Minimizers of the variable exponent, non-uniformly convex Dirichlet energy

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Abstract

We study energy minimizing properties of the function $u = \lim_{\lambda_j \to 1^+} u_{\lambda_j}$, where u_{λ_j} is the solution to the $p_{\lambda_j}(\cdot)$-Laplacian Dirichlet problem with prescribed boundary values. Here $p\colon \Omega \to [1,\infty)$ is a variable exponent and $p_{\lambda_j}(x) = \max\{p(x), \lambda_j\}$ for $\lambda_j > 1$. This problem leads in a natural way to a mixture of Sobolev and total variation norms. The main results are obtained under the assumption that p is strongly log-Hölder continuous and bounded. To motivate our approach we also consider the one-dimensional case and give examples which justify our assumptions. The results can be applied in the analysis of a model for image restoration combining total variation and isotropic smoothing.

Résumé

Nous étudions des propriétés de la fonction $u = \lim_{\lambda_j \to 1^+} u_{\lambda_j}$, où u_{λ_j} est la solution du $p_{\lambda_j}(\cdot)$-Laplacien problème de Dirichlet aux limites. Ici, $p\colon \Omega \to [1,\infty)$ est un exposant variable et $p_{\lambda_j}(x) = \max\{p(x), \lambda_j\}$, quand $\lambda_j > 1$. Ces conditions conduisent naturellement à une norme combinée de la variation totale et de la norme de Sobolev. Nos résultats principaux sont obtenus sous la supposition que l'exposant est fortement Hölderien et borné. Afin de justifier les suppositions nous examinons aussi le cas unidimensionnel. Les résultats peuvent être appliqués dans l'analyse d'une méthode de restauration d'image, qui combine du lissage basé sur la variation totale et du lissage isotropique.

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1. Introduction

During the last decade, function spaces with variable exponent have attracted a lot of interest, and substantial research efforts are ongoing with the aim of understanding these structures. A survey of the

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history of the field with a comprehensive bibliography is provided by Diening, Hästö and Nekvinda [17]; a further survey is due to Samko [51]. Apart from interesting theoretical considerations, these investigators were motivated by a proposed application of variable exponent spaces to modeling electrorheological fluids, see Růžička [48,49] and Acerbi & Mingione [2,3].

Very recently, another application emerged, as Chen, Levine & Rao [12,44] (see also [10]) proposed a variable exponent formulation for the problem of image restoration. The problem is the following: we are given an input signal I which equals the true signal u plus an additive, random noise ϵ (on a two dimensional rectangle, say). From I we must recover u. Since the noise is random, an obvious thing to do is to smooth the signal, which will lead the high frequency noise to cancel out. The problem with this approach is that it also looses critical information about object boundaries in the image. This problem can be overcome by smoothing only perpendicular to the direction of the gradient, so called total variation smoothing, a method proposed by Rudin, Osher & Fatemi [47]; see, e.g., Chambolle & Lions [11] or Esedoglu & Osher [21] for some problems and newer results. The central problem with the second approach is that it too readily introduces boundaries, even when none exist in the true image, an effect which has been termed stair-casing.

To understand the role of the variable exponent in the image restoration problem we look at the variational formulation of the previous two approaches. Isotropic smoothing corresponds to finding the minimum of the energy

$$E_p(u) = \int_\Omega |\nabla u|^p + \lambda |u - I|^2 \, dx, \tag{1.1}$$

with $p \equiv 2$, where $\lambda > 0$ is a parameter indicating the strength of the smoothing. Total variation smoothing, on the other hand, corresponds to minimizing the energy (1.1) with $p \equiv 1$.

The first minimization problem is naturally solved in the Sobolev space $W^{1,2}(\Omega)$, whereas the second is solved in the space $\mathrm{BV}(\Omega)$ of functions of bounded variation (for the definition of this space see Section 4). Since we would like to combine the strengths of these two approaches, it is natural to formulate the minimization problem (1.1) for an exponent $p = p(x)$ varying in the interval $[1, 2]$. This is the essence of the model proposed in [12].

