Overview of differential equations with non-standard growth

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Abstract

Differential equations with non-standard growth have been a very active field of investigation in recent years. In this survey we present an overview of the field, as well as several of the most important results. We consider both existence and regularity questions. Finally, we provide a comprehensive list of papers published to date.

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I. Introduction

1. Overview of the history and the article

Differential equations with non-standard growth and corresponding function spaces with variable exponents have been a very active field of investigation in recent years. We counted over 250 publications from more than 100 authors this decade. The theory of variable exponent Lebesgue and Sobolev spaces has been surveyed in [6, 24], see also the upcoming monograph [5]. Mingione [137] has written an extensive exposition of regularity theory, including the non-standard growth case; Fan [70] summarized some results of his research group on the existence and multiplicity of solutions of eigenvalue problems. However, there has so far not been any comprehensive survey of differential equations with nonstandard growth including and comparing results on existence and regularity. It is the purpose of the present article to fill this gap. Although we were obviously forced to make choices about what to include in detail, we have tried to include in the bibliography all works published to-date (in international fora), with an indication in the text of where they fit in.

We start by sketching the development of the field. Initial interest was on function spaces. Variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by Orlicz [21]. However, investigations then focused on spaces with modular of the form

$$\varrho(f) = \int_\Omega \varphi(|f(x)|)\, dx,$$

which are now called Orlicz spaces. Notice that since φ does not depend explicitly on x, the case $|f(x)|^{p(x)}$ is not included. From here, the theory took a more abstract turn to spaces with modulars not given by a specific function, so-called modular spaces. They were first systematically studied by Nakano [19, 20]; later, a more explicit version of these spaces, modular function spaces, was investigated by Polish mathematicians, like Hudzik, Kamińska and Musielak, cf. [18]. Variable exponent Lebesgue spaces on the real line have been independently developed by the Russian researchers Sharapudinov and Tsenov.

In the mid-80s, Zhikov [161] started a new line of investigation that was to become intimately related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions. Kováčik [116, 117] also had a few results in the 80s and 90s, but they appear not to have been very influential. Zhikov's work was continued by Fan from around 1995 [68, 69, 83] and by Alkhutov since 1997 [35]. On a separate path, researchers in Italy, e.g. Marcellini [17], studied minimization problems with (p, q)-growth, i.e. minimizing

$$\int_\Omega F(x, |\nabla u|)\, dx$$

where $z^p \leqslant F(x, z) \leqslant z^q + 1$ for all $z \geqslant 0$. Note that we recover standard growth conditions if $p = q$. As a special case of (p, q)-growth we have $p(x)$-growth: here $F(x, z) \approx z^{p(x)}$ for some bounded function $p\colon \Omega \to (1, \infty)$. Regularity properties of such functionals were extensively studied by Acerbi, Mingione and their collaborators starting at the end of the 90s. Finally, Růžička and his collaborators [22, 23] studied equations with non-standard growth in the modeling of so-called electrorheological fluids, see also [29, 31, 40]. As another application, Chen, Levine, and Rao [51] suggested a model for image restoration, see also [26, 119].

In the remainder of the article we flesh out this sketch of the development of the field but focus mostly on newer developments. The structure of the presentation is as follows. We start by recalling the well-known p-growth differential equation and show its relation to the $p(\cdot)$-growth condition considered here. We divide the main content of the survey into two parts: existence and regularity.

Within the existence part, we first consider the one-dimensional case. Then we move on to $p(\cdot)$-type Laplace equations

$$-\Delta_{p(\cdot)}u = B(x,u)$$

with increasingly complex right hand side B, namely 0, then a power $|u|^{q(x)-2}u$ of the function, then a general function $f(x,u)$ in Sections 4, 5 and 6, respectively. In the last section of the part we deal with systems of two $p(\cdot)$-type equations.

The regularity theory for equations with non-standard growth seems more complete than the existence theory. In the first section of Part III we deal with harmonic functions and Harnack inequalities. Then we consider elliptic equations and corresponding minimizers, which cover almost all cases of existence from Part II. In the subsequent sections we deal with quasimimimizers, parabolic equations and weaker notions of solutions.

It should be emphasized that we do not aim at giving every detail about the results we cover. In order to give a clearer picture of the field we have standardized notation and expressed conditions and conclusions of theorems with appropriate modifications from the original sources where necessary. Sometimes, the assumptions that we present are not quite as general as they could be, and at other times we omit certain technical conditions (but we do explicitly state this). Therefore, we strongly suggest that the reader consult the original sources when using the results for further research.

There are a variety of related problems that we could not include in this paper; in particular, problems studied only in a few papers have not been included, since it is possible to get an overview of that research from the cited papers.

- To a minimization problem we can add an additional requirement that our minimizer be larger than a given function. In such *obstacle problems* we minimize, say, the energy

  $$\int_\Omega F(x,u,\nabla u)\,dx\,,$$

  or solve some equation, with the additional condition that $u \geqslant \psi$ with ψ given. In the variable exponent setting obstacle problems have been studied by Eleuteri and Habermann [66], Harjulehto, Hästö, Koskenoja, Lukkari and Marola [103], as well as Rodrigues, Sanchón and Urbano [143].

- The standard parabolic *porous medium equation* is

  $$\partial_t u - \operatorname{div}\left(|u|^\gamma \nabla u\right) = 0\,.$$

  If γ is a function, this may be considered a variable exponent problem. Such problems have been studied by Antontsev and Shmarev [41] and Henriques and Urbano [112]. Since the variable exponent acts on $|u|$ and not $|\nabla u|$, the problems are not of variable exponent type with respect to the Sobolev space.

- The following group includes equations where the variable exponent is not connected to differential operator. This includes the result by Diening and Růžička [62] on the divergence equation $-\operatorname{div} u = f$ where $f \in L^{p(\cdot)}$. A parabolic version of this problem was considered by Pinasco [142]. Also included in this category is the study by Calotă [49] of the equation

  $$-\Delta_q u = \lambda|u|^{p(x)-2}u + |u|^{q^*-2}u$$

for constant q. The same applies to *large solutions* with variable exponent right-hand side. Two such models,

$$\Delta u = u^{q(x)} \quad \text{and} \quad \Delta u = e^{q(x)u}$$

with infinite boundary values have been considered by Garcia Melian, Rossi, and Sabina [97, 98]. Similar boundary blow-up solutions have been studied by Zhang, Liu and Qiu [154, 157, 159] for equations with $\Delta_{p(\cdot)}$ on the left hand side.

- Habermann has studied *higher order problems* with variable exponent. In [99] the Hölder continuity of the highest derivative is proved, while [100] deals with partial regularity and higher integrability.
- Adamowicz and Hästö [32] took the strong form of the equation $-\operatorname{div}(|\nabla u|^{q-2}\nabla u) = 0$ as a starting point for a variable exponent theory. This leads to a more complicated PDE, which is not in divergence form, but the solutions seem to have some desirable geometric properties.

Definitions and notation. For a vector $\xi \in \mathbb{R}^n$, we denote by $(\xi_1, \dots, \xi_n)$ the co-ordinates, similarly for vector-valued functions, etc. By c we denote a generic constant, whose value may change between appearances even within a single line. The notation $f \approx g$ means that $\frac{1}{c} f \leqslant g \leqslant cf$. The distance from $x \in \mathbb{R}^n$ to $E \subset \mathbb{R}^n$ is denoted by $\operatorname{dist}(x, E)$.

An important convention is that *by $\Omega \subset \mathbb{R}^n$ we always denote a bounded domain.* A measurable function $p\colon \Omega \to [1, \infty]$ is called a *variable exponent*, and we denote for $A \subset \Omega$

$$p_A^+ := \operatorname*{ess\,sup}_{x \in A} p(x), \quad p_A^- := \operatorname*{ess\,inf}_{x \in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-.$$

The Hölder and Sobolev conjugates are defined point-wise, hence

$$p'(x) := \frac{p(x)}{p(x)-1} \quad \text{and} \quad p^*(x) := \frac{np(x)}{n-p(x)},$$

the latter for $p(x) < n$. When $p(x) \geqslant n$, we set $p^*(x) := +\infty$.

We say that p satisfies the *local* log-*Hölder continuity* condition if

$$|p(x) - p(y)| \leqslant \frac{c}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. If

$$|p(x) - p_\infty| \leqslant \frac{c}{\log(e + |x|)}$$

for some $p_\infty \geqslant 1$, $c > 0$ and all $x \in \Omega$, then we say p satisfies the log-*Hölder decay condition* (*at infinity*). If both conditions are satisfied, we simply speak of log-*Hölder continuity.* We denote by $\mathcal{P}^{\log}(\Omega)$ the class of variable exponents whose reciprocal is log-Hölder continuous.

We define a *(semi)modular* on the set of measurable functions by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx;$$

here we use the convention $t^\infty = \infty \chi_{(1,\infty]}(t)$ in order to get a left-continuous modular, see [5, Chapter 2] for details. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{L^{p(\cdot)}(\Omega)}(u/\lambda)$ is finite for some $\lambda > 0$. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \Big\{ \lambda > 0 : \ \varrho_{L^{p(\cdot)}(\Omega)}\big(\tfrac{u}{\lambda}\big) \leqslant 1 \Big\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Following [9], we define *the Sobolev space with zero boundary values*, $W_0^{1,p(\cdot)}(\Omega)$, as the closure of the set of compactly supported $W^{1,p(\cdot)}(\Omega)$-functions with respect to the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$. When smooth functions are dense, we could equivalently use the the closure of $C_0^\infty(\Omega)$.

2. The differential equation and our conventions

For a constant $q \in (1,\infty)$, the Dirichlet energy integral is

$$\int_\Omega |\nabla u|^q \, dx.$$

We minimize the energy integral among all Sobolev functions with the given boundary values. The Euler–Lagrange equation of this problem is the q-Laplace equation,

$$\operatorname{div}(|\nabla u|^{q-2}\nabla u) = 0,$$

which has to be understood in the weak sense. Note that we regain the classical Laplace equation $\Delta u = 0$ when $q = 2$. The energy integral and q-Laplace equation have been widely studied, see for example the monographs [10, 13, 16]. The q-Laplace equation is the prototype of a class of non-linear elliptic equations.

As we noted in the introduction, this problem has been generalized to a variable exponent case, which gives rise to differential equations with non-standard growth. If we change the exponent in the minimization problem from q to $p(x)$, we arrive at the energy

$$\int_\Omega |\nabla u(x)|^{p(x)} \, dx \quad \text{with Euler–Lagrange equation} \quad \operatorname{div}(p(x)|\nabla u(x)|^{p(x)-2}\nabla u) = 0.$$

Replacing q by $p(x)$ in the differential equation instead leads to the minimization problem

$$\int_\Omega \tfrac{1}{p(x)}|\nabla u(x)|^{p(x)} \, dx \quad \text{with Euler–Lagrange equation} \quad \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u) = 0.$$

We note that both of these formulations have been used by several researchers. Since we almost exclusively deal with bounded exponents p, the difference between the two forms is often of no importance.

To structure the presentation of the plethora of studies, we use the framework based on a more general equation with $p(\cdot)$-type growth. In other words, we consider weak solutions of the quasi-linear, second order differential equation

$$-\operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u) \tag{$\star$}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where A and B are suitable Carathéodory functions. Recall that $A : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a *Carathéodory function* if it is measurable in the first argument and continuous in the latter two arguments, similarly for B. The definition of a weak solution, means, as usual, that equality holds after we multiply the formal differential equation by a test function and integrate by parts. This leads to the following definition.

