfies property (α) if there is a constant $C < \infty$ such that

$$\mathsf{E}\Big|\sum_{k,\ell=1}^{N} \varepsilon_k \varepsilon'_\ell \alpha_{k\ell} x_{k\ell}\Big|_X \leq C\mathsf{E}\Big|\sum_{k,\ell=1}^{N} \varepsilon_k \varepsilon'_\ell x_{k\ell}\Big|_X$$

for all $N \in \mathbf{Z}_+$, all vectors $x_{k\ell} \in X$ and scalars $|\alpha_{kl}| \leq 1$ $(1 \leq k, \ell \leq N)$.

These properties relate to the multiplier theorems as follows:

1.3. **Theorem.** *The Mihlin condition $|\xi|^{|\theta|}\left|D^\theta m(\xi)\right| \leq C$ for all $\theta \in \{0,1\}^n$ is sufficient for $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$, $n \geq 1$, if and only if X is a UMD-space.*

The theorem as stated is a combination of several results proved in the 80's by J. Bourgain [2, 3], D. L. Burkholder [4], T. R. McConnell [19] and F. Zimmermann [26].

1.4. **Theorem.** *The Marcinkiewicz–Lizorkin condition $\left|\xi^\theta\right|\left|D^\theta m(\xi)\right| \leq C$ for all $\theta \in \{0,1\}^n$ is sufficient for $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$, $n > 1$, if and only if X is a UMD-space with property (α).*

Predecessors of the sufficiency part of this theorem with more restrictive geometric assumptions were obtained by D. L. Fernandez [9] assuming X to be a Banach *lattice* with the UMD property, and by Zimmermann [26] requiring UMD and *local unconditional structure*; he also showed that UMD alone does not suffice. These conditions were recently weakened to UMD and the property (α) by Ž. Štrkalj and L. Weis [23] (a different proof is due to R. Haller, H. Heck and A. Noll [13]), and the optimality of this condition was observed by Weis and the author [17].

In the last few years, Theorems 1.3 and 1.4 have also been generalized to the operator-valued situation where the symbol m takes values in $\mathscr{L}(X)$, bounded linear operators on X. In this case the multiplier boundedness conditions have to be rephrased with the notion of R-boundedness (see (3.1) below), but otherwise the statements remain unchanged. Such results were pioneered by Weis [24], who also applied them successfully to the long-standing maximal regularity question for the abstract Cauchy problem. Further developments of the operator-valued multiplier techniques in the functional analytic approach to elliptic and parabolic PDE's are found in [7, 18], where many more references are also given.

Although both Theorems 1.3 and 1.4 have been established in the respective maximal classes of Banach spaces, it is the purpose of this paper to show that there is still some place for improvement. In fact, all the examples (that the author is aware of) of Marcinkiewicz–Lizorkin multipliers whose boundedness is actually known to imply the property (α), are relatively artificial and possess a more complicated structure than many of the naturally occurring multipliers which are traditionally handled with the help of the Marcinkiewicz–Lizorkin theorem. One of the standard examples (related to parabolic equations; cf. [22], p. 110) is

$$m(\xi_1, \xi_2) = \mathbf{i}\xi_1 \cdot (\mathbf{i}\xi_1 + \xi_2^2)^{-1}. \tag{1.5}$$

While Theorems 1.3 and 1.4 cannot be used to deduce that $T_m \in \mathscr{L}(L^p(\mathbf{R}^2, X))$ if X is a UMD-space without property (α), it is not very difficult to show this by an *ad hoc* argument, using the techniques from [7] or [18], for instance. These observations suggest that there should be a condition intermediate between those of Mihlin and

Marcinkiewicz–Lizorkin, which still guarantees the multiplier boundedness on all UMD spaces without extra geometric assumptions.

In this paper we obtain such a general result, an *anisotropic* Mihlin-type multiplier theorem, which covers in particular the example (1.5) and other multipliers with a similar dilation structure. Note that the Mihlin multiplier class is invariant under the radial dilations $m(\xi) \mapsto m(\lambda\xi)$, $\lambda > 0$, whereas the Marcinkiewicz–Lizorkin class allows more general n-parameter dilations

$$m(\xi) \mapsto m(\lambda_1\xi_1, \ldots, \lambda_n\xi_n).$$

The multiplier in (1.5), on the other hand, is preserved by the *anisotropic one-parameter* dilations $m(\xi) \mapsto m(\lambda^2\xi_1, \lambda\xi_2)$, and this is the characteristic feature of the multiplier classes that are covered by our Theorem 3.2 below. It implies in particular that every m which is sufficiently smooth on the unit-sphere and satisfies

$$m(\lambda^{a_1}\xi_1, \ldots, \lambda^{a_n}\xi_n) = m(\xi) \tag{1.6}$$

for some $a_1, \ldots, a_n > 0$, all $\lambda > 0$ and $\xi \in \mathbf{R}^n$, induces a bounded $T_m \in \mathscr{L}(L^p(\mathbf{R}^n, X))$ for $1 < p < \infty$ and all UMD spaces X. We should note that this result remains true for operator-valued m without an explicit condition of R-boundedness (although, as required by a theorem of Clément–Prüss [6], the R-boundedness of the range of m will in fact follow from the assumptions).

A parallel theorem is also proved for anisotropic singular integrals, again for operator-valued kernels and on all UMD spaces, generalizing some work of V. S. Guliev [11, 12], who has obtained such results for scalar-kernels on UMD lattices.