As was mentioned above, Lebesgue and Sobolev spaces with variable exponent have been intensively investigated and are now quite well understood. The paradigmatic Dirichlet minimization problem,

$$\int_\Omega |\nabla u|^{p(x)} \, dx$$

for $u - w \in W_0^{1,p(\cdot)}(\Omega)$, where $w \in W^{1,p(\cdot)}(\Omega)$ gives the boundary values, has been investigated e.g. in [1,13,20,25,37,39] and the corresponding Euler-Lagrange equation

$$\operatorname{div}\Big(p(x)|\nabla u|^{p(x)-2}\nabla u\Big) = 0$$

e.g. in [4–7,23,28,29,38,52]. However, all of these investigations have been limited to the case when p is bounded away from 1; usually it is also assumed that p is log-Hölder continuous, but also stronger continuity conditions have been used, for instance in [1,4,13,28].

The problem is that once we let $p \to 1$ we leave the realm of reflexive Sobolev spaces and require a new space with some crucial properties of the space $\mathrm{BV}(\Omega)$ of function of bounded variation. An obvious idea is simply to patch together the two different spaces; it turns out that this works only under additional assumptions on the exponent, whereas a more general approach is available in limited other cases. To avoid these problems when investigating (1.1) with $p = p(x)$, Chen, Levine & Rao [12] had to resort to various approximation procedures and auxiliary minimization problems. The purpose of the present article is to provide the foundations for solving the Dirichlet problem in the case $p \to 1$ so that the the *ad hoc* measures of [12] can be avoided in the future.

Previous investigations of limit cases of the variable exponent (e.g. [19,34] on Sobolev inequalities when $p \nearrow n$, [30,33] on Sobolev inequalities when $p \searrow n$, and [16,31,41,46] on maximal inequalities when $p \searrow 1$) suggest that it will be quite arduous to carry out this plan. This turns out to be the case, as we encounter several new difficulties.

Our approach to the minimization problem is loosely based on the article [42] by P. Juutinen which recovers functions of least perimeter (minimizers of the 1-Dirichlet problem) as limits of minimizers of the

p-Dirichlet problem when p is a constant exponent approaching 1. In order to apply these techniques we must first define an appropriate function space of BV-Sobolev type. We believe, however, that it will be easier to investigate other, more complicated differential equations of BV-$W^{1,p(\cdot)}$ type once the foundations are laid properly in place.

The structure of this article is as follows. We start by reviewing the preliminaries of variable exponent spaces. Then we try to figure out the right definition for a BV-$W^{1,p(\cdot)}$ space. To do this we first consider minimizers in the one-dimensional case in Section 3. In Section 4 we make a supposition of the definition, and prove that functions in the BV-Sobolev space can be approximated by smooth functions in the appropriate sense (Theorem 4.6). We then move on to the general minimization problem. In Section 5 we study Caccioppoli type estimates and in Section 6 we prove some auxiliary existence results for the problem at hand. Section 7 contains our main result, that is, the existence of minimizers of the Dirichlet energy integral in the variable exponent case for strongly log-Hölder continuous exponents which need not be bounded away from 1 (Theorem 7.1). In the appendix we consider some examples of what goes wrong when we omit some conditions from our theorems.

2. Preliminaries

Conventions

The following notation will be used throughout the rest of this article, often without further mention. By $\Omega \subset \mathbb{R}^n$ we denote a bounded open set. A measurable function $p\colon \Omega \to [1,\infty)$ is called a *variable exponent*, and we denote for $A \subset \Omega$

$$p_A^+ := \operatorname*{ess\,sup}_{x\in A} p(x), \quad p_A^- := \operatorname*{ess\,inf}_{x\in A} p(x), \quad p^+ := \operatorname*{ess\,sup}_{x\in \Omega} p(x) \quad \text{and} \quad p^- := \operatorname*{ess\,inf}_{x\in \Omega} p(x).$$

For $\lambda > 1$ we denote $p_\lambda(x) = \max\{\lambda, p(x)\}$. By the symbol Y (for "yksi", meaning one in Finnish) we always denote the set where p equals one, $Y := \{x \in \Omega : \, p(x) = 1\}$.