Definition 2.1. The function $u \in W^{1,p(\cdot)}(\Omega)$ is a *weak solution* of $(\star)$ if

$$\int_\Omega A(x,u,\nabla u)\cdot\nabla\varphi \, dx = \int_\Omega B(x,u,\nabla u)\varphi \, dx$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Note that we always consider weak solutions in the sense of the previous definition, also when considering some special cases of the equations, such as $B = 0$. Let us then define formally what we mean by the equation being $p(\cdot)$-type.

Definition 2.2. The Carathéodory function $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is *of $p(\cdot)$-type*, or *has $p(\cdot)$-type growth* if there exist $k \geqslant 0$ and $\lambda, \Lambda > 0$ such that the following conditions are satisfied for all $(x, u, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$:

1. $A(x, u, z) \cdot z \geqslant \lambda\,(k + |z|^2)^{\frac{p(x)-2}{2}}\,|z|^2$
2. $|A(x, u, z)| \leqslant \Lambda\,(k + |z|^2)^{\frac{p(x)-2}{2}}\,|z|$

The following condition is a special case of the previous one in the sense that every function with derivative $p(\cdot)$-type growth is also itself of $p(\cdot)$-growth.

Definition 2.3. The Carathéodory function $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with $A = (A_1, \ldots, A_n)$ is *of derivative $p(\cdot)$-type*, or *has derivative $p(\cdot)$-type growth* if there exist $k \geqslant 0$ and $\lambda, \Lambda > 0$ such that the following conditions are satisfied for all $(x, u, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$:

1. $A(x, u, 0) = 0$.
2. $\displaystyle\sum_{i,j} \frac{\partial A_j}{\partial z_i}(x, u, z)\xi_i\xi_j \geqslant \lambda\,(k + |z|^2)^{\frac{p(x)-2}{2}}\,|\xi|^2$ for all $\xi \in \mathbb{R}^n$.
3. $\displaystyle\sum_{i,j} \left|\frac{\partial A_j}{\partial z_i}(x, u, z)\right| \leqslant \Lambda\,(k + |z|^2)^{\frac{p(x)-2}{2}}$.

The most important example of a function A satisfying the $p(\cdot)$-growth conditions is $A(x, u, z) = |z|^{p(x)-2}z$. In this case we denote the differential operator by $\Delta_{p(\cdot)}$, i.e. $\Delta_{p(\cdot)}u := \operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u)$. Due to its importance, the differential equation with this left-hand side deserves a special tag:

$$-\Delta_{p(\cdot)}u = B(x, u, \nabla u). \tag{$\star\star$}$$

Throughout the article, we always use A and B to refer to the Carathéodory functions appearing in Equations ($\star$) and ($\star\star$), and by weak solution we always refer to a weak solution of these equations, unless specified otherwise.

A differential equation is usually related to a set of boundary value conditions. After a review of the non-standard growth literature, we found that the vast majority of results concern homogeneous Dirichlet or Neumann boundary conditions. Recall that the *Dirichlet boundary condition* with *boundary data* g means that $u - g \in W_0^{1,p(\cdot)}(\Omega)$, whereas *Neumann boundary condition* with boundary data h means that

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla\varphi\, dx = \int_\Omega B(x, u, \nabla u)\varphi\, dx + \int_{\partial\Omega} h\varphi\, dS$$

for all $\varphi \in C^\infty(\overline{\Omega})$. Also, *homogeneous* Dirichlet or Neumann boundary conditions mean that $g = 0$ or $h = 0$, respectively.

II. Existence and uniqueness

3. The one-dimensional case: $\big(A(x, u, u')\big)' = B(x, u, u')$

Let us start by stating the Dirichlet energy integral problem on an interval in the simplest possible case, that is, Equation ($\star\star$) with $B = 0$. Even in this simple case, where we obtain an explicit formula for the solution, we are able to give several examples which demonstrate the limitations on the generalizability of constant exponent results to the variable exponent case.

The first results are mainly from [102]. We assume that the interval under consideration is $(0,1)$. Since every element in the space $W^{1,p(\cdot)}(0,1)$ has a continuous representative, we assume that every Sobolev function is continuous. In this case $u \in W_0^{1,p(\cdot)}(0,1)$ if it can be continuously continued by 0 outside $(0,1)$. The extension is again denoted by u. Thus $u \in W_0^{1,p(\cdot)}(0,1)$ if and only if $u(0) = u(1) = 0$.

Definition 3.1. A function $u \in W^{1,p(\cdot)}(0,1)$ is a *minimizer* with boundary values 0 and $a > 0$ if $u(0) = 0$, $u(1) = a$ and

$$\int_0^1 |u'(y)|^{p(y)}\, dy \leqslant \int_0^1 |v'(y)|^{p(y)}\, dy$$

for every $v \in W^{1,p(\cdot)}(0,1)$ with $v(0) = 0$ and $v(1) = a$.

If p is a constant, then the minimizer is linear, $u(x) = ax$. The next example shows that the variable exponent adds some interest to this minimization question.

Example 3.2. We define $p(x) := 3\chi_{(0,1/2)} + 2\chi_{[1/2,1)}$. Suppose that $u \in W^{1,p(\cdot)}(0,1)$ is the minimizer for the boundary values 0 and $a > 0$. Denote $u(1/2) = b$. Then $u|_{(0,1/2)}$ is the solution to the classical energy integral problem with boundary values 0 and b, and $u|_{(1/2,1)}$ is the solution with boundary values b and a. Therefore these functions are linear. This u has Dirichlet energy $4b^3 + 2(a-b)^2$. The function $b \mapsto 2b^3 + (a-b)^2$ has a minimum at $b = (\sqrt{1+12a} - 1)/6$, which determines the minimizer of the variable exponent problem. A calculation shows that the minimizer is convex if $a > 2/3$, concave if $a < 2/3$ and linear for $a = 2/3$.

In general, by solving the Euler–Lagrange equation, we find that the possible minimizers are of the form

$$u(x) = \int_0^x \left(\frac{m}{p(y)}\right)^{\frac{1}{p(y)-1}} dy, \tag{3.3}$$

for some $m \geqslant 0$. It was shown in [102, Theorem 3.2] that there exists a unique minimizer for the boundary values 0 and a if and only if there exists $\tilde{m} \geqslant 0$ such that

$$a \leqslant \int_0^1 \left(\frac{\tilde{m}}{p(x)}\right)^{\frac{1}{p(x)-1}} dx < \infty. \tag{3.4}$$

This implies that we always obtain a solution to the equation if $p^- > 1$.

Proposition 3.5. *If $\Omega = (0,1)$ and $1 < p^- \leqslant p^+ < \infty$, then for every $a \geqslant 0$ there exists a unique minimizer with boundary values 0 and a.*

The following example shows that the Dirichlet energy integral does not always have a minimizer.

Example 3.6. For $p(x) := 1 + x$ the minimizer does not exists for large a. Indeed, if $m > 1$, then

$$\int_0^1 m^{\frac{1}{p(x)-1}}\, dx = \int_0^1 m^{\frac{1}{x}}\, dx \geqslant \max\{1, \log m\} \int_0^1 \frac{dx}{x} = \infty,$$

so that no minimizer exists for $a > \int_0^1 p(x)^{1/(1-p(x))} dx$ by (3.4).

When $A(x,u,z) = -|z|^{p(x)-2} z$ and $B(x,u,z) = g(x,u)$, Fan and Fan [78] considered the boundary value problem of ordinary differential systems

$$\begin{cases} \left(|u'|^{p(x)-2} u'\right)' = g(x,u), \ x \in I, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{3.7}$$

where $I := [0,T] \subset \mathbb{R}$. They assume that

(F1) $g \in C(I \times \mathbb{R}^N, \mathbb{R}^N)$,

(F2) $\exists r > 0$ such that $\langle g(x,z), z\rangle \geqslant 0$ for all $x \in I$ and $z \in \mathbb{R}^N$ with $|z| = r$.

Theorem 3.8 (Theorem 1.1, [78])**.** *If $p \in C(I)$, $p^- > 1$ and (F1)–(F2) hold, then problem (3.7) has at least one weak solution $u \in C^1(I, \mathbb{R}^N)$ such that $|u(x)| \leqslant r$ for all $x \in I$.*

More recently, Wang and Yuan [146] obtained some existence theorems for problem (3.7) in the case then B is of potential-type, i.e. $B(x,u) = \nabla g(x,u)$.

A more general problem, associated with the weight function w through $B(x,u,z) = w(x)g(x,u)$ and $A(x,z) = -w(x)|z|^{p(x)-2}z$, is

$$\begin{cases} \left(w(x)|u'|^{p(x)-2}u'\right)' = w(x)g(x,u) \text{ in } (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{3.9}$$

If (F1)–(F2) hold for $I = [a,b]$ and

$$w \in C^1(I, \mathbb{R}^N),\ w > 0,\ w^{\frac{1}{1-s}} \in L^1(I),\ s \in (1, p^-),$$

then this problem has a solution in $C^1(I, \mathbb{R}^N)$ [80, Theorem 1.1]. In particular, the solvability of (3.9) with $w(x) = x^{N-1}$ is applied for the existence of solutions of systems of $p(\cdot)$-Laplacian partial differential equations [80, Theorem 1.2].

In [152], for $A(x,z) = |z|^{p(x)-2}z$ and $B(x,u,z) = \frac{1}{x^{\theta(x)}}g(x,u)$, Zhang obtained sufficient conditions for the existence of oscillatory of solutions of the $p(\cdot)$-Laplacian equation:

$$-\left(|u'|^{p(x)-2}u'\right)' = \frac{1}{x^{\theta(x)}}g\left(x,u\right),\ x \in \mathbb{R}_+. \tag{3.10}$$

Compared with the previous results, the exponent p is here supposed to be more regular, namely $p \in C^1(\mathbb{R})$. Additionally, $g \in C(\mathbb{R}_+ \times \mathbb{R})$ is increasing with respect to the second variable, $g(x,u)u > 0$ and

$$0 < \liminf_{x\to+\infty} g(x,u)u \leqslant \limsup_{x\to+\infty} g(x,u)u < +\infty,\ u \in \mathbb{R}\setminus\{0\};$$

$\theta \in C(\mathbb{R}^+)$ and $\limsup_{x\to+\infty}\theta(x) < \liminf_{x\to+\infty}p(x)$. Then the positive solutions of (3.10), if they exist, are increasing when x is sufficiently large and tend to $+\infty$ as $x \to +\infty$ [152, Theorem 1.1]. In the special case when $g(x,u) = |u|^{q(x)-2}u$, for q with

$$\limsup_{x\to+\infty}\theta(x) < \liminf_{x\to+\infty}q(x)$$

and

$$1 < \limsup_{x\to+\infty}q(x) < \liminf_{x\to+\infty}p(x) \quad \text{or} \quad \lim_{x\to+\infty}q(x) = \lim_{x\to+\infty}p(x),$$

it is shown in [152, Theorem 1.2] that all of solutions of (3.10) are oscillatory provided p satisfies the log-Hölder decay condition.