We note that while the applications of the UMD theory to PDE's typically deal with lattices like L^p, or reflexive Sobolev or related spaces which still satisfy the property (α), this is not the case with the significant applications in *non-commutative* analysis. In fact, the Schatten–von Neumann classes $\mathscr{C}^p$ are UMD spaces for all $1 < p < \infty$ but fail property (α) for all $p \neq 2$ (cf. [5]). While the general Marcinkiewicz–Lizorkin theorem is false for these interesting spaces, our anisotropic multiplier theorem can be used instead to deal with many of the concrete multipliers which fall outside the scope of the classical Mihlin theorem.

Let us finally mention that it is an interesting open problem to find the largest class of spaces for which our anisotropic multiplier theorem remains valid if we allow for homogeneity as in (1.6) but with dilation exponents a_i of different signs. Such "hyperbolic" multipliers naturally occur in estimating a norm $\left\|D^\beta u\right\|_p$ by other partial derivatives $\|D^\alpha u\|_p$ (cf. [1]). The author has been unable to find a general result in this setting, but the particular question of partial derivatives is treated in [15]. In the scalar case, very general dilations have been handled by F. Ricci and E. M. Stein [21], but their method relies on a reduction to the product or n-parameter theory which is unavailable unless one is willing to assume property (α).

2. Set-up for anisotropic analysis

Let us fix a positive vector $\bar{a} = (a_1, \ldots, a_n)$. The anisotropic size $\varrho(x)$ of $x \in \mathbf{R}^n \setminus \{0\}$, which was already introduced in the pioneering work on anisotropic singular integrals by E. B. Fabes and N. M. Rivière [8], is defined as the unique positive solution ϱ of

$$\sum_{i=1}^{n} x_i^2 \varrho^{-2a_i} = 1, \tag{2.1}$$

and $\varrho(0) := 0$. Observe that $\varrho(x) = 1$ if and only if the Euclidean norm $|x| = 1$, i.e., x is on the unit-sphere S^{n-1}.

For $\lambda > 0$, we denote by $\lambda^{\bar{a}}$ the vector $(\lambda^{a_1}, \ldots, \lambda^{a_n})$, and we define the product of two vectors component-wise; thus $\lambda^{\bar{a}} x = (\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n)$. Observe that $\varrho(\lambda^{\bar{a}} x) = \lambda \varrho(x)$.

For every $x \in \mathbf{R}^n \setminus \{0\}$, there is a unique representation in anisotropic polar coordinates (λ, u), $\lambda > 0$ and $u \in S^{n-1}$, so that $x = \lambda^{\bar{a}} u$. Here $\lambda = \varrho(x)$ and $u = \varrho(x)^{-\bar{a}} x$.

A standard calculus computation gives the change-of-variables formula for integration:

$$\mathrm{d}x = \lambda^{|\bar{a}|-1} \,\mathrm{d}\lambda \sum_{i=1}^{n} a_i u_i^2 \,\mathrm{d}\sigma(u), \tag{2.2}$$

where $\mathrm{d}\sigma$ is the surface measure on S^{n-1} and $|\bar{a}| = a_1 + \ldots + a_n$.

Next we are going to define the appropriate Littlewood–Paley decomposition for the present purposes. When $E \subset \mathbf{R}^n$, we denote by $\Delta[E]$ the Fourier multiplier T_m with $m = 1_E$. For $b > 1$, let $I_j^b := \big[-b^j, -b^{j-1}\big[\cup \big] b^{j-1}, b^j\big]$. Bourgain [3] extended the Littlewood–Paley inequality to the UMD-valued L^p spaces in the form

$$\mathsf{E}\Big\| \sum_j \varepsilon_j \Delta[I_j^b] f \Big\|_{L^p(\mathbf{R},X)} \simeq \|f\|_{L^p(\mathbf{R},X)}, \qquad 1 < p < \infty,$$

where, we recall, the ε_j are independent random signs on some probability space $(\Omega, \mathscr{A}, \mathsf{P})$ satisfying $\mathsf{P}(\varepsilon_j = +1) = \mathsf{P}(\varepsilon_j = -1) = 1/2$, and E is the expectation on Ω. (In fact, this was proved in [3] in the periodic setting and only for $b = 2$, but the transference to $\mathbf{R}$ can be found in [26], and the case of a general b is an easy consequence of the special case.) The Littlewood–Paley estimate says that $(\Delta[I_j^b])_{j \in \mathbf{Z}}$ is an unconditional Schauder decomposition, for short a u.c. decomposition, of the space $L^p(\mathbf{R}, X)$. By simple application of Fubini's theorem it follows that $(\Delta_k[I_j^b])_{j \in \mathbf{Z}}$, where $\Delta_k[E] := \Delta[\mathbf{R}^{k-1} \times E \times \mathbf{R}^{n-k}]$, is a u.c. decomposition of the mixed-norm space $L^{\bar{p}}(\mathbf{R}^n, X)$, $\bar{p} = (p_1, \ldots, p_n)$ with $\bar{1} < \bar{p} < \bar{\infty}$ (short hand for $1 < p_i < \infty$, $i = 1, \ldots, n$).