Variable exponent spaces

Usually it is assumed that $p^+ < \infty$, since this condition is known to imply many desirable features for $L^{p(\cdot)}(\Omega)$. Spaces with $p^+ = \infty$ have been investigated in [16,18,43]. In the general case we denote $\Omega_\infty = \{x \in \Omega\colon p(x) = \infty\}$ and define a *modular* by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}\,dx + \operatorname*{ess\,sup}_{\Omega_\infty} |u|.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{L^{p(\cdot)}(\Omega)}(u/\mu)$ is finite for some $\mu > 0$. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{\mu > 0 : \, \varrho_{L^{p(\cdot)}(\Omega)}(\tfrac{u}{\mu}) \leqslant 1\Big\}.$$

In the case of norms and modulars taken over the whole set Ω we also use an abbreviated notation where $L^{p(\cdot)}(\Omega)$ in the subscript is replaced simply by $p(\cdot)$. Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For a constant function p the variable exponent Lebesgue space coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

We use repeatedly the following fact: if E is a measurable set with a finite measure, and p and q are variable exponents satisfying $q \leqslant p$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular this implies that every function $u \in W^{1,p(\cdot)}(\Omega)$ also belongs to $W^{1,p_B^-}(B)$, $B \subset \Omega$. The variable exponent Hölder inequality takes the form

$$\int_\Omega fg\,dx \leqslant 3\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where p' is the pointwise conjugate exponent, $1/p(x) + 1/p'(x) \equiv 1$. For all these results we refer to Kováčik & Rákosník [43].

Since $L^{p(\cdot)}(\Omega)$ is a normed space, a well known fact of the functional analysis implies that the norm is weakly lower semicontinuous: if $u_i \rightharpoonup u$ in $L^{p(\cdot)}(\Omega)$ then

$$\|u\|_{p(\cdot)} \leqslant \liminf_{i\to\infty} \|u_i\|_{p(\cdot)}.$$

If p is constant, this directly gives the same property for the modular. For the variable exponent we can still reduce to this general function analytic result, but some more work is needed.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ and let p be a bounded exponent. If u_i converges weakly to u in $L^{p(\cdot)}(\Omega)$, then*

$$\int_\Omega |u|^{p(x)}dx \leqslant \liminf_{i\to\infty} \int_\Omega |u_i|^{p(x)}dx.$$

Proof. Fix $\mu > 1$ and let $M \in \mathbb{N}$ be the smallest integer satisfying $\mu^M > p^+$. Define $\Omega_j := p^{-1}([\mu^j, \mu^{j+1}))$ for $j = 0, \ldots, M-1$. We use the relationship between the modular and norm, and the lower semi-continuity of the norm in each Ω_j:

$$\begin{aligned}\int_\Omega |u|^{p(x)}dx &= \sum_j \int_{\Omega_j} |u|^{p(x)}dx \\ &\leqslant \sum_j \max\Big\{\|u\|^{p_j^+}_{L^{p(\cdot)}(\Omega_j)}, \|u\|^{p_j^-}_{L^{p(\cdot)}(\Omega_j)}\Big\} \\ &\leqslant \liminf_{i\to\infty} \sum_j \max\Big\{\|u_i\|^{p_j^+}_{L^{p(\cdot)}(\Omega_j)}, \|u_i\|^{p_j^-}_{L^{p(\cdot)}(\Omega_j)}\Big\} \\ &\leqslant \liminf_{i\to\infty} \sum_j \max\Big\{\varrho_{L^{p(\cdot)}(\Omega_j)}(u_i)^{\frac{p_j^+}{p_j^-}}, \varrho_{L^{p(\cdot)}(\Omega_j)}(u_i)^{\frac{p_j^-}{p_j^+}}\Big\},\end{aligned} \tag{2.1}$$