Zhang, Qiu and Liu [160] considered a more general case of B including also the derivative of the solution: $B(x,u,u') = -f\left(x,u,(w(x))^{1/(p(x)-1)}u'\right)$ where $w(x)$ is the weight of operator $A(x,u,u') = w(x)|u'|^{p(x)-2}u'$. By Leray–Schauder's degree method, they studied the existence of solution of

$$\begin{cases} -\left(w(x)|u'|^{p(x)-2}u'\right)' + f\left(x,u,(w(x))^{1/(p(x)-1)}u'\right) = 0 \text{ in } (a,b), \\ u(a) = u(b) = 0 \end{cases} \tag{3.11}$$

in C^1 such that $w(x)|u'(x)|^{p(x)-2}u'(x)$ is absolutely continuous and has one-sided limits at the end-points.

Their assumptions for p, w and f are:

(Q1) $p, w \in C([a,b])$, $p > 1$, $w > 0$ on (a,b); and $w^{1/(1-p(\cdot))} \in L^1(a,b)$.

(Q2) $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is Caratéodory and $\sup_{y,z\in K} f(\cdot, y, z)$ is integrable on $[a,b]$ for every $K \Subset \mathbb{R}^2$.

The next theorem deals with sub- and supersolutions, which are functions satisfying the differential equation with an inequality rather than an equality, see Definition 8.4.

Theorem 3.12 (Theorem 1.1, [160])**.** *Let f satisfy (Q1)–(Q2), $v_1 \leqslant v_2$ be subsolutions, and $y_1 \leqslant y_2$ be supersolutions such that*

- $v_i(a), v_i(b) \leqslant 0$ *and* $0 \leqslant y_i(a), y_i(b)$, $i = 1, 2$,
- $v_i \leqslant y_i$, $i = 1, 2$, *and* $v_2 \nleqslant y_1$.

Suppose that there exists an increasing function $h \in C([1,\infty), [1,\infty))$ with $\int_1^\infty h(z^{1/(p^- -1)})^{-1}\, dz = +\infty$ such that

$$|f(x,u,z)| \leqslant c\,(1 + |z|)h(|z|) + c$$

for every $x \in [a,b]$ and $u \in [v_1(x), y_2(x)]$.

Then problem (3.11) *has at least three solutions u_1, u_2 and u_3 satisfying*

$$v_1 \leqslant u_1 \leqslant y_1, v_2 \leqslant u_2 \leqslant y_2, \text{ and } v_1 \leqslant u_3 \leqslant y_2.$$

In addition, it was shown that if a solution of problem (3.11) fulfills some more conditions, then this problem has at least three other solutions [160, Theorem 1.2]. The case of which there exist $(2N-1)$ solutions of problem (3.11) was also studied [160, Theorem 1.3]. It is worth noting out that in [160] the solutions are real-valued, whereas others have considered vector-valued solutions.

4. The $p(\cdot)$-Laplace equation: $-\Delta_{p(\cdot)} u = 0$

We start our investigation of the higher dimensional case by considering the simplest possible non-linear elliptic equation of $p(\cdot)$-type, viz. the $p(\cdot)$-Laplace equation. In other words, we consider the equation $(\star\star)$ with $B = 0$. It is easy to see that the solutions of the equation and minimizers of the $p(\cdot)$-Dirichlet energy coincide. We have two existence results, depending on whether the boundary value function is bounded [111, Theorem 2] or not [104, Theorem 5.2].

Theorem 4.1. *Let $1 < p^- \leqslant p^+ < \infty$ and $w \in W^{1,p(\cdot)}(\Omega)$. Then the Dirichlet $p(\cdot)$-energy minimizer with boundary values w exists if either of the following conditions holds:*

1. $w \in L^\infty(\Omega)$.
2. $p^+ < (p^-)^*$ *or* $p \in C(\overline{\Omega})$.

To the best of our knowledge, the only counter-example to existence in the case $p^- > 1$ is:

Proposition 4.2 (Theorem 3, [111])**.** *Let $n \geqslant 3$, $q_1 \in (1, n')$ and $q_2 \in (q_1^*, n)$. Then there exist a bounded domain Ω, a continuous exponent p with $p^- = q_1$ and $p^+ = q_2$, and a boundary value function $w \in W^{1,p(\cdot)}(\Omega)$ such that there does not exist a Dirichlet $p(\cdot)$-energy minimizer for the boundary value w.*

Note that if we had $q_2 \leqslant q_1^*$ in the previous proposition, then a minimizer would always exist, by Theorem 4.1; so in this sense Theorem 4.1 is optimal. Also worth noting is the fact that there thus far does not exist a single counter-example to existence in the two-dimensional case when $p^- > 1$.

One may also wonder whether the restrictions $1 < p^- \leqslant p^+ < \infty$ are necessary for existence. This question has been investigated in a few papers recently. If $p^- = 1$ or $p^+ = \infty$, then already the question of the appropriate definition of solutions is non-trivial. The approaches used have been based on approximation by exponents bounded away from the critical values.

Assume that p is a bounded coutinuous variable exponent with $p^- = 1$. For $\lambda > 1$ we set $p_\lambda := \max\{p, \lambda\}$; we can find a $p_\lambda(\cdot)$-solution or equivalently Dirichlet $p_\lambda(\cdot)$-energy minimizers u_λ for the given bounded boundary value function $f \in W^{1,p_\delta(\cdot)}(\Omega)$ when $\lambda \leqslant \delta$. We denote $Y := \{x \in \Omega : p(x) = 1\}$.

Theorem 4.3 (Theorem 7.1, [105]). *Let* p *be a bounded continuous exponent with* $p^- = 1$ *and let* (λ_j) *be a sequence decreasing to* 1. *Let* (u_{λ_j}) *be a sequence of* $p_{\lambda_j}(\cdot)$*-solutions in* Ω *with boundary value function* $f \in W^{1,p_\delta(\cdot)}(\Omega) \cap L^\infty(\Omega)$, *for some* $\delta > 1$. *Then there exists a subsequence* (λ_j) *and* $u \in L^\infty(\Omega)$ *such that*

1. $u_{\lambda_j} \to u$ *in* $L^{p(\cdot)}_{loc}(\Omega)$;
2. $u_{\lambda_j} \rightharpoonup u$ *in* $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$; *and*
3. u *is a* $p(\cdot)$*-solution in* $\Omega \setminus Y$.

If, in addition, p *is* log*-Hölder continuous and*

$$\lim_{x\to y} |p(x) - 1| \log \frac{1}{|x-y|} = 0$$

for every $y \in Y$, *then the limit function* u *belongs to a variable exponent mixed* BV*-Sobolev space in* Ω *and it minimizes a* BV*-Sobolev energy among all functions with the same boundary values.*

At the opposite limit, when $p \to \infty$, we have the following result, where $p^\lambda := \min\{p, \lambda\}$:

Theorem 4.4 (Theorem 6.2, [106]). *Let* $p\colon \Omega \to (n, \infty]$. *Assume that* $f \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *with* $\int_\Omega |\nabla f|^{p(x)} dx < \infty$. *Let* u_λ *be the Dirichlet* $p^\lambda(\cdot)$*-energy minimizer for the boundary value function* f. *Then there exist a sequence* (λ_i) *converging to infinity and a function* $u \in W^{1,p(\cdot)}(\Omega)$ *such that* (u_{λ_i}) *converges locally uniformly to* u *in* Ω. *Moreover,* $\int_\Omega |\nabla u|^{p(x)}\, dx$ *is finite and* $|\nabla u| \leqslant 1$ *almost everywhere in* $\{p = \infty\}$.

Manfredi, Rossi and Urbano [127] considered the case when $p = \infty$ in $D \subset \Omega$ and $p^+_{\Omega\setminus D} < \infty$. Using an increasing sequence of exponents, they found a sequence of solutions and showed that, under certain conditions, the limit function satisfies the infinity-Laplace equation $\Delta_\infty u = 0$ on D.

The infinity-Laplacian can be recovered as the limit $p \to \infty$ of the p-Laplace equation $\Delta_p u = 0$. Using this idea, Lindqvist and Lukkari [120] consider the equation which is the limit as $k \to \infty$ of the non-standard growth Laplacian $\Delta_{kp(\cdot)} u = 0$ which leads to an equation similar to, but not identical with, the infinity-Laplacian. Another paper along similar lines is [128].

We note here some studies which are similar in spirit to the above mentioned problems [56, 93, 161]. They deal with a sequence of problems, say,

$$-\Delta_{p_k(\cdot)} u_k = f_k(x, u_k)$$

where the functions p_k, f_k converge. The convergence of solutions to the solutions of the limit problem, and other related questions are considered.

5. The eigenvalue problem: $-\Delta_{p(\cdot)} u = \lambda\, |u|^{q(x)-2} u$

In this section we consider the *eigenvalue problem* by which we mean $(\star\star)$ with $B(x,u) = \lambda\, |u|^{q(x)-2} u$, explicitly,

$$-\Delta_{p(\cdot)} u = \lambda\, |u|^{q(x)-2} u \quad \text{in } \Omega.$$

We say that $\lambda \in \mathbb{R}^+$ is an *eigenvalue*, if this equation has a solution. The set of eigenvalues is denoted by Λ. We consider first the special case when the exponent function is the same on both sides, $q = p$.

Theorem 5.1 (Theorem 2.2, [82]). *Suppose that* $\partial\Omega$ *is smooth,* $p \in C(\overline{\Omega})$ *and* $p^- > 1$. *Then the set* Λ *of eigenvalues of the homogeneous Dirichlet eigenvalue problem is infinite and unbounded.*

An important question is whether the 0 is an accumulation point of Λ or not. In fact,

$$\inf \Lambda = \inf_{u \in W^{1,p(\cdot)}(\Omega)} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx},$$

so $\inf \Lambda > 0$ if and only if the modular version of the Poincaré inequality holds. However, this is almost never the case, cf. [82, Theorem 3.1–3.4] and [38]. A version of the of the Rayleigh quotient minimization problem better suited to the variable exponent context was considered in [74]

In [71, Theorem 3.1–3.3], Fan considered a corresponding homogeneous Neumann eigenvalue problem.

Theorem 5.2. *Suppose that* $\partial\Omega$ *is smooth,* $p \in C(\overline{\Omega})$ *and* $p^- > 1$. *Then the set* Λ *of eigenvalues of the homogeneous Neuman eigenvalue problem is infinite and unbounded, and* $0 \in \Lambda$.

Deng [59] considered the eigenvalue problem with $\lambda = -1$, and non-homogeneous Neumann boundary conditions.

$$|\nabla u|^{p(x)-2}\tfrac{\partial u}{\partial \gamma} = \mu\, |u|^{p(x)-2}u, \quad x \in \partial\Omega, \tag{5.3}$$

where γ is the outward unit normal of $\partial\Omega$ and $\mu \in \mathbb{R}$. He gave some sufficient conditions for when the infimum of all "eigenvalues" μ of the problem is zero and positive. He also proved the following.

Theorem 5.4. *Suppose that* $\partial\Omega$ *is smooth,* $p \in C(\overline{\Omega})$ *and* $p^- > 1$. *Then the eigenvalue equation with* $\lambda = -1$ *and boundary conditions* (5.3) *has has a solution for infinitely many values of* μ.

In [60], Deng considers a more general problem, where the right-hand side of the differential equation and the boundary condition contain additional functions of u.