For $k, \ell \in \{1, \ldots, n\}$, the projections $\Delta_k[I_{j(k)}^{b(k)}]$ and $\Delta_\ell[I_{j(\ell)}^{b(\ell)}]$ clearly commute, and so according results of H. Witvliet [25] we can build a new u.c. decomposition of $L^{\bar{p}}(\mathbf{R}^n, X)$ by an appropriate *blocking* of the product decomposition $(\prod_{i=1}^n \Delta_i[I_{j(i)}^{b(i)}])_{j_1, \ldots, j_n \in \mathbf{Z}}$ (which, without property (α), is not unconditional as such). This u.c. decomposition is given by $(\Delta[E_j^{\bar{b}}])_{j \in \mathbf{Z}}$, where

$$\begin{aligned} E_{kn+r}^{\bar{b}} &:= \prod_{i<r} \big[-b_i^{k+1}, b_i^{k+1}\big] \times \big(\big[-b_r^{k+1}, -b_r^k\big[\cup \big] b_r^k, b_r^{k+1}\big]\big) \times \prod_{i>r} \big[-b_i^k, b_i^k\big] \\ &=: E_{kn+r}^{\bar{b},-1} \cup E_{kn+r}^{\bar{b},+1}, \end{aligned}$$

for $k\in\mathbf{Z}$ and $1\leq r\leq n$, i.e., with $S^{b(i)}_j:=\sum_{k\leq j}\Delta_i[I^{b(i)}_k]$,

$$\Delta[E^{\bar b}_{kn+r}]=\prod_{i<r}S^{b(i)}_{k+1}\cdot\Delta_r[I^{b_r}_{k+1}]\cdot\prod_{i>r}S^{b(i)}_k.$$

This construction, with $b_i\equiv 2$, has been used in [13, 23, 26] as a step towards proving the Mihlin Theorem 1.3. For the purposes of the anisotropic analogue, we exploit the freedom to take different b_i's, and in fact we shall use $b_i=\lambda^{a_i}$, some $\lambda>1$. With this choice, it is easy to verify that $\varrho(x)\simeq\lambda^k$ for $x\in E_{kn+r}$.

3. The anisotropic Mihlin theorem

For the statement of our first main result, let us recall the central notion of R-boundedness: $\mathscr{T}\subset\mathscr{L}(X)$ is called R-bounded if

$$\mathsf{E}\Big|\sum_{k=1}^N\varepsilon_kT_kx_k\Big|_X\leq C\mathsf{E}\Big|\sum_{k=1}^N\varepsilon_kx_k\Big|_X \tag{3.1}$$

for all $N\in\mathbf{N}$, $T_k\in\mathscr{T}$ and $x_k\in X$. The smallest admissible C is in (3.1) is denoted by $\mathscr{R}(\mathscr{T})$. We often drop extra parentheses and write $\mathscr{R}[F(k):k\in\mathscr{K}]:=\mathscr{R}(\{F(k):k\in\mathscr{K}\})$.

3.2. Theorem. *Let X be a UMD space and $\bar 1<\bar p<\bar\infty$. Let $m:\mathbf{R}^n\setminus\{0\}\to\mathscr{L}(X)$ have locally integrable distributional derivatives $D^\theta m$ such that, for some $\lambda>1$ and $\bar a>0$,*

$$\mathscr{R}[\lambda^{k\bar a\cdot\theta}(D^\theta m)(\lambda^{k\bar a}\xi):k\in\mathbf{Z}]\leq B<\infty \tag{3.3}$$

for all $\xi\in\mathbf{R}^n$ with $1\leq\varrho(\xi)\leq\lambda$, and all $\theta\in\{0,1\}^n$. Then $T_m\in\mathscr{L}(L^{\bar p}(\mathbf{R}^n,X))$.

In particular, the conclusion follows under the condition that

$$\{\varrho(\xi)^{\bar a\cdot\theta}D^\theta m(\xi):\xi\in\mathbf{R}^n\setminus\{0\}\} \tag{3.4}$$

be R-bounded for all $\theta\in\{0,1\}^n$.

Note that for $\bar a=\bar 1$, we have $\varrho(\xi)=|\xi|$ and $\bar a\cdot\theta=|\theta|$, so that the isotropic Mihlin Theorem 1.3 (and its operator-valued version from [13, 23]) is obtained as a special case. On the other hand, $\varrho(\xi)^{a_i}\geq|\xi_i|$ so that $\varrho(\xi)^{\bar a\cdot\theta}\geq|\xi^\theta|$, and Theorem 3.2 reduces to a special case of the Marcinkiewicz–Lizorkin multiplier theorem when the space X enjoys property (α). In particular, despite the fact that the statement may seem a little exotic, our anisotropic theorem is nothing new in the scalar case $X=\mathbf{C}$.

Proof. The proof is modeled after the isotropic case in [7]. We keep using the notation of the previous section, in particular $\bar b:=\lambda^{\bar a}$.

The operator T_m commutes with the projections $\Delta[E]$, and hence

$$\begin{aligned}\|T_mf\|_{L^{\bar p}(\mathbf{R}^n,X)}&\leq C\mathsf{E}\Big\|\sum_j\varepsilon_jT_m\Delta[E^{\bar b}_j]f\Big\|_{L^{\bar p}(\mathbf{R}^n,X)}\\&\leq\sum_{r=1}^n\sum_{\sigma=\pm1}\mathsf{E}\Big\|\sum_k\varepsilon_kT_m\Delta[E^{\bar b,\sigma}_{kn+r}]f\Big\|_{L^{\bar p}(\mathbf{R}^n,X)}.\end{aligned}$$