where $p_j^\pm = p_{\Omega_j}^\pm$. Note that $p_j^+/p_j^- \in [1, \mu]$. For a sequence $(a_j)_{j=0}^{M-1}$ of non-negative reals we denote by J the set of indices j for which $a_j \leqslant 1$. Then, using the power-mean inequality for the first estimate in the set J, we derive

$$\begin{aligned}\sum_j \max\Big\{a_j^{\frac{p_j^+}{p_j^-}}, a_j^{\frac{p_j^-}{p_j^+}}\Big\} &\leqslant \sum_j \max\Big\{a_j^\mu, a_j^{\frac{1}{\mu}}\Big\} \leqslant \Big(\sum_{j\notin J} a_j\Big)^\mu + |J|^{1-\frac{1}{\mu}}\Big(\sum_{j\in J} a_j\Big)^{\frac{1}{\mu}} \\ &\leqslant (2M)^{1-\frac{1}{\mu}} \max\Big\{\Big(\sum_j a_j\Big)^\mu, \Big(\sum_j a_j\Big)^{\frac{1}{\mu}}\Big\}.\end{aligned} \tag{2.2}$$

Here the last step follows from the inequality

$$a^\mu + b^{\frac{1}{\mu}} \leqslant 2^{1-\frac{1}{\mu}} \max\Big\{(a+b)^\mu, (a+b)^{\frac{1}{\mu}}\Big\},$$

which is proved as follows: if $a + b \geqslant 1$, then $2^{1-\frac{1}{\mu}}(a+b)^\mu - a^\mu$ is increasing in a hence we may assume that $a = 0$ or $a + b \leqslant 1$. The case $a = 0$ is clear, so we need only consider the case $a \leqslant 1$. Then $a^\mu \leqslant a^{1/\mu}$, and $a^\mu + b^{1/\mu} \leqslant 2^{1-1/\mu}(a+b)^{1/\mu}$, so the inequality holds.

Next we set $a_j = \varrho_{L^{p(\cdot)}(\Omega_j)}(u_i)$ and use (2.2) to estimate the sum in (2.1). Taking into consideration that $\sum_j \varrho_{L^{p(\cdot)}(\Omega_j)}(u_i) = \varrho_{L^{p(\cdot)}(\Omega)}(u_i)$, we get

$$\int_\Omega |u|^{p(x)}dx \leqslant \liminf_{i\to\infty} (2M)^{1-\frac{1}{\mu}} \max\Big\{\varrho_{L^{p(\cdot)}(\Omega)}(u_i)^\mu, \varrho_{L^{p(\cdot)}(\Omega)}(u_i)^{\frac{1}{\mu}}\Big\}.$$

Since $M \sim \frac{\log p^+}{\mu-1}$, we see that $M^{1-1/\mu} \to 1$ as $\mu \to 1$. Thus the claim follows as we let $\mu \to 1$. □

The variable exponent p is said to be log-*Hölder continuous* if there is a constant $C > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$. A bounded exponent p is log-Hölder continuous in Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p_B^- - p_B^+} \leqslant C$$

for every ball $B \subset \Omega$ [15, Lemma 3.2]. Under the log-Hölder condition smooth functions are dense in variable exponent Sobolev spaces, [50], and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$.

We write $u_\delta = \varphi_\delta * u$ to denote the standard mollification of u. Here $\varphi \in C_0^\infty(B(0,1))$ has unit mass and $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. If p is a bounded log-Hölder continuous exponent, then $u_\delta \to u$ in $L^{p(\cdot)}(\Omega)$, see for example the proof of [26, Theorem 2.6].