We return now to the general eigenvalue equation with exponents $p \neq q$. Mihăilescu and Rădulescu [134] showed that the homogeneous Dirichlet eigenvalue problem ($\star\star$) has a continuous family of eigenvalues in a neighborhood of the origin:

Theorem 5.5. *Let* $\partial\Omega$ *be smooth. Suppose that* $p, q \in C(\overline{\Omega})$ *satisfy*

$$1 < q^- < p^- < q^+,\ p^+ < n \text{ and } (p^* - q)^- > 0.$$

Then there exists $\lambda_* > 0$ *such that every* $\lambda \in (0, \lambda_*)$ *is an eigenvalue.*

Similar results for more general elliptic equations were proved in [135]. Although it might seem that the assumption $(p^* - q)^- > 0$ would be necessary for existence, this is in fact not the case. Kurata and Shioji [118] have shown that in some cases the equation $\Delta u = u^{q(\cdot)}$ has a positive solution even if $(2^* - q)^- = 0$. Recently the problem was revisited by Fan [75]. With p as in Theorem 5.1 and $(p^* - q)^- > 0$, he proved that for every $t > 0$ the eigenvalue problem has arbitrarily large eigenvalues for eigenfunctions with $\varrho_{p(\cdot)}(\frac{1}{p(\cdot)}u) = t$.

Zhang [151] considered positive solutions of the eigenvalue problem in the case that $q < 1$. His result reads as follows:

Theorem 5.6. *Suppose that* $\partial\Omega$ *is smooth and that* $p, q \in C^1(\overline{\Omega})$, $1 < p^- \leqslant p^+ < +\infty$ *and* $q^+ < 1$. *Then the homogeneous Dirichlet eigenvalue problem* ($\star\star$) *has a unique positive weak solution* u_λ *for large enough* λ*; also,* u_λ *is increasing with respect to* λ.

He also considers the asymptotic behavior of the solutions. First of all, there exists a positive constant c such that $u_\lambda(x) \geqslant c\operatorname{dist}(x, \partial\Omega)$ as $x \to \partial\Omega$. Let us denote $d(x) := \operatorname{dist}(x, \partial\Omega)$ and $\Omega_\varepsilon := \overline{\{x \in \Omega \,|\, d(x) < \varepsilon\}}$ for $\varepsilon > 0$. In the following cases we have better estimates of the asymptotic behavior:

- If $q^+_{\Omega_{2\varepsilon}} \leqslant 0$, then $c\, d(x)^{\theta_1} \leqslant u(x) \leqslant c\, d(x)^{\theta_2}$ as $x \to \partial\Omega$, where

$$\theta_1 := \max_{x\in\Omega_\varepsilon} \frac{p(x)}{p(x) - q(x)} \quad \text{and} \quad \theta_2 := \min_{x\in\Omega_\varepsilon} \frac{p(x)}{p(x) - q(x)}.$$

- If $q^-_{\Omega_\varepsilon} < 0 < q^+_{\Omega_\varepsilon}$, then $c\, d(x) \leqslant u(x) \leqslant c\, d(x)^{\theta_1}$ as $x \to \partial\Omega$.
- If $0 < q^-_{\Omega_\varepsilon}$, then $c\, d(x) \leqslant u(x) \leqslant c\, d(x)^{\theta}$ as $x \to \partial\Omega$, for any fixed $\theta \in (0, 1)$.

6. The general equation: $-\operatorname{div}(A(x,u,\nabla u)) = B(x,u)$

We now turn to existence results in the most general case, which to date means functions B depending on x and u, but not ∇u. Although more than two dozen papers deal with equations of this type, progress has been far from incremental: essentially none of the results includes any of the other, earlier ones.

In our opinion the clearest result is by Fan and Zhang [81], dating to 2003. As a preliminary result we mention that the equation with homogeneous Dirichlet boundary data has a unique weak solution if $B = B(x) \in L^{(p^*)'(1+\varepsilon)}(\Omega)$ is independent of u [81, Theorem 4.2]. Sanchón and Urbano [145] have shown that the same conclusion holds for entropy solutions even if only $B \in L^1(\Omega)$ (see Section 11 for more details on their investigation).

An easy modification of the proof of [81, Theorem 4.3] gives the following result (the original result being for the $p(\cdot)$-Laplacian).

Theorem 6.1. *Suppose that A satisfies $p(\cdot)$-growth conditions for $p \in C(\overline{\Omega})$ and suppose that $|B(x,u)| \leqslant c + c|u|^{p^- - \varepsilon - 1}$. Then $(\star)$ has a weak solution for Dirichlet boundary values $g \in W^{1,p(\cdot)}(\Omega)$.*

To the best of our knowledge, this is the most general result which does not require a largeness assumption on B. Newer results, by contrast, place restrictions on the growth of B at the origin or at ∞; in particular, these results do not include as a special case $B = 0$. Also in this case, the old paper by Fan and Zhang [81] provide the most transparent result. We quote here Theorem 4.7 of the paper.

Theorem 6.2. *Let $p \in C(\overline{\Omega})$ with $1 < p^- \leqslant p^+ < \infty$. Suppose that the following three conditions hold.*

1. *$|B(x,u)| \leqslant c + c|u|^{p^*(x)-1-\varepsilon}$ for some $\varepsilon > 0$.*
2. *There exist $R > 0$ and $\theta > p^+$ such that $0 < \theta \int_0^u B(x,v)\,dv \leqslant u\,B(x,u)$ for all $u \in \mathbb{R} \setminus (-R,R)$ and $x \in \Omega$.*
3. *$B(x,u) = o\big(|u|^{p^+-1}\big)$ as $u \to 0$ uniformly in x.*

Then $(\star\star)$ has a weak solution in $W_0^{1,p(\cdot)}(\Omega)$.

In Theorem 4.8 of the same paper it is shown that there exist infinitely many solutions if the third condition is replaced by the assumption that B is odd in the second argument, see also [113]. A variant of this result was proved in [50]: there it is assumed that $B(x,u) = -\lambda(x)|u|^{p(x)-2}u + b(x,u)$, where $\lambda \approx 1$ and b satisfies the same conditions as B in the previously stated theorem.

It seems that in general the operator $u \mapsto -\Delta_{p(\cdot)}u + \lambda(\cdot)|u|^{p(\cdot)-2}u$, with $\lambda \approx 1$, is in some cases easier to deal with than the $p(\cdot)$-Laplacian itself. Antontsev and Shmarev [42] proved the existence of a weak solutions of $(\star)$ in $W_0^{1,p(\cdot)}(\Omega)$ when A satisfies $p(\cdot)$-type growth conditions and

$$B(x,u) = -\lambda(x)|u|^{r(x)-2}u + b(x), \ \lambda > c > 0.$$

Their paper is based on the assumption that p is log-Hölder continuous, although it is not clear whether this is needed for the existence part. They considered a parabolic version of the equation in [43], see Section 10 below for details. (Also in the parabolic case, it is not clear whether log-Hölder continuity is needed for existence.)

There is a wide selection of other papers dealing with this case; we feel the results are not mature and unifiable enough for a reasonable treatment here at this moment in time. Therefore we content ourselves with presenting the list of such papers: [57, 58, 73, 76, 77, 79, 87, 88, 91, 92, 113, 114, 123, 131, 148, 156].

It seems that more recent and advanced results, like those mentioned in thef previous paragraph, suffer from the short-coming that they do not connect naturally to the more restricted results because of their assumptions on B. Moreover, the conditions employed are sometimes so untransparent that even the authors do not see the inclusions. For instance, in [150] condition 2. of the above theorem is replaced by two new conditions, upon which existence is again shown. However, the new conditions of [150] directly imply 2., so there are no new cases included in this result.

The methods employed in papers along this line of investigation often yield strange restrictions. For instance, [34, 48, 130] deal only with the case $p \geqslant 2$ and $n \geqslant 3$ for various functions B. Similar results with more understandable assumptions were derived by Yao and Wang [149]. Their main result can be expressed as follows

Theorem 6.3. *Suppose that* $b \in C(\overline{\Omega} \times \mathbb{R})$ *is odd in the second argument and that*

$$\inf_{x} \liminf_{u \to 0} \frac{b(x,u)}{|u|^{p^- -1}} > -\infty.$$

Let $B(x,u) = \lambda(x)|u|^{r(x)-2}u + b(x,u)$ *and assume that* $\lambda, p, r \in C(\overline{\Omega})$*,* $a \approx 1$ *and* $r^+ < p^-$*. Then* $(\star\star)$ *has infinitely many weak solutions in* $W_0^{1,p(\cdot)}(\Omega)$ *converging to* 0 *.*

A somewhat similar result was proved by Papageorgiou and Rocha [140]: in their result r in the previous theorem is a constant, but it may be larger, namely, $r \in (p^+, (p^-)^*)$; similarly, the p^- in the $\liminf$ condition is larger. This paper is a generalization of [129].

Remark 6.4. All the conditions in the previous theorem can be localized. For instance, it suffices that $r(x_0) < p(x_0)$ and $a(x_0) > 0$. This is obtained by solving the problem in a subdomain. The assumptions on B may be restricted to the set $\overline{\Omega} \times (-\varepsilon, \varepsilon)$ by a truncation procedure.

This result has many interesting corollaries. For instance, we obtain infinitely many weak solutions of $(\star\star)$ in $W_0^{1,p(\cdot)}(\Omega)$ when

$$B(x,u) = \lambda\, |u|^{r(x)-2}u, \ \lambda > 0,$$

provided again $r(x_0) < p(x_0)$ at some point. A similar result with $\lambda = 1$ was proved for more general boundary conditions in [122]; a somewhat similar result is proved in [115].

Next we consider a result by Galewski [94]. He considered $(\star)$ in $W_0^{1,p(\cdot)}(\Omega)$ when

$$A(x,z) = a(x)|z|^{p(x)-2}z \quad \text{and} \quad B(x,u) = \tfrac{\partial}{\partial u}F(x,u) - b(x)|u|^{p(x)-2}u,$$

with $a, b \approx 1$. The function F is assumed to be convex in u; both the function and the partial derivative are assumed to be Carathéodory. The exponent is assumed to be uniformly continuous and the problem is studied in a bounded Lipschitz domain. An additional significant restriction is that $p^- > n$. We refer to [94] for the exact details on the conditions required. An important special case covered is $B(x,u) = f(x) - b(x)|u|^{p(x)-2}u$ in which case any $f \in L^\infty$ satisfies the conditions. In [95, 96], the same author studies other equations with slightly different functions A. A variant of this with the same A and

$$B(x,u) = k(x)u^{q(x)-2}u - |u|^{p(x)-2}u$$

is studied in [39], while positive solutions of $(\star\star)$ with singular coefficient

$$B(x,u) = k(x)u^{q(x)} + h(x)u^{-r(x)}$$

is considered in [54].

Let us mention some results on $(\star)$ when A is not of $p(\cdot)$-growth but the right-hand side instead contains the sum of two $p(\cdot)$-type Laplacian operators. Note that for constant exponents, this is a typical example of an equation with (p,q)-growth.

In [132], continuing on the path of [133, 136], Mihăilescu studied a homogeneous Dirichlet problem given by

$$\Delta_{p_1(\cdot)}u + \Delta_{p_2(\cdot)}u = s\left(-\lambda\, |u|^{m(x)-2}u + |u|^{q(x)-2}u\right), \tag{6.5}$$

where $p_1, p_2 \in C(\overline{\Omega})$, $m := \max\{p_1, p_2\}$ and q are variable exponents and $\lambda, s \in \mathbb{R}$. By a $\mathbb{Z}_2$-symmetric version for even functionals of the Mountain Pass Lemma, he proved the following result.