For fixed r and σ, let us denote by $\eta=\eta(r,\sigma)$ and $\eta'=\eta'(r,\sigma)$ two opposite corner points of the rectangle $E^{\bar b,\sigma}_r$; whence $\lambda^{k\bar a}\eta$ and $\lambda^{k\bar a}\eta'$ will be two opposite corners of $E^{\bar b,\sigma}_{kn+r}$. For $\xi\in E^{\bar b}_{kn+r}$, some calculus shows that

$$\begin{aligned} m(\xi) &= \sum_{\theta\in\{0,1\}^n}\int_{[\lambda^{k\bar a}\eta,\xi]^\theta} D^\theta m(\zeta_\theta,(\lambda^{k\bar a}\eta)_{\bar 1-\theta})\,\mathrm{d}\zeta_\theta \\ &= \sum_{\theta\in\{0,1\}^n}\int_{[\lambda^{k\bar a}\eta,\lambda^{k\bar a}\eta']^\theta} D^\theta m(\zeta_\theta,(\lambda^{k\bar a}\eta)_{\bar 1-\theta}) 1_{[(\zeta_\theta,(\lambda^{k\bar a}\eta')_{\bar 1-\theta}),\lambda^{k\bar a}\eta']}(\xi)\,\mathrm{d}\zeta_\theta \\ &= \sum_{\theta\in\{0,1\}^n}\int_{[\eta,\eta']^\theta} \lambda^{k\bar a\cdot\theta} D^\theta m(\lambda^{k\bar a}(\zeta_\theta,\eta_{\bar 1-\theta})) 1_{[\lambda^{k\bar a}(\zeta_\theta,\eta'_{\bar 1-\theta}),\lambda^{k\bar a}\eta']}(\xi)\,\mathrm{d}\zeta_\theta. \end{aligned}$$

where the short-hand notation for integrals is interpreted as follows: On the first line, for instance, we integrate with respect to ζ_i from $\lambda^{ka_i}\eta_i$ to ξ_i for all those i with $\theta_i=1$, the other variables of the integrand being fixed at the values $\lambda^{ka_i}\eta_i$; for $\theta=0$, the expression is simply $m(\lambda^{\bar a}\eta)$.

It follows that

$$\begin{aligned} &T_m\Delta[E^{\bar b,\sigma}_{kn+r}] \\ &\qquad= \sum_{\theta\in\{0,1\}^n}\int_{[\eta,\eta']^\theta} \lambda^{k\bar a\cdot\theta} D^\theta m(\lambda^{k\bar a}(\zeta_\theta,\eta_{\bar 1-\theta}))\Delta[\lambda^{k\bar a}(\zeta_\theta,\eta'_{\bar 1-\theta}),\lambda^{k\bar a}\eta']\,\mathrm{d}\zeta_\theta. \end{aligned}$$

We substitute this into the computation in the beginning of the proof, to the result

$$\begin{aligned} &\|T_m f\| \le C\sum_{r=1}^n\sum_{\sigma=\pm1}\sum_{\theta\in\{0,1\}^n}\int_{[\eta(r,\sigma),\eta'(r,\sigma)]^\theta}\mathrm{d}\zeta_\theta\times \\ &\times\mathsf{E}\Big\|\sum_k\varepsilon_k\lambda^{k\bar a\cdot\theta}D^\theta m(\lambda^{k\bar a}(\zeta_\theta,(\eta(r,\sigma))_{\bar 1-\theta}))\Delta[\lambda^{k\bar a}(\zeta_\theta,(\eta'(r,\sigma))_{\bar 1-\theta}),\lambda^{k\bar a}\eta']f\Big\| \\ &\le \sum_{r,\sigma,\theta}\int_{[\eta(r,\sigma),\eta'(r,\sigma)]^\theta} B\times K\times\Big\|\sum_k\varepsilon_k\Delta[E^{\bar b}_{kn+r}]f\Big\|\,\mathrm{d}\zeta_\theta\le C\,\|f\|\,. \end{aligned}$$

Here B is the R-bound from the assumptions and K is the (finite) R-bound of the projections $\Delta[R]$, R a rectangle, on $L^{\bar p}(\mathbf{R}^n,X)$. (This R-boundedness has been established in [26] for $\bar p=p\cdot\bar 1$, and follows easily from this special case with some use of Fubini.). We also used the fact that $\Delta[F]=\Delta[F]\Delta[E]$ for $F\subseteq E$, and the final estimate used the unconditionality of the decomposition $(\Delta[E^{\bar b}_j])_{j\in\mathbf{Z}}$ together with the boundedness of the domains of summation and integration. □

For multipliers with an anisotropic homogeneity we have:

3.5. **Corollary.** *Let $m:\mathbf{R}^n\setminus\{0\}\to\mathscr{L}(X)$ have continuous derivatives $D^\theta m$ for $\theta\in\{0,1\}^n$, and let there be $\lambda>1$ and $\bar a>0$ such that $m(\lambda^{\bar a}\xi)=m(\xi)$ for all $\xi\in\mathbf{R}^n\setminus\{0\}$. Then $T_m\in\mathscr{L}(L^{\bar p}(\mathbf{R}^n,X))$ when X is a UMD space and $\bar 1<\bar p<\bar\infty$.*

Proof. Observe first that the assumed homogeneity implies $m(\lambda^{k\bar a}\xi)=m(\xi)$ for all $k\in\mathbf{Z}$, and after differentiation

$$\lambda^{k\bar a\cdot\theta}(D^\theta m)(\lambda^{k\bar a}\xi)=D^\theta m(\xi). \tag{3.6}$$

The continuity of $D^\theta m$ implies that its norm attains a finite maximum value on the compact set $\{\xi:1\le\varrho(\xi)\le\lambda\}$. The equation (3.6) shows that the set which is required to be R-bounded in (3.3) consists of the single bounded operator $D^\theta m(\xi)$, where we may take $1\le\varrho(\xi)\le\lambda$. Thus we have the required uniform R-boundedness by the previous compactness remark, and Theorem 3.2 implies the claim. □

The absence of R-boundedness in the assumptions of the Corollary may seem a little surprising at the first sight, but not so much once one realizes that also in Theorem 3.2 one only needs R-bounds in the radial direction, whereas uniform boundedness in the angular direction is sufficient. This idea has appeared earlier in the isotropic context (e.g. [10, 14]), but it seems that the implication for homogeneous multipliers has not been explicitly stated before.

