Due on Monday March 25 by 14.15.

1. **Dropping a beacon into a black hole.** Consider a stationary observer hovering at constant spatial coordinates $(r_0, \theta_0, \varphi_0)$ outside ($r_0 > r_\mathrm{s}$) a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation with constant wavelength λ_em in the beacon rest frame.

   a) Calculate the coordinate velocity $\mathrm{d}r/\mathrm{d}t$ of the beacon as a function of r.

   b) Calculate the beacon three-velocity measured by another observer sitting still at fixed r as the beacon passes by. What does it approach as r approaches r_s?

   c) Calculate the wavelength λ_obs, measured by the observer at r_0, as a function of the radius r_em at which the radiation was emitted.

   d) Calculate the time t_obs at which a beam emitted by the beacon at radius r_em will be observed at r_0.

   e) Show that at late times, the redshift grows exponentially: $\lambda_\mathrm{obs}/\lambda_\mathrm{em} \propto e^{t_\mathrm{obs}/T}$. Give an expression for the time constant T in terms of the black hole mass M.

2. **Acceleration of a stationary observer.** Consider the same observer as in the previous problem.

   a) What is their four-acceleration $a^\alpha \equiv u^\beta \nabla_\beta u^\alpha$? (Take the connection coefficients, listed in the lecture notes, as given. No need to derive them.)

   b) What is the three-acceleration they measure?

   c) Assume that the observer is in a rocket that generates the force by producing perfectly collimated downward moving photons. How does the rocket's total energy depend on proper time?

3. **Horizon of an accelerated observer.** Consider an observer moving in Minkowski space with constant acceleration $a \equiv \sqrt{a^\alpha a_\alpha} > 0$.

   a) Solve for the position of the observer $x^\alpha(\tau)$ in Cartesian coordinates.

   b) Draw a picture of the observer's trajectory. Show that there are regions of Minkowski space from where the observer cannot receive signals, and regions to which they cannot send signals. (You can do this graphically.) Compare the causal structure to the case of a stationary observer in the Schwarzschild metric.

4. **Naked singularities.** Consider the Kerr–Newman metric given in section 7.2.4. Find the values of a and Q for which the above metric is smooth all the way to the singularity, meaning the singularity is visible to the outside, for

   a) $a = 0$ (Reissner–Nordström solution), and

   b) $Q = 0$ (Kerr solution).