Theorem 6.6. *Suppose that $\partial\Omega$ is a smooth, $p_1^-, p_2^- > 1$, $m^+ < q^-$ and $q^+ < (m^-)^*$.*

- *When $s = -1$, (6.5) has a non-trivial weak solution in $W_0^{1,p(\cdot)}(\Omega)$ for all sufficiently large positive λ.*
- *When $s = 1$, (6.5) has infinitely many weak solutions in $W_0^{1,p(\cdot)}(\Omega)$ provided $p_1, p_2 \geqslant 2$.*

Remark 6.7. The mathematical content of [64] and [132] is the same, and even the formating is nearly identical. We contacted the authors of the two papers about this issue to ask how this is possible. Mihăilescu claimed precedence and offered a variety of supporting evidence for his claim, whereas the other author did not reply to our inquiry.

Finally we mention that Mihăilescu [130] and Boureanu and Mihăilescu [47] have studied existence of weak solutions of $(\star\star)$ when $B = B(u)$ under homogeneous Neumann boundary condition. On a similar note, Rovenţa [144] has studied asymptotic boundary behaviour estimates for the equation $(\star\star)$ with $B(x,u) = g(x)f(u)$ and infinity boundary values.

Remark 6.8. The paper [90], written already 2002, is to date only one to address the general equation $(\star)$ with $B(x,u,\nabla u)$ depending also on ∇u. Unfortunately, this result is based on the modular Poincaré inequality. It is claimed in [90, Lemma 2.14] that this inequality holds for all bounded exponents, but this is false: in fact, the modular Poincaré inequality is equivalent to the positivity

$$\inf_u \frac{\varrho_{p(\cdot)}(|\nabla u|)}{\varrho_{p(\cdot)}(u)} > 0,$$

of the Rayleigh quotient of the eigenvalue problem. But it was shown by Allegretto [38] that this holds essentially only if p is monotone. Therefore, the result of [90] is of limited interest.

7. Systems of equations

In this section we consider systems of equations; most of the existence results to date deal only with $p(\cdot)$-Laplace type equations, generalizing $(\star\star)$. The general form of such equations is

$$\begin{cases} -\Delta_{p(\cdot)} u = B_1(x,u,v), \\ -\Delta_{q(\cdot)} v = B_2(x,u,v). \end{cases} \tag{7.1}$$

The problem is accompanied by some boundary conditions; in fact, all the investigations so far deal only with homogeneous Dirichlet boundary conditions. Weak solutions of the system are defined in the obvious way. We also mention the significant result by Diening and Růžička [63, Theorem 21] which says that the system governing the flow of electrorheological fluids has a strong solution on a short time scale. We say a few words more about this problem in the context of parabolic equations in Section 10.

We first consider (7.1) in the case $p = q$. Zhang [153, 155, 158] considered systems with the following structure:

(Z1) $B_1(x,u,v) = f(v)$ and $B_2(x,u,v) = g(u)$.

(Z2) $f, g \in C^1([0,+\infty))$ are non-decreasing functions with

$$f(u), g(v) \to +\infty \text{ as } u, v \to +\infty.$$

(Z3) There exists $M > 0$ such that

$$\frac{f\big(Mg(w)^{\frac{1}{p^- -1}}\big)}{w^{p^- -1}} \to 0 \text{ as } w \to +\infty.$$

His newest result from [158] includes the older ones and is as follows:

Theorem 7.2. *Let Ω have C^2 boundary and let $p \in C^1(\overline{\Omega})$ with $p^- > 1$. If (Z1)–(Z3) hold, then problem* (7.1) *has a weak positive solution in $W_0^{1,p(\cdot)}(\Omega)$.*

Zhang also studied problem (7.1) with respect to more complicated right-hand sides, namely

$$B_1(x,u,v) = \lambda^{p'(x)} f(v) \quad \text{and} \quad B_2(x,u,v) = \lambda^{p'(x)} g(u)$$

in $\Omega = (-1,1)$, or

$$B_1(x,u,v) = \lambda\,(v^{q_1(x)} + f(x)) \quad \text{and} \quad B_2(x,u,v) = \lambda\,(u^{q_2(x)} + g(x)).$$

Let us then consider problem (7.1) without the assumption $p = q$. For Ω a ball in $\mathbb{R}^n$ Afrouzi and Ghorbani [33] considered a system with the following structural conditions:

(A1) $B_1(x,u,v) = \lambda\,(h(x)a(u) + f(v))$ and $B_2(x,u,v) = \lambda\,(h(x)b(v) + f(u))$;

(A2) $h \in C(\overline{\Omega}, \mathbb{R}^+)$;

(A3) $a, b \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ are non-decreasing such that

$$\frac{a(u)}{u^{p^- -1}}, \frac{b(v)}{v^{p^- -1}} \to 0 \text{ as } u, v \to +\infty;$$

as well as (Z2) and (Z3) from before. They prove the existence of a positive weak solution with p as in Theorem 7.2.

El-Hamidi [67] considered (7.1) with the following structure:

(H1) $B_1(x,u,v) = \frac{\partial F}{\partial u}(x,u,v)$ and $B_2(x,u,v) = \frac{\partial F}{\partial v}(x,u,v)$.

(H2) $F(\cdot,u,v) \in C(\overline{\Omega}) \cap C^1(\Omega)$ for $u, v \in \mathbb{R}$.

(H3) $F(x,u,v) \leqslant C_1 + C_2|u|^{p_1(x)} + C_3|v|^{q_1(x)} + C_4|u|^{\alpha(x)}|v|^{\beta(x)}$ in $\Omega \times \mathbb{R}^2$, where $p_1, q_1, \alpha, \beta \in C_+(\overline{\Omega})$ satisfy

$$p_1 < p^*,\ q_1 < q^*,\ \frac{\alpha}{p^*} + \frac{\beta}{q^*} < 1 \text{ in } \overline{\Omega},$$
$$p^+ < p_1^-,\ p^+ < \alpha^-,\ q^+ < q_1^-,\ q^+ < \beta^-.$$

(H4) There exist $\theta_1 > p^+$, $\theta_2 > q^+$ and $K \Subset \mathbb{R}^2$ such that

$$0 < F(x,u,v) \leqslant \frac{u}{\theta_1}\frac{\partial F}{\partial u}(x,u,v) + \frac{v}{\theta_2}\frac{\partial F}{\partial v}(x,u,v)$$

for all $x \in \Omega$ and $(u,v) \in \mathbb{R}^2 \setminus K$.

(H5) $F(x,u,v) = O(|u|^{p^+} + |v|^{q^+})$ as $(u,v) \to (0,0)$, uniformly with respect to $x \in \Omega$.

(H5') F is even in the second and in the third variables.

Theorem 7.3. *Suppose that $\partial\Omega$ is smooth and $p, q \in C(\overline{\Omega})$ with $p^-, q^- > 1$, and that F satisfies (H1)–(H4).*

- *If (H5) is satisfied, then* (7.1) *has at least one weak solution in $W_0^{1,p(\cdot)}(\Omega)$.*
- *If (H5') is satisfied, then* (7.1) *has infinitely many weak solutions in $W_0^{1,p(\cdot)}(\Omega)$.*

A variant of El-Hamidi's theorem was considered in [147]. There the Laplacian $-\Delta_{p(\cdot)}u$ is replaced by $-\Delta_{p(\cdot)}u + |u|^{p(\cdot)-2}u$ in a change which we saw was also frequent in the case of only one equation. Another variant is due to Ogras, Mashiyev, Avci and Yucedag [139]: they considered $\Omega = \mathbb{R}^n$ without boundary conditions; and assumed stronger growth conditions on the right-hand sides of the system and more regularity of F than (H2).

III. Regularity theory

8. Harmonic and superharmonic functions

We again start with the simple case of one-dimensional minimizers in the sense of Definition 3.1, but now consider regularity. Using the explicit formula for the derivative from Section 3 we obtain:

Theorem 8.1. *If $1 < p^- \leqslant p^+ < \infty$, then the minimizers are bi-Lipschitz continuous. The derivative of the minimizer is α-Hölder continuous if and only if the exponent p is α-Hölder continuous.*

The next result shows that if we relax the assumption $p^- > 1$ then we are liable to lose regularity of the minimizer.

Example 8.2. Let $p(x) := 1 + (\log(1/x))^{-1}$ on $(0,1)$. We show that the minimizers are α-Hölder continuous for some α depending on $a > 0$. The derivatives of the minimizers are not even uniformly continuous for large a.

We have

$$\int_0^1 m^{\frac{1}{p(x)-1}}\,dx = \int_0^1 x^{-\log m}dx = \frac{1}{1-\log m},$$

provided $m < e$, so condition (3.4) is satisfied. Therefore the derivative of the minimizer is $(m/p(x))^{1/(p(x)-1)}$ for some $m > 0$; we notice that the derivative is unbounded when $m > 1$, and hence not uniformly continuous. Also,

$$\begin{aligned} |u(x) - u(y)| &= \Big| \int_y^x x^{-\log m} p(x)^{-1/(p(x)-1)} dx \Big| \approx \frac{x^{1-\log m} - y^{1-\log m}}{1-\log m} \\ &\leqslant \frac{(x-y)^{1-\log m}}{1-\log m} \end{aligned}$$

for $0 < y < x < 1$. We see that u is $(1-\log m)$-Hölder continuous; u is of course also locally Lipschitz continuous in $(0,1)$, but it is not Lipschitz on $[0,1]$.

In the one-dimensional context we can easily derive the following Harnack inequality from our formula for the solution:

Theorem 8.3 (Harnack's inequality)**.** *Let $1 < p^- \leqslant p^+ < \infty$. If $u \in W^{1,p(\cdot)}(0,1)$ is a non-negative minimizer, then*

$$\sup_{y\in B(x,r)} u(y) \leqslant \Big(1 + c(p^-,p^+) \max\Big\{ m^{\frac{1}{p^--1} - \frac{1}{p^+-1}}, m^{\frac{1}{p^+-1} - \frac{1}{p^--1}} \Big\}\Big) \inf_{y\in B(x,r)} u(y)$$

for every $x \in (0,1)$ and every r with $B(x,2r) \subset (0,1)$, where the constant m is from (3.3).

In [107] it is shown that even in case of the very nice exponent

$$p(x) := \begin{cases} 3 & \text{for } 0 < x \leqslant \frac{1}{2}; \\ 3 - 2(x - \frac{1}{2}) & \text{for } \frac{1}{2} < x < 1 \end{cases}$$

the constant in the Harnack inequality depends on the minimizer, i.e. the inequality $\sup u \leqslant c \inf u$ does not hold with absolute constant c.

We then move on to the higher dimensional case, and consider regularity in $\Omega \subset \mathbb{R}^n$; we start by considering Equation ($\star$) with $A = A(x, z)$ and $B = 0$. We assume that A has $p(\cdot)$-type growth with $k = 0$ and

$$\big(A(x,z) - A(x,\xi)\big) \cdot (z - \xi) > 0$$

for all $x \in \Omega$ and $z, \xi \in \mathbb{R}^n$ with $z \neq \xi$. The most general Harnack inequality to-date, for A-solutions, was proved in [5, Chapter 12]; in what follows we quote also the original sources which dealt with the case $A(x,z) = |z|^{p(x)-2}z$, i.e. Equation ($\star\star$). It is often convenient to relax the concept of solution by replacing the equality in the definition by an inequality. This leads to the following:

Definition 8.4. A function $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a (weak) *A-supersolution* in Ω, if

$$\int_\Omega A(x, \nabla u(x)) \cdot \nabla \psi \, dx \geqslant 0 \tag{8.5}$$

for every non-negative test function $\psi \in W^{1,p(\cdot)}_0(\Omega)$. A function u is a *A-subsolution* in Ω if $-u$ is a A-supersolution in Ω.