3. The one-dimensional case

We next introduce the key ideas of this paper in the one-dimensional context. This will provide us with guidelines of how to handle the more general cases, and, more importantly, it will provide us with examples which show that our assumptions are essentially the best ones possible.

Throughout this section we assume that the variable exponent is bounded on the interval $[-1, 1]$. Consider minimizing

$$\varrho_{p(\cdot)}(u') := \int_{-1}^{1} |u'(x)|^{p(x)}\, dx \tag{3.1}$$

where u is absolutely continuous with boundary values $-a$ and $a > 0$. If we assume that the minimizer exists and is absolutely continuous, then one easily finds (by the Euler–Lagrange equation) that it is of the form

$$u'(x) = \Big(\frac{c}{p(x)}\Big)^{\frac{1}{p(x)-1}}, \tag{3.2}$$

where the constant $c > 0$ is chosen so that $u(\pm 1) = \pm a$. The next theorem answers the question of when this is the case.

Theorem 3.1 (Theorem 3.2, [35])**.** *Let p be bounded and strictly greater than one almost everywhere. Then the minimization problem* (3.1) *with boundary values $\pm a$ has a unique absolutely continuous minimizer if and only if there exists $\tilde{c} \geqslant 1$ such that*

$$2a \leqslant \int_{-1}^{1} \Big(\frac{\tilde{c}}{p(x)}\Big)^{\frac{1}{p(x)-1}}\, dx < \infty.$$

In this case the derivative of the minimizer is given by (3.2)*, for appropriate $c \in (0, \tilde{c}]$.*

Remark 3.2. The original formulation of this result included a small mistake, so we reformulated it here. The proof is as in [35].

We see that there are potential problems with existence in the previous theorem if $p^- = 1$. As a corollary Harjulehto, Hästö and Koskenoja concluded only that the minimizer exists if p approaches 1 sufficiently slowly.

The starting point for our investigation of minimizers in the case $p^- = 1$ is the following. If $p_i \to p$ uniformly, where $p^- > 1$, then we directly see that the corresponding minimizers satisfy $u_i \to u$ uniformly and in Sobolev space. We want to investigate whether the analogous limit taking process can serve as a reasonable definition for a minimizer in the case $p^- = 1$. We restrict our attention to a special type of exponents in the one-dimensional case: based on the results we make certain hypotheses, which will be shown in coming sections to be reasonable also in higher dimensions.

We consider next an exponent p which is decreasing on $[-1, 0]$ and even. Further we assume that $p(0) = 1$. Recall also that $p_\lambda = \max(p, \lambda)$. For every $\lambda > 1$, we get a p_λ-minimizer by formula (3.2). We denote this minimizer by u_λ and the corresponding constant by c_λ. It is easy to see that c_λ is bounded as $\lambda \to 1$, since

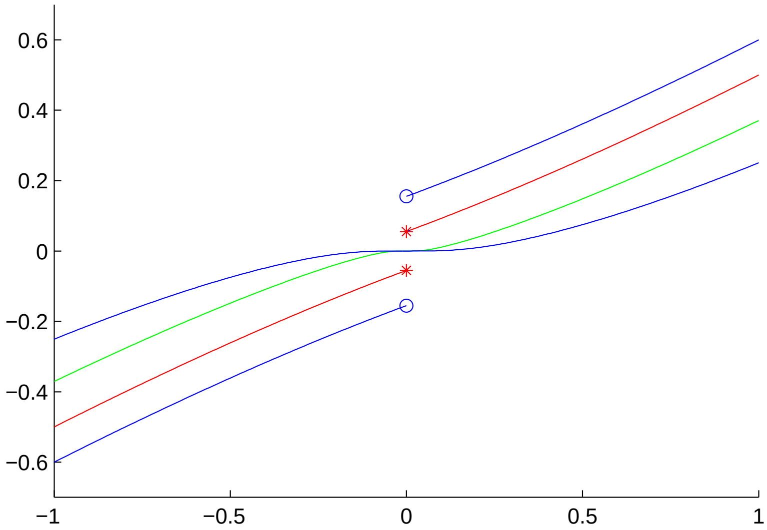

Fig. 1. Four minimizers for $p(x) = 1 + |x|$ with different boundary values.