From results of Štrkalj and Weis [23] concerning the R-boundedness of sufficiently smooth functions, one can see that the assumptions of the Corollary imply that $\{m(\xi) : 1 \leq \varrho(\xi) \leq \lambda\}$, which by dilation invariance is the whole range of m, is R-bounded. (This would also follow from the conclusion of the Corollary and the theorem of Clément and Prüss [6].) Despite this implicit R-boundedness of m itself, it is worth noting that still no R-bounds are imposed on its derivatives.

In the isotropic situation, it has been shown that one can somewhat reduce the number of the required derivatives in Theorem 1.3 and its operator version by taking into account the *Fourier-type* of the Banach space X, see [10, 14]. One could easily adapt those arguments to the present situation, but we will not develop this idea here in detail.

4. Boundedness condition for convolution operators

We next derive a rather general sufficient condition for the boundedness of a convolution operator $f \mapsto K * f$ on $L^{\bar{p}}(\mathbf{R}^n, X)$, where K is an $\mathscr{L}(X)$-valued distribution. It is an anisotropic version of a result of Weis and the author [16], which in turn was inspired by some work of M. Girardi and Weis [10]. This general result will be applied in the next section to anisotropically homogenenous principal value distributions to deduce a generalization of Guliev's theorem.

Let us first introduce a smooth anisotropic Littlewood–Paley decomposition as follows: Given $\lambda > 1$ and $\bar{a} > 0$, we choose $\hat{\varphi} := \mathscr{F}\varphi \in \mathscr{D}(\mathbf{R}^n)$ which is the constant 1 on $\varrho(\xi) \leq 1$ and is supported where $\varrho(\xi) \leq \lambda$. We then define

$$\hat{\phi}_0(\xi) := \hat{\varphi}(\xi) - \hat{\varphi}(\lambda^{\bar{a}}\xi), \quad \hat{\phi}_j(\xi) := \hat{\phi}_0(\lambda^{-j\bar{a}}\xi), \quad \hat{\chi}_j := \hat{\phi}_{j-1} + \hat{\phi}_j + \hat{\phi}_{j+1},$$

so that

$$\begin{aligned} \operatorname{supp} \hat{\phi}_j &\subseteq \{\xi : \lambda^{j-1} \leq \varrho(\xi) \leq \lambda^{j+1}\} \\ &\subseteq \{\xi : \hat{\chi}_j(\xi) = 1\} \subseteq \operatorname{supp} \hat{\chi}_j \subseteq \{\xi : \lambda^{j-2} \leq \varrho(\xi) \leq \lambda^{j+2}\}. \end{aligned}$$

Let us also denote

$$\hat{\mathscr{D}}_0(\mathbf{R}^n) := \{\psi \in \mathscr{S}(\mathbf{R}^n) : \operatorname{supp} \hat{\psi} \subset \mathbf{R}^n \setminus \{0\} \text{ compact}\}.$$

With these notations at hand, we are going to prove:

4.1. **Proposition.** *Let X be a UMD space, $\bar{1} < \bar{p} < \bar{\infty}$, and let K be an $\mathscr{L}(X)$-valued tempered distribution, $K \in \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X))$. If*

$$\begin{aligned} \mathsf{E} \int_{\mathbf{R}^n} \Big\| \sum_j \varepsilon_j \Big(\phi_0 * \lambda^{-j|\bar{a}|} K(\lambda^{-j\bar{a}} \cdot)\Big)(y) \chi_j * f \Big\|_{L^{\bar{p}}(\mathbf{R}^n, X)} \log^n(e + \varrho(y)) \, \mathrm{d}y& \\ \leq B \|f\|_{L^{\bar{p}}(\mathbf{R}^n, X)}& \end{aligned}$$

for all $f \in \hat{\mathscr{D}}_0(\mathbf{R}^n) \otimes X$, then $[f \mapsto K * f] \in \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))$.

Proof. Given $f \in \hat{\mathscr{D}}_0(\mathbf{R}^n) \otimes X$, $g \in \hat{\mathscr{D}}_0(\mathbf{R}^n) \otimes X'$, we have

$$\langle g, K * f\rangle = \left\langle \tilde{K}' g, f \right\rangle = \sum_j \left\langle (\phi_j * \tilde{K}') * (\chi_j * g), \chi_j * f \right\rangle,$$

where the summation is finite, and $\tilde{K}'(\psi) := K(\tilde{\psi})'$, where $\tilde{\psi}(x) := \psi(-x)$. (Formally "$\tilde{K}'(x) = K(-x)'$", but K may not be pointwise defined.)

A computation shows that

$$(\phi_j * \tilde{K}') * (\chi_j * g)(x) = \int (\phi_0 * \lambda^{-j|\bar{a}|} \tilde{K}'(\lambda^{-j\bar{a}} \cdot))(y)(\chi_j * g)(x - 2^{-j\bar{a}} y)\,\mathrm{d}y$$

(the integrals here and below being over all $\mathbf{R}^n$), and hence

$$\begin{aligned}\langle g, K * f\rangle &= \int \sum_j \left\langle \chi_j * g(\cdot + \lambda^{-j\bar{a}} y), \phi_0 * \lambda^{-j|\bar{a}|} K(\lambda^{-j\bar{a}} \cdot)(y)\chi_j * f \right\rangle \mathrm{d}y \\ &= \int \mathsf{E} \left\langle \sum_i \varepsilon_i (\chi_i * g)(\cdot + \lambda^{-i\bar{a}}), \sum_j \varepsilon_j (\phi_0 * \lambda^{-j|\bar{a}|} K(\lambda^{-j\bar{a}} \cdot))(y)\chi_j * f \right\rangle \mathrm{d}y.\end{aligned}$$