Note that u is a weak solution in Ω if it is both a super- and a subsolution in Ω. For instance, minimizers of the Dirichlet energy integral with an obstacle are supersolutions [103]. Alkhutov has shown in [35] that bounded supersolutions satisfy Harnack's weak inequality and solutions satisfy Harnack's inequality. Harjulehto, Kinnunen and Lukkari [107] extend his result to unbounded supersolutions. Moser's iteration is used in both papers. The key estimate in Moser's iteration is the Caccioppoli estimate. Several different versions of the Caccioppoli estimate can be found from the literature, see for example [35, Lemma 1.1], or [37, Proposition 6.1], or [105, Lemma 5.3].

Lemma 8.6 ([107], Caccioppoli estimate)**.** *Let B be a ball and $1 < p^- \leqslant p^+ < \infty$. Suppose that u is a nonnegative A-supersolution in B. Let $\eta \in C^\infty_0(B)$ with $0 \leqslant \eta \leqslant 1$. For every $\gamma < 0$ there exists $c > 0$ such that*

$$\int_\Omega u^{\gamma-1} |\nabla u|^{p(x)} \eta^{p^+} \, dx \leqslant c \int_\Omega u^{\gamma + p(x) - 1} |\nabla \eta|^{p(x)} \, dx.$$

In the proof of Harnack's inequality the Caccioppoli estimate is used for the function $u + R$, where R is a radius of a ball. The extra term R is used to handle negative powers which come from putting together the variable exponent Caccioppoli estimate and a constant exponent modular form Sobolev inequality.

Theorem 8.7 ([107], The weak Harnack inequality)**.** *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$ and let B be a ball with $4B \Subset \Omega$. Assume that u is a nonnegative A-supersolution and $s > p^+_{4B} - p^-_{4B}$. Then there exists q_0 such that*

$$\left(\frac{1}{|2B|} \int_{2B} u^{q_0} \, dx \right)^{\frac{1}{q_0}} \leqslant c \left(\operatorname*{ess\,inf}_B u(x) + R \right),$$

where c depends on n, $p(\cdot)$ and $L^{ns}(4B)$-norm of u.

Since the exponent p is uniformly continuous, we can take for example $s = \frac{1}{n} p^-_{B(x,4R)}$ by choosing R small enough. Thus the constants in the estimates are finite for all supersolutions u on a scale depending only on p.

In a recent paper, Harjulehto, Hästö and Latvala [106], combining techniques from [35] and [107], prove the following result which includes the case $p^- = 1$.

Theorem 8.8. *Let $p \in \mathcal{P}^{\log}(\Omega)$ be bounded and let B be a ball such that $4B \Subset \Omega$ and let u be a A-solution in Ω. If $s > p^+_{4B} - p^-_{4B}$, then*

$$\operatorname*{ess\,sup}_B |u| \leqslant c \left(\left(\frac{1}{|2B|} \int_{2B} |u|^t \, dx \right)^{\frac{1}{t}} + R \right)$$

for every $t > 0$. The constant c depends only on n, p, t and $L^{ns}(4B)$-norm of u.

Theorems 8.7 and 8.8 yield the following full version of Harnack's inequality.

Theorem 8.9 ([35, 107], Harnack's inequality)**.** *Let* $p \in \mathcal{P}^{\log}(\Omega)$ *with* $1 < p^- \leqslant p^+ < \infty$. *Let* u *be a* A*-solution which is nonnegative in* $B(x, 4R) \Subset \Omega$. *Then*

$$\sup_{x \in B(x,R)} u(x) \leqslant c \Big(\inf_{x \in B(x,R)} u(x) + R \Big),$$

where the constant c *depends on* n, p *and the* $L^{ns}(B(x, 4R))$*-norm of* u.

This Harnack inequality implies, as pointed in [35], that solutions are locally Hölder continuous. Since there is the extra term R on the right-hand side, the inequality does not imply the strong maximum principle (by the *strong maximum principle* we mean that a solution does not attain its minimum or maximum). It is, however, possible to obtain this principle by other means, and this was done by Fan, Zhao and Zhang [86] when $p \in C^1(\overline{\Omega})$ with $1 < p^- \leqslant p^+ < \infty$. Their proof is based on choosing a suitable test function. Very recently, a maximum principle for more general equations of type ($\star$) was proved by Fortini, Mugnai and Pucci in [89]. There are also some results limiting the regularity that can be expected in general: examples in [110, 162] show that minimizers of the Dirichlet energy integral need not even be continuous in the variable exponent case if the exponent is not sufficiently regular.

In [37], Alkhutov and Krashennikova study boundary regularity of solutions. They prove a Wiener type capacity condition for boundary regularity and study other geometric conditions for boundary regularity in the variable exponent case. They also study capacity potentials. We also mention the article [36] which deals with the case of the exponent p having point-wise irregularities.

Superharmonic functions are needed to develop a potential theory.

Definition 8.10. We say that a function $u : \Omega \to (-\infty, \infty]$ is $p(\cdot)$*-superharmonic* in Ω if

1. u is lower semicontinuous;
2. u is finite almost everywhere; and
3. The comparison principle holds: if h is a solution in $D \Subset \Omega$, continuous in $\overline{D}$ and $u \geqslant h$ on ∂D, then $u \geqslant h$ in D.

Every supersolution in Ω which satisfies

$$u(x) = \operatorname{ess} \liminf_{y \to x} u(y)$$

for every $x \in \Omega$ is superharmonic in Ω. On the other hand if u is a superharmonic function then $\min\{u, \lambda\}$ is a supersolution for every λ [103]. In [124], Lukkari showed that if μ is a finite Radon measure, then weak solutions of

$$-\operatorname{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u) = \mu$$

are superharmonic. Higher integrability properties of superharmonic functions and their point-wise defined "gradient" is studied in [103]. It is also known that the infinity set of a superharmonic function is small [107]. Fine continuity of superharmonic functions is studied by Harjulehto and Latvala [109]. Lukkari, Maeda and Marola [126] have derived point-wise inequalities for superharmonic functions in terms of Wolff's potential and Lukkari [125] has studied singular solutions.

We finally mention a paper by Pastukhova and Zhikov [141] which deals with higher integrability of the gradient $|\nabla u|$ in the case of less regular exponent than $p \in \mathcal{P}^{\log}$.

9. General elliptic problems of $p(\cdot)$-Laplace type

In this section we deal with regularity results for the general equation ($\star$), in particular in the special case where also the right-hand side is of "divergence form", i.e. $B(x) := -\operatorname{div}(|F(x)|^{p(x)-2}F(x))$ Additionally, we study minimizers for energies of the form

$$\int_\Omega F(x,u,\nabla u)\,dx \quad \text{or} \quad \int_\Omega F(x,\nabla u)\,dx\,. \tag{9.1}$$

The most general and widest-grasping result, as far as the form of the equation is concerned, is given by Fan [72]. This is for equations of the quasi-linear form ($\star$). In addition to this, however, we shall present some less general versions since:

- Some of the existence questions related to Fan's formulation are to date open problems.
- Fan needs to make the additional assumption of boundedness, which might not be necessary, but this seems to date also an open problem.
- Fan's assumptions on the exponent p are stricter than in some other results, in particular, the usual log-Hölder assumption is not sufficient.

Let us set the structural conditions. We assume A has derivative $p(\cdot)$-type growth (cf. Definition 2.3) and $|B(x,u,z)| \leqslant c(1+|z|^{p(x)})$. We also assume A has continuity modulus

$$\begin{aligned}|A(x_1,u_1,\eta) &- A(x_2,u_2,\eta)| \\ &\leqslant \Lambda\left(|x_1-x_2|^{\beta_1}+|u_1-u_2|^{\beta_2}\right)\left(\left(k+|\eta|^2\right)^{\frac{p(x_1)-2}{2}}+\left(k+|\eta|^2\right)^{\frac{p(x_2)-2}{2}}\right)\left(1+\log\left(k+|\eta|^2\right)\right)|\eta|,\end{aligned} \tag{9.2}$$

with $\beta_i > 0$ and for all $(x_i,u_i,\eta) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n\setminus\{0\}$.

Fan provides the following three theorems for Equation ($\star$). The first theorem considers local behaviour, the remaining two are for boundary value problems. By a *local weak solution* we mean that the function satisfies the other condition of a weak solution, but may belong to $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ instead of $W^{1,p(\cdot)}(\Omega)$. Moreover, Fan considers *generalized solutions* in which one tests the equation only with bounded test functions.

Theorem 9.3. *Suppose* $u \in W^{1,p(\cdot)}_{loc}(\Omega)\cap L^\infty(\Omega)$ *is a local weak solution of Equation* ($\star$). *If* A *has derivative* $p(\cdot)$*-type growth,* $|B(x,u,z)| \leqslant c\,(1+|z|^{p(x)})$, *condition* (9.2) *holds, and* p *is Hölder continuous, then* $u \in C^{1,\alpha}(\Omega)$ *for some* $0<\alpha<1$.

We stress that we indeed mean Hölder continuity here, not log-Hölder. The same applies for Theorems 9.4 and 9.5.

Theorem 9.4. *Suppose* $u \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ *is a weak solution of Equation* ($\star$) *with Dirichlet boundary data* $g \in W^{1,p(\cdot)}_0(\Omega)$. *If* A *has derivative* $p(\cdot)$*-type growth,* $|B(x,u,z)| \leqslant c\,(1+|z|^{p(x)})$, *condition* (9.2) *holds,* p *is Hölder continuous, and* $g \in C^{1,\gamma}$ *for some* $0<\gamma<1$, *then* $u \in C^{1,\alpha}(\overline{\Omega})$ *for some* $0<\alpha<1$.

Theorem 9.5. *Suppose* $u \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ *is a weak solution of Equation* ($\star$) *with Neumann boundary data*

$$\int_\Omega A(x,u,\nabla u)\nabla\varphi\,dx = \int_\Omega B(x,u,\nabla u)\varphi\,dx + \int_{\partial\Omega} h(x,u)\varphi(x)\,dS \ \text{ for all } \varphi \in C^\infty(\overline{\Omega})\,.$$

If A *has derivative* $p(\cdot)$*-type growth,*$|B(x,u,z)| \leqslant c\,(1+|z|^{p(x)})$, *condition* (9.2) *holds,* p *is Hölder continuous, and* h *is Hölder continuous with respect to* x *and* u, *then* $u \in C^{1,\alpha}(\overline{\Omega})$ *for some* $0<\alpha<1$.

In all cases α depends on some of the data. For details, see [72].

Fan and Zhao [84] provide Hölder continuity for minimizers of (9.1) and solutions of ($\star$). Here p is only assumed to be, in essence, log-Hölder continuous and the crucial extra assumption of boundedness is actually proved. The regularity result is based on careful analysis of De Giorgi type function classes.