the energy of u_λ has to be smaller than or equal to the energy of the linear function, $x \mapsto ax$, which is given by

$$\int_{-1}^{1} a^{p_\lambda(x)}\,dx \leqslant 2\max\{a^{p^+}, 1\} < \infty. \tag{3.3}$$

Thus we can find a sequence $\lambda_i \to 1$ such that $c_{\lambda_i} \to c \in (0, \infty)$.

For the sequence (c_{λ_i}) we find by dominated convergence that

$$u_{\lambda_i}(x) = a - \int_x^1 \left(\frac{c_{\lambda_i}}{p_{\lambda_i}(t)}\right)^{\frac{1}{p_{\lambda_i}(t)-1}} dt \to a - \int_x^1 \left(\frac{c}{p(t)}\right)^{\frac{1}{p(t)-1}} dt =: u(x) \tag{3.4}$$

and $u(-x) = -u(x)$ for every $x \in (0, 1)$. Moreover, (3.4) implies that u is increasing and it attains the boundary values $\pm a$ at ± 1, respectively. The last property follows from the pointwise convergence since u'_{λ_i} is uniformly bounded close to the endpoints ± 1. The previous formula also easily implies that u_{λ_i} is a Cauchy sequence in the space $\mathrm{BV}(-1, 1)$ of functions of bounded variation. Recall that in general a BV-function on the real line is the difference of two non-decreasing functions, so that its derivative is a signed measure. In our case the limit function is obviously increasing, so the function u extended to 0 by 0 will have as its derivative a (positive) measure. Moreover, the measure is absolutely continuous with respect to the Lebesgue measure in the set $(-1, 0) \cup (0, 1)$. Depending on p and a, u' may or may not have a singular part at the origin (see Figure 1).

We want to define a suitable modular which measures the "energy" of the function u. Since $p(0) = 1$ the immediate choice would seem to be to consider the contribution of the singular part like in the BV-modular. Since u attains the boundary values $\pm a$ and is symmetric, we know that $2(a - \int_{(0,1)} u'(x)\,dx)$ gives the height of the jump at origin. Therefore the singular part of the measure is given by

$$2\left(a - \int_{(0,1)} u'(x)\,dx\right)\delta_0,$$

where δ_0 is the Dirac delta measure at the origin. It turns out that the BV measure of the singularity does not adequately capture the effect of the varying exponent on the energy.

We evaluate the contribution of the possible singularity to the energy of u. Suppose that u has a jump of height $b > 0$ at the origin. The exponent p_λ equals λ in the set $K_\lambda = \{x\colon p(x) \leqslant \lambda\}$. The minimizer will grow by $b + \epsilon(\lambda)$ on this interval, where $\epsilon(\lambda)$ is some function tending to 0 as $\lambda \to 1$. Since the exponent is constant, the minimizer will be linear, hence $u'_\lambda \equiv (b + \epsilon(\lambda))/|K_\lambda|$ on K_λ. Thus the contribution to the modular from this interval is

$$\int_{K_\lambda} |u'_\lambda(x)|^{p_\lambda(x)}\,dx = \Big(\frac{b+\epsilon(\lambda)}{|K_\lambda|}\Big)^\lambda |K_\lambda|. \tag{3.5}$$

As $\lambda \to 1$, we see that the contribution of the jump at the origin tends to $b \lim_{\lambda\to 1} |K_\lambda|^{1-\lambda}$, if this limit exists. It turns out that there exist log-Hölder continuous exponents for which this limit does not exist. In Appendix A we construct such an exponent, and show that some of the following conclusions do not hold for this exponent. For the rest of this argument we add the following assumption on the exponent: the limit

$$\alpha := \