By the inequalities of Hölder and Kahane, it follows that

$$\begin{aligned}|\langle g, K * f\rangle| \leq \int_{\mathbf{R}^n} &\mathsf{E}\Big\| \sum_j \varepsilon_j \chi_j * g(\cdot + \lambda^{-j\bar{a}} y)) \Big\|_{L^{\bar{p}'}(\mathbf{R}^n, X')} \times \\ &\times \mathsf{E}\Big\| \sum_j \varepsilon_j (\phi_0 * \lambda^{-j|\bar{a}|} K(\lambda^{-j\bar{a}} \cdot))(y)\chi_j * f \Big\|_{L^{\bar{p}}(\mathbf{R}^n, X)} \mathrm{d}y.\end{aligned}$$

The proof is finished by showing that

$$\begin{aligned}\mathsf{E}\Big\| \sum_j \varepsilon_j \chi_j * g(\cdot + \lambda^{-j\bar{a}} y)) \Big\| &\leq C \log^n(e + \varrho(y)) \mathsf{E}\Big\| \sum_j \varepsilon_j \chi_j * g \Big\| \\ &\leq C \log^n(e + \varrho(y)) \|g\|,\end{aligned}$$

where the norms are in $L^{\bar{p}'}(\mathbf{R}^n, X')$.

The second inequality follows from Theorem 3.2 upon making the easy verification that $m = \sum \varepsilon_j \hat{\chi}_j$ is an anisotropic multiplier of the kind treated there. The first estimate, in turn, follows from an n-fold application of

$$\mathsf{E}\Big\| \sum_j \varepsilon_j \chi_j * g(\cdot + \lambda^{-j a_i} y_i e_i)) \Big\| \leq C \log(e + \varrho(y)) \mathsf{E}\Big\| \sum_j \varepsilon_j \chi_j * g \Big\|,$$

where e_i is the ith standard unit vector. But the support of $\mathscr{F}[\chi_j * g] = \hat{\chi}_j \hat{g}_j$ in the ith variable is contained in $\lambda^{2a_i}[-\lambda^{j a_i}, \lambda^{j a_i}]$, so it follows from Lemma 10 of Bourgain [3] that the above inequality holds with $|y_i|$ in place of $\varrho(y)$ (cf. [10] for the transference of this result from Bourgain's periodic setting to the Euclidean space). Since $\log(e + |y_i|) \leq \log(e + \varrho(y)^{a_i}) \leq C \log(e + \varrho(y))$, we are done. □

4.2. **Corollary.** *Let X be a UMD space, and $K \in \mathscr{S}'(\mathbf{R}^n, \mathscr{L}(X))$. A sufficient condition for $[f \mapsto K * f] \in \mathscr{L}(L^{\bar{p}}(\mathbf{R}^n, X))$ for all $\bar{1} < \bar{p} < \bar{\infty}$ is*

$$\int_{\mathbf{R}^n} \mathscr{R}\Big[\Big(\phi_0 * \lambda^{-j|\bar{a}|} K(\lambda^{-j\bar{a}} \cdot)\Big)(x) : j \in \mathbf{Z}\Big] \log^n(e + \varrho(x))\,\mathrm{d}y < \infty.$$

In particular, if K has the homogeneity property $\lambda^{|\bar{a}|}K(\lambda^{\bar{a}}\cdot) = K$, then a sufficient condition is

$$\int_{\mathbf{R}^n} \|\phi_0 * K(x)\|_{\mathscr{L}(X)} \log^n(e + \varrho(x))\,\mathrm{d}x < \infty.$$

Proof. If the first assumption of the Corollary is satisfied, then the condition of Corollary 4.1 follows from the definition of R-boundedness and the $L^{\bar{p}}(\mathbf{R}^n, X)$-boundedness of the multipliers $m = \sum \varepsilon_j \hat{\chi}_j$ (Theorem 3.2). The second claim is an obvious special case of the first one. □

5. Extension of Guliev's theorem

We now consider a class of singular integral kernels, treated earlier in UMD lattices by Guliev [11, 12]. Let $K(x) = \Omega(x)\varrho(x)^{-|\bar{a}|}$, where the characteristic $\Omega \in C(\mathbf{R}^n \setminus \{0\}, \mathscr{L}(X))$ satisfies the following homogeneity, cancellation and continuity conditions: $\Omega(\lambda^{\bar{a}}x) = \Omega(x)$ for all $\lambda > 0$ and $x \in \mathbf{R}^n \setminus \{0\}$, and

$$\begin{aligned} &\int_{S^{n-1}} \sum_{i=1}^n a_i u_i^2 \Omega(u)\,\mathrm{d}\sigma(u) = 0, \qquad \int_0^1 \omega(t)\frac{\mathrm{d}t}{t} < \infty, \\ &\omega(t) := \sup\{\|\Omega(u) - \Omega(v)\|_{\mathscr{L}(X)} : u, v \in S^{n-1}, |u - v| \le t\}. \end{aligned} \tag{5.1}$$

The cancellation condition is better understood by comparing with the change-of-variable formula (2.2).

Associated to such a kernel is the principal-value distribution $\operatorname{pv} K$, which in turn defines a singular integral operator

$$\begin{aligned} Tf(x) &:= \operatorname{pv}\int_{\mathbf{R}^n} K(y)f(x-y)\,\mathrm{d}y := \lim_{\epsilon\downarrow 0}\int_{\varrho(y)>\epsilon} K(y)f(x-y)\,\mathrm{d}y \\ &= \lim_{\epsilon\downarrow 0}\int_\epsilon^\infty \frac{\mathrm{d}\lambda}{\lambda}\int_{S^{n-1}} \mathrm{d}\sigma(u)\sum_{i=1}^n a_i u_i^2 \Omega(u) f(x - \lambda^{\bar{a}}u). \end{aligned}$$

We are going to prove the following theorem by verifying the assumptions of Corollary 4.2 for this class of operators:

5.2. **Theorem.** *Let X be a UMD space and $\bar{1} < \bar{p} < \bar{\infty}$. Then the operators T described above are bounded on $L^{\bar{p}}(\mathbf{R}^n, X)$.*

For this we need an anisotropic version of a lemma from [16]:

5.3. **Lemma.** *Let $\varphi \in \mathscr{S}(\mathbf{R}^n)$ with $\int \varphi\,\mathrm{d}x = 0$. Then, there exist $(\psi_m)_{m=0}^\infty \subset \mathscr{D}(\mathbf{R}^n)$ so that $\varphi = \sum_{m=0}^\infty \psi_m$ with convergence in $\mathscr{S}(\mathbf{R}^n)$, and $\operatorname{supp}\psi_m \subseteq \{x : \varrho(x) \le 2^m\}$, $\int \psi_m\,\mathrm{d}x = 0$, and $\|\psi_m\|_{\alpha,\beta} = \mathscr{O}(2^{-mM})$ for all $M > 0$ and all Schwartz semi-norms $\|\psi\|_{\alpha,\beta} := \left\|x \mapsto x^\beta D^\alpha \psi(x)\right\|_\infty$.*

Proof. Since the construction is fairly standard (cf. [16]), only an outline is given here. We fix an $\eta \in \mathscr{D}(\mathbf{R}^n)$ which equals 1 on $\{x : \varrho(x) \le 2^{-1}\}$ and is supported on $\{x : \varrho(x) \le 1\}$. For $\lambda > 0$, let

$$\varphi_\lambda(x) := \eta(\lambda^{-\bar{a}}x)\Big(\varphi(x) - \frac{1}{\lambda^{|\bar{a}|}\int \eta\,\mathrm{d}y}\int_{\mathbf{R}^n} \varphi(y)(\eta(\lambda^{-\bar{a}}y) - 1)\,\mathrm{d}y\Big).$$

Then one may verify that $\|\varphi_\lambda - \varphi\|_{\alpha,\beta} = \mathscr{O}(\lambda^{-M})$ as $\lambda \to \infty$, and that $\psi_0 := \varphi_1$, $\psi_m := \varphi_{2^m} - \varphi_{2^{m-1}}$ for $m \ge 1$ will do. □

5.4. **Lemma.** *For $\varrho(x) \geq B\varrho(y)$, where B is an appropriate constant, we have*

$$\begin{aligned}\|K(x-y)-K(x)\| &= \Big\|\frac{\Omega(x-y)-\Omega(x)}{\varrho(x-y)^{|\bar{a}|}} + \Omega(x)\frac{\varrho(x)^{|\bar{a}|}-\varrho(x-y)^{|\bar{a}|}}{\varrho(x)^{|\bar{a}|}\varrho(x-y)^{|\bar{a}|}}\Big\| \\ &\leq \frac{1}{\varrho(x)^{|\bar{a}|}}\tilde{\omega}\Big(\frac{B\varrho(y)}{\varrho(x)}\Big),\end{aligned}$$

where $\tilde{\omega}$ is another increasing function like ω in (5.1).

Proof. Write first $\Omega(x-y)-\Omega(x) = \Omega(\varrho(x-y)^{-\bar{a}}(x-y)) - \Omega(\varrho(x)^{-\bar{a}}x)$, and

$$\begin{aligned}&\varrho(x-y)^{-\bar{a}}(x-y) - \varrho(x)^{-\bar{a}}x \\ &= \Big(\int_0^1 [a_i\varrho(x-ty)^{-a_i-1}y\cdot\nabla\varrho(x-ty)(x_i-ty_i) - \varrho(x-ty)^{-a_i}y_i]\,\mathrm{d}t\Big)_{i=1}^n.\end{aligned}$$

Differentiating the defining equality (2.1), we find that

$$\partial_j\varrho(x) = \frac{x_j\varrho(x)^{-2a_j}}{\sum_{i=1}^n a_i x_i^2\varrho(x)^{-2a_i}}\varrho(x).$$

Observe that the denominator here is between $a_{\min}$ and $a_{\max}$. Consequently,

$$\begin{aligned}|y\cdot\nabla\varrho(x)| &\leq \frac{\varrho(x)}{a_{\min}}\Big|\sum_{i=1}^n y_i x_i\varrho(x)^{-2a_i}\Big| \\ &\leq \frac{\varrho(x)}{a_{\min}}\cdot 1\cdot\Big(\sum_{i=1}^n(y_i\varrho(x)^{-a_i})^2\Big)^{1/2} \leq \frac{\sqrt{n}}{a_{\min}}\varrho(x)\Big(\frac{\varrho(y)}{\varrho(x)}\Big)^{a_{\min}},\end{aligned}$$

provided that $\varrho(y) \leq \varrho(x)$.

One can easily check that $\varrho(ty) \leq \varrho(y)$ for $t \in [0,1]$ and

$$\varrho(x+y) \leq 2^{1/a_{\min}}\max\{\varrho(x),\varrho(y)\};$$

hence $\varrho(x-ty) \leq 2^{1/a_{\min}}\varrho(x)$ if $\varrho(y) \leq \varrho(x)$ and $\varrho(x) \leq 2^{1/a_{\min}}\varrho(x-ty)$ if $\varrho(y) \leq 2^{-1-1/a_{\min}}\varrho(x)$.

Substituting these estimates, we obtain

$$\begin{aligned}&\big|\varrho(x-y)^{-\bar{a}}(x-y) - \varrho(x)^{-\bar{a}}x\big| \\ &\qquad\leq C\sum_{i=1}^n\Big[\varrho(x)^{-a_i-1}\varrho(x)\Big(\frac{\varrho(y)}{\varrho(x)}\Big)^{a_{\min}}|x_i| + \varrho(x)^{-a_i}|y_i|\Big] \leq C\Big(\frac{\varrho(y)}{\varrho(x)}\Big)^{a_{\min}},\end{aligned}$$

recalling that $|x_i| \leq \varrho(x)^{a_i}$. It follows that

$$\|\Omega(x-y)-\Omega(x)\| \leq \omega\Big[\Big(\frac{B\varrho(y)}{\varrho(x)}\Big)^{a_{\min}}\Big], \qquad \varrho(x) \geq B\varrho(y).$$

Using similar estimates we also deduce that

$$\begin{aligned}\Big|\varrho(x-y)^{|\bar{a}|} - \varrho(x)^{|\bar{a}|}\Big| &\leq \int_0^1 |\bar{a}|\,\varrho(x-ty)^{|\bar{a}|-1}\,|y\cdot\nabla\varrho(x-ty)|\,\mathrm{d}t \\ &\leq C\varrho(x)^{|\bar{a}|-1}\varrho(x)\Big(\frac{\varrho(y)}{\varrho(x)}\Big)^{a_{\min}},\end{aligned}$$

and the assertion follows with $\tilde{\omega}(t) = C[\omega(t^{a_{\min}}) + t^{a_{\min}}]$. □

Proof of Theorem 5.2. We estimate $\psi_m * K(x)$ in different ways for large and small x. We start from the case $\varrho(x) \geq B2^m$, where B is the constant from Lemma 5.4. Now

$$\begin{aligned}\|\psi_m * K(x)\| &= \Big\| \int_{\varrho(y)\leq 2^m} \psi_m(y)[K(x-y) - K(x)]\,\mathrm{d}y \Big\| \\ &\leq \int_{\varrho(y)\leq 2^m} |\psi_m(y)|\, \frac{1}{\varrho(x)^{|\bar{a}|}} \tilde{\omega}\Big(\frac{B\varrho(y)}{\varrho(x)}\Big)\,\mathrm{d}y \leq \frac{C2^{-mM}}{\varrho(x)^{|\bar{a}|}} \tilde{\omega}\Big(\frac{B2^m}{\varrho(x)}\Big).\end{aligned}$$

For $\varrho(x) \leq B2^m$ we write, exploiting the cancellation assumption,

$$\begin{aligned}\psi_m * K(x) = \lim_{\epsilon\downarrow 0} \int_\epsilon^1 \frac{\mathrm{d}\lambda}{\lambda} \int_{S^{n-1}} \mathrm{d}\sigma(u) \sum_{i=1}^n a_i u_i^2 \Omega(u) [\psi_m(x - \lambda^{\bar{a}} u) - \psi_m(x)] \\ + \int_1^\infty \frac{\mathrm{d}\lambda}{\lambda} \int_{S^{n-1}} \mathrm{d}\sigma(u) \sum_{i=1}^n a_i u_i^2 \Omega(u) \psi_m(x - \lambda^{\bar{a}} u).\end{aligned}$$

We observe that for $\lambda \in \,]0,1]$

$$\begin{aligned}\big|\psi_m(x-\lambda^{\bar{a}}u) - \psi_m(x)\big| &= \Big| \int_0^1 \sum_{i=1}^n \partial_i \psi_m(x - (t\lambda)^{\bar{a}} u)(-a_i t^{a_i - 1} \lambda^{a_i} u_i)\,\mathrm{d}t \Big| \\ &\leq C\, \|\nabla \psi_m\|_\infty \lambda^{a_{\min}},\end{aligned}$$

whereas for $\lambda > 1$

$$\begin{aligned}\big|u_i \psi_m(x - \lambda^{\bar{a}} u)\big| &= \lambda^{-a_i} \big|(\lambda^{a_i} u_i - x_i + x_i)\psi_m(x - \lambda^{\bar{a}} u)\big| \\ &\leq \lambda^{-a_{\min}} [\|y \mapsto |y|\, \psi_m(y)\|_\infty + \varrho(x)^{a_i} \|\psi_m\|_\infty].\end{aligned}$$

Substituting these estimates, we have for $\varrho(x) \leq B2^m$

$$\begin{aligned}\|\psi_m * K(x)\| &\leq C2^{-mM} \int_0^1 \lambda^{a_{\min}-1}\,\mathrm{d}\lambda \\ &\quad + C2^{-m(M+a_{\max})}(1 + \varrho(x)^{a_{\max}}) \int_1^\infty \lambda^{-a_{\min}-1}\,\mathrm{d}\lambda \leq C2^{-mM}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\mathbf{R}^n} \|\psi_m * K(x)\| \log^n(e + \varrho(x))\,\mathrm{d}x &\leq C2^{-mM} 2^{m|\bar{a}|} \log^n(e + 2^m) + \\ &\quad + C2^{-mM} \int_{B2^m}^\infty \tilde{\omega}\Big(\frac{B2^m}{\lambda}\Big) \frac{\mathrm{d}\lambda}{\lambda} \leq C2^{-m(M-|\bar{a}|-\epsilon)}.\end{aligned}$$

This is summable for $m = 0, 1, \ldots$ provided we take $M > |\bar{a}| + \epsilon$, and shows that $\|\phi_0 * K\| \in L^1(\log^n(e+\varrho(x))\,\mathrm{d}x)$. The theorem now follows from Corollary 4.2. □

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