Let us present the result for solutions first. We again consider equation $(\star)$, now with the structure conditions

$$|A(x,u,z)| \leqslant |z|^{p(x)-1} + b\,|u|^{\sigma(x)} + c, \tag{9.6}$$

$$A(x,u,z)\cdot z \geqslant \tfrac{1}{a}|z|^{p(x)} - b\,|u|^{q(x)} - c, \tag{9.7}$$

$$|B(x,u,z)| \leqslant a_2|z|^{\tau(x)} + b|u|^{q(x)-1} + c\,. \tag{9.8}$$

Here $a, b, c > 0$ are constants, $p, q \in C(\overline{\Omega})$, $(p^* - q)^- > 0$, and

$$\sigma(x) := \frac{p(x)-1}{p(x)}q(x),\ \tau(x) := \frac{q(x)-1}{q(x)}p(x),\ \text{for } x \in \overline{\Omega}\,.$$

Theorem 9.9. *Let $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ be a local weak solution of equation $(\star)$. If conditions (9.6)–(9.8) hold and $p \in \mathcal{P}^{\log}(\overline{\Omega})$, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some α.*

Remark 9.10. Fan and Zhao note that if the complement of Ω is "thick enough" near $\partial\Omega$ and we consider a boundary value problem with Hölder continuous boundary values, then actually $u \in C^{0,\alpha}(\overline{\Omega})$. See the article for details.

The result for minimizers is as follows. Note that we generalize the concept of minimzation to quasiminimization later on, towards the end of this section. Let $F = F(x,u,z)$ be a Carathéodory function with the $p(\cdot)$-like growth

$$\tfrac{1}{a}|z|^{p(x)} - b\,|u|^{q(x)} - c \leqslant F(x,u,z) \leqslant a|z|^{p(x)} + b\,|u|^{q(x)} + c\,. \tag{9.11}$$

Here $a, b, c > 0$ are constants and $(p^* - q)^- > 0$.

Theorem 9.12. *Let $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ be a local minimizer for the energy (9.1). If condition (9.11) holds, $p \in \mathcal{P}^{\log}(\Omega)$, and q is continuous, then $u \in C^{0,\alpha}(\Omega)$ for some α.*

A finer regularity result is given by Acerbi and Mingione [28]. The result is formulated in the paper for minimizers only. However, see our notes presented after the theorem.

Let $F = F(x,z)$ be a Carathéodory function satisfying

$$\tfrac{1}{L}\left(\mu^2 + |z|^2\right)^{\frac{p(x)}{2}} \leqslant F(x,z) \leqslant L\left(\mu^2 + |z|^2\right)^{\frac{p(x)}{2}}\,, 0 \leqslant \mu \leqslant 1\,, \tag{9.13}$$

$$\int_{(0,1)^n} F(x_0, z_0 + \nabla\varphi(x)) - f(x_0,z_0)\,dx \geqslant \frac{1}{L}\int_{(0,1)^n}\left(\mu^2 + |z_0|^2 + |\nabla\varphi(x)|^2\right)^{\frac{p(x_0)-2}{2}}|\nabla\varphi(x)|^2\,dx \tag{9.14}$$

for all test functions $\varphi \in C_0^\infty$,

$$|F(x_1,z) - F(x_2,z)| \leqslant L\omega(|x_1 - x_2|)\left(\left(\mu^2 + |z|^2\right)^{\frac{p(x_1)}{2}} + \left(\mu^2 + |z|^2\right)^{\frac{p(x_2)}{2}}\right)\left(1 + \left|\log\left(\mu^2 + |z|^2\right)\right|\right)\,. \tag{9.15}$$

Here ω is the modulus of continuity of p. Note that these conditions are quite like variational versions of $p(\cdot)$-type growth and condition (9.2).

In addition to log-Hölder continuity, we now need to introduce *strong log-Hölder continuity*, that is

$$\lim_{R\to 0}\omega(R)\log\frac{1}{R} = 0\,,$$

instead of simply $\omega(R)\log\frac{1}{R} \leqslant c$. We denote the class of strongly log-Hölder continuous functions by $\mathcal{P}^{\text{s-}\log}$.

Previous reported theorems have asserted Hölder continuity for *some* α. Now we have an actual improvement.

Theorem 9.16. *Let $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ be a local minimizer for the energy (9.1). If conditions (9.13)–(9.15) hold and $p \in \mathcal{P}^{s\text{-}\log}(\Omega)$, then $u \in C^{0,\alpha}(\Omega)$ for any $0 < \alpha < 1$.*

Per [28], we may also have better differentiability. For historical reference, we mention the earlier paper [55] with stricter assumptions on p. We also mention a new proof by Habermann and Zatorska-Goldstein [101]. Their article deals with Hölder continuous p. For the result from [28], we present some additional structural conditions:

$$F(x,\cdot) \in C^2(\mathbb{R}^n \setminus \{0\}) \text{ for all } x \in \Omega\,, \tag{9.17}$$

$$\frac{1}{L}\left(\mu^2+|z|^2\right)^{\frac{p(x)-2}{2}}|\lambda|^2 \leqslant D_z^2F(x,z) : \lambda \otimes \lambda \leqslant L\left(\mu^2+|z|^2\right)^{\frac{p(x)-2}{2}}|\lambda|^2 \text{ for all } \lambda \in \mathbb{R}^n\,. \tag{9.18}$$

Here $\xi : \zeta$ denotes the Frobenius matrix inner product.

Theorem 9.19. *Let $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ be a local minimizer for the energy* (9.1)*. If conditions* (9.13)–(9.18) *hold and $p \in \mathcal{P}^{\log}(\Omega)$, then $u \in C^{1,\alpha}(\Omega)$ for some $0 < \alpha < 1$.*

Acerbi and Mingione state that the results could be extended to the case $F = F(x, u, \nabla u)$, that is (9.1). This is explicitly presented in a paper by Eleuteri, see [65]. We note that even though the results are only presented for minimizers, the conditions are general enough to allow for the energy

$$F(x,z) = \frac{|z|^{p(x)}}{p(x)}\,,$$

so we readily have the results for at least the elliptic $p(\cdot)$-Laplace equation. However, some more general equations with quasi-linear structure might not be the Euler-Lagrange equation of any energy functional.

Acerbi and Minione also extend the results to the vector-valued case, i.e. $u \in W^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$, $N \geqslant 2$, see [27]. The conditions are simple analogues of the above, except that one now takes $F(x,\cdot) \in C^2$ but does not need to take any growth condition akin to (9.18). We only briefly report the result here.

Theorem 9.20. *Let $u \in W^{1,p(x)}(\Omega, \mathbb{R}^N)$ be a minimizer, $N \geqslant 2$. If analogues of conditions* (9.13)–(9.17) *hold, $\mu = 1$, and p is Hölder continuous, then there is an exceptional set E of measure zero such that $u \in C^{1,\alpha}(\Omega \setminus E)$.*

Acerbi and Mingione state that the result could be extended to the case $F = F(x, u, \nabla u)$.

The proofs of all previous theorems rely on some kind of higher integrability result for the gradient $|\nabla u|$. Acerbi and Mingione [30] also give further refined higher integrability results. Let $A = A(x, z)$ be C^1 in z, let A have derivative $p(\cdot)$-like growth in the sense

$$\nu\left(\mu^2+|z|^2\right)^{\frac{p(x)-2}{2}}|\lambda|^2 \leqslant D_zA(x,z) : \lambda \otimes \lambda \leqslant L\left(\mu^2+|z|^2\right)^{\frac{p(x)-2}{2}}|\lambda|^2 \text{ for all } \lambda \in \mathbb{R}^n\,, \tag{9.21}$$

and modulus of continuity, somewhat similar to (9.2),

$$|A(x_1,z) - A(x_2,z)| \leqslant L\omega(|x_1-x_2|)\left(\mu^2+|z|^2\right)^{\frac{p(x_1)-1}{2}}\left|\log\left(\mu^2+|z|^2\right)\right|\,. \tag{9.22}$$

Here $0 \leqslant \mu \leqslant 1$, ν, L constants, and ω is the modulus of continuity for p. The next results apply to Equations ($\star$) and ($\star\star$) with

$$B(x) := -\operatorname{div}\left(|F(x)|^{p(x)-2}F(x)\right). \tag{9.23}$$

Theorem 9.24. *Let $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution of Equation ($\star$) with* (9.23)*. If $F \in L^{qp(\cdot)}(\Omega, \mathbb{R}^n)$, $q \in (1,\infty)$, conditions* (9.21)–(9.22) *hold, and $p \in \mathcal{P}^{s\text{-}\log}(\Omega)$, then $|\nabla u| \in L^{qp(\cdot)}_{loc}(\Omega)$.*

There is also a vector-valued version, but it is given only for the $p(\cdot)$-Laplace system.

Theorem 9.25. *Let $u \in W^{1,p(\cdot)}(\Omega, \mathbb{R}^N)$, $N \geqslant 1$ be a weak solution of Equation ($\star\star$) with* (9.23)*. If $F \in L^{qp(\cdot)}(\Omega, \mathbb{R}^{Nn})$, $q \in (1,\infty)$, and $p \in \mathcal{P}^{s\text{-}\log}(\Omega)$, then $\nabla u \in L^{qp(\cdot)}_{loc}(\Omega, \mathbb{R}^{Nn})$.*

Acerbi and Mingione comment on how the assumption of strong log-Hölder continuity contrasts to assuming simply log-Hölder continuity, see their Remark 2 in [30]. There are also two older works on higher integrability by Zhikov [163, 164].

To our understanding, all the widely considered important cases have thus been reported. However, one more problem also deserves a mention in this section. Acerbi and Mingione [29] considered the somewhat more specialized case of stationary electro-rheological fluid problem. The problem in question is of the type

$$\begin{cases} \operatorname{div} u = 0\,, \\ -\operatorname{div} A\left(x, \mathcal{E}u)\right) + D\pi = B(x, u, Du)\,. \end{cases}$$

The assumptions are very similar to the ones presented before and the result is very familiar, $u \in C^{1,\alpha}$.

Quasiminimizers

We consider briefly Q-*quasiminimizers*, which are functions u satisfying

$$\int_\Omega F(x, u, \nabla u)\, dx \leqslant Q \int_\Omega F(x, v, \nabla v)\, dx$$

for all suitable v, where $Q \geqslant 1$. Note that with $Q = 1$ we regain the standard concept of a minimizer. As historical reference, we mention the constant exponent studies [7, 8].

Basic results for quasiminimizers are given both by Fan and Zhao [85] and by Chiadò Piat and Coscia [53]. We report the results of [85] here, as they are slightly more general.

Theorem 9.26. *Let $p \in C(\Omega)$ and let u be a local Q-quasiminimizer with energy given by the Carathéodory function F satisfying*

$$\tfrac{1}{a}|z|^{p(x)} - b\,|u|^{q(x)} - g(x) \leqslant F(x, u, z) \leqslant a_1 |z|^{p(x)} + b\,|u|^{q(x)} + g(x) \quad \text{and} \quad (p^* - q)^- > 0.$$

If $p \in \mathcal{P}^{\log}(\Omega)$ and $g \in L^{s(\cdot)}(\Omega)$ with $(sp)^- > n$, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some α.

Fan and Zhao also discuss the relation of local quasiminimizers with general F of the previous theorem and the case $J(x, u, z) = a\,|z|^{p(x)} + b\,|u|^{q(x)} + g(x)$. Every local quasiminimizer of F, in particular every minimizer, is a quasiminimizer of J. This showcases the flexibility of the notion of quasimimizer, since similar results are not available in the case of solutions or minimizers. See [85, Section 2] for details.

Harjulehto, Kuusi, Lukkari, Marola and Parviainen [108] provide different versions of Harnack inequalities for quasiminimizers. One can also arrive at Hölder continuity based on Harnack estimates. Finally we mention the article [87] which generalizes the results to a class of discontinuous exponents: one basically divides Ω into parts, assumes log-Hölder continuity in each part, and adds geometric conditions.

10. Parabolic problems of $p(\cdot)$-Laplace type

In this section, we consider the parabolic versions of ($\star$) and ($\star\star$), which have been up to this point considered in the elliptic setting. The regularity theory of parabolic problems is, even in the constant p case, somewhat harder than elliptic problems. Thus we have fewer results to report than in Sections 8–9. We consider the parabolic $p(\cdot)$-Laplace equation

$$\partial_t u - \Delta_{p(\cdot)} u = 0, \text{ in } \Omega_T\,. \tag{10.1}$$

Here Ω_T is a parabolic cylinder, often $\Omega_T = \Omega \times (0, T)$, but other definitions exist, e.g. $\Omega_T = \Omega \times (0, T]$. We also consider the more general equation

$$\partial_t u - \operatorname{div} A(x, t, \nabla u) = 0, \text{ in } \Omega_T\,. \tag{10.2}$$

In the setting of parabolic equations, we understand the Sobolev space $W^{1,p(\cdot)}$ with the differentiation with respect to only the spatial variable x. By an *energy solution* we mean a weak solution u with the energy

$$\int_{\Omega_T} |\nabla u(x,t)|^{p(x,t)}\,dx\,dt < \infty.$$

We stress that we always require this energy to be finite over the whole of Ω_T even if we speak of local solutions. Note also that we mean this form of energy even for the general equation (10.2).

We note that the weak solutions of parabolic equations are usually assumed to belong to certain parabolic spaces, for instance often

$$u \in C(0,T;L^2(\Omega)) \cap L^{p(\cdot)}(0,T;W^{1,p(\cdot)}(\Omega)).$$

For the definition of parabolic spaces in the constant exponent case, see [15, Section 2.1.2].

The most complete result for equation (10.1) has been presented few years ago by Chen and Xu [52].

Theorem 10.3. *Assume u is a local weak energy solution of Equation* (10.1) *in the cylinder Ω_T and let $n \geqslant 2$. If p is log-Hölder continuous with respect to x and t, and*

$$\frac{2n}{n+2} < p^- \leqslant p^+ < \infty,$$

then u is locally bounded in Ω_T. If moreover $p^-_{\Omega_T} > 2$, then $u \in C^{0,\alpha}_{loc}(\Omega_T)$ for some $\alpha > 0$.

The proof follows the De Giorgi type measure theoretical approach in the constant p case [3, 4, 25]. The oscillation of u is reduced in suitably nested cylinders. The method results in u having a modulus of continuity that makes u Hölder continuous.

We now present another regularity result by Acerbi, Mingione and Seregin [31]. The result has stricter conditions on p, but it is valid for a family of equations with full quasi-linear structure, for both the scalar and vector valued cases.

Let $A : \Omega_T \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ with $N \geqslant 1$. We assume that $A \in C\left(\Omega_T \times \mathbb{R}^{nN}\right)$, $D_z A \in C\left(\Omega_T \times \mathbb{R}^{nN}\right)$, A has derivative $p(\cdot)$-type growth, and A has familiar-looking continuity modulus

$$\begin{aligned}|A(x_1,t_1,z) - A(x_2,t_2,z)| \leqslant L &\left(\left(1+|z|^2\right)^{\frac{p(x_1,t_1)-1}{2}} + \left(1+|z|^2\right)^{\frac{p(x_2,t_2)-1}{2}}\right) \\ &\times \log(2+|z|)\left(|x_1-x_2| + |t_1-t_2|^{\frac{\beta}{2}}\right).\end{aligned} \tag{10.4}$$

Theorem 10.5. *Assume u is a weak energy solution of the system* (10.2). *If $0 < \gamma < \beta \leqslant 1$, A has derivative $p(\cdot)$-type growth, condition* (10.4) *holds,*

$$\frac{2n}{n+2} < p^- \leqslant p^+ < \infty,$$

p is Lipschitz continuous with respect to x, and $\frac{\beta}{2}$-Hölder continuous with respect to t, then there is an exceptional set $E \subset \Omega_T$ of zero measure, such that $\nabla u(\cdot,t) \in C^{0,\gamma}(\Omega_T \setminus E)$ and $\nabla u(x,\cdot) \in C^{0,\frac{\gamma}{2}}(\Omega_T \setminus E)$.

Remark 10.6. The result means that the modulus of continuity for ∇u is of the form

$$|x_1-x_2|^\gamma + |t_1-t_2|^{\frac{\gamma}{2}}.$$

We would also like to remark that there are obvious similarities between the parabolic conditions of Theorem 10.5 and the elliptic conditions presented in Section 9.

In their introduction to [31], Acerbi, Minginoe and Seregin note that their studies relate to the modeling of electro-rheological fluids, see also the end of our Section 9. These are systems of the form

$$\begin{cases} \operatorname{curl} E = 0\,, \\ \operatorname{div} E = 0\,, \\ \partial_t u - \operatorname{div} A(x,t,E,D(u)) + D\pi = -\operatorname{div}(u \otimes u) + f\,, \\ \operatorname{div} u = 0\,. \end{cases}$$

These systems have been more thoroughly studied by Růžička et al. For the most part, Růžička deals with existence and stability questions, but there are also some regularity type results. These are mainly instances in which the solutions belong to slightly better function spaces than initially assumed. We do not duplicate the results here, but see [22, Theorems 1.1, 1.2] and [23, Chapter 4.1].

A higher integrability type result plays a part in the proof of Theorem 10.5. Antontsev and Zhikov [44] provide a Meyers type higher integrability result in a some sense more general setting, loosening the assumptions on p back to the log-Hölder condition.

Theorem 10.7. *Assume u is a local weak energy solution of the equation* (10.1). *Let $n \geqslant 2$. If p is log-Hölder continuous and*

$$\frac{2n}{n+2} < p^- \leqslant p^+ < \infty\,,$$

then u is locally bounded and ∇u is locally higher integrable: $|\nabla u| \in L^{(1+\varepsilon)p(\cdot)}_{loc}(\Omega_T)$ for a small $\varepsilon > 0$.

The proof relies on a Gehring type lemma.

Remark 10.8. We note that the local boundedness result of Theorem 10.7 holds for slightly more irregular exponents p. We need log-Hölder continuity only with respect to the time variable, density of smooth functions in the Sobolev space, and "jump conditions" that are automatically true for e.g. p uniformly continuous in all variables [138].

To close this subsection, we would like to note that the regularity theory of parabolic variable exponent equations is an ongoing project of interest to the authors.

11. More general notions of solution

We have thus far dealt with weak solutions. We now consider two more general notions of solution, namely *very weak solutions* and *entropy solutions.* Closely related to the entropy solutions are renormalized solutions, which deal with L^1 data as well. These are not further discussed here, but see [45] for recent results in the variable exponent case.

Bögelein and Zatorska-Goldstein [46] consider very weak solutions of the elliptic problem ($\star$) with divergence-type right-hand side (9.23); their study deals also with the case of systems, i.e. $u : \Omega \to \mathbb{R}^N$, $N \geqslant 1$. We assume A has $p(\cdot)$-type growth.

We note that the familiar weak formulation

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega |F(x)|^{p(x)-2} F(x) \cdot \nabla \varphi \, dx \text{ with } \varphi \text{ a suitable test function}$$

actually makes sense already with $|\nabla u| \in L^{p(\cdot)-\varepsilon}$ for some $0 \leqslant \varepsilon \leqslant 1$ instead of the traditional $|\nabla u| \in L^{p(\cdot)}$. Solution u for which we only assume, a priori, that $|\nabla u| \in L^{p(\cdot)-\varepsilon}$ is called a very weak solution. For a discussion in the constant exponent case, see [11, 14].

The usual problem is the question whether a very weak solution is actually a weak solution, i.e. whether u already has some higher integrability properties, and thus more regularity than assumed a priori. The following from [46, Theorem 1] provides a partial answer.

Theorem 11.1. *Let* $F \in L^{\sigma p(\cdot)}(\Omega), \sigma > 1$ *a constant,* $p \in \mathcal{P}^{\log}(\Omega)$*, and consider system* $(\star)$ *with* (9.23)*. If* A *has* $p(\cdot)$*-type growth, then very weak solutions* $u \in W^{1,p(\cdot)-\varepsilon}\left(\Omega, \mathbb{R}^N\right)$, $N \geqslant 1$ *are in fact weak solutions for some* $\varepsilon \in (0,1)$ *depending on data.*

The above result obviously extends to all $\widetilde{\varepsilon}$ with $0 \leqslant \widetilde{\varepsilon} \leqslant \varepsilon$.

It should be stressed that addressing higher integrability problems is much harder for very weak solutions than for weak solutions. Thus the proof needs many more ingredients. We would also like to note that an analogous result for the parabolic case is, to our best knowledge, an open problem. See [2, 12] for the parabolic constant exponent case.

Another generalization, the entropy solution, is considered by Sanchón and Urbano [145]. This generalization arises from the need to accommodate data that is only in L^1. Entropy solutions were introduced for the constant exponent case in [1].

We consider the homogeneous Dirichlet problem $(\star)$ with

$$A(x,u,z) = A(x,z) \quad \text{and} \quad B(x,u,z) = B(x). \tag{11.2}$$

Let us assume A has $p(\cdot)$-type growth. We might actually generalize a little and assume the upper bound

$$|A(x,z)| \leqslant \beta \left(j(x) + |z|^{p(x)-1}\right),$$

with $j \in L^{p'(\cdot)}(\Omega)$ a nonnegative function.

We only assume $B \in L^1$, so we need the concept of an entropy solution. Define the truncation operator

$$T_t(s) := \begin{cases} t, & s > t\,, \\ s, & |s| \leqslant t\,, \\ -t, & s < -t\,, \end{cases}$$

with $t > 0$. The function u is called an entropy solution of the problem $(\star)$ if for every $t > 0$ we have $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega A(x,\nabla u)\cdot T_t(u-\varphi)\,dx \leqslant \int_\Omega B(x) T_t(u-\varphi)\,dx$$

for all $\varphi \in W_0^{1,p(\cdot)} \cap L^\infty$.

Theorem 11.3. *Let* $B \in L^1$ *and let* u *be an entropy solution to* $(\star)$ *with* (11.2)*. Define the variable exponents*

$$q_0(x) := \frac{p^*(x)}{(p')^+} \text{ and } q_1(x) := \frac{q_0(x)}{q_0(x)+1}p(x), \text{ both with } x \in \Omega\,.$$

If A *has* $p(\cdot)$*-type growth and* p *is log-Hölder continuous with* $1 < p^- \leqslant p^+ < n$*, then* $u \in L^{q_0(\cdot)-\varepsilon}(\Omega)$ *and* $|\nabla u| \in L^{q_1(\cdot)-\varepsilon}(\Omega)$ *for every* $\varepsilon > 0$.

The following is a refinement. The theorem shows that the idea of an entropy solution is actually a well-chosen generalization.

Theorem 11.4. *Assume that the definitions and conditions of Theorem 11.3 hold. If* $q_1^- > 1$ *and*

$$p^- > 2 - \frac{1}{n}\,,$$

then $u \in W_0^{1,q_1(\cdot)-\varepsilon}(\Omega)$ *for every* $\varepsilon > 0$*. If, in addition,* $(p-q_1)^+ < 1$*, then entropy solution* u *is actually a weak solution.*

The techniques in the proofs rely on estimates in variable exponent Marcinkiewicz spaces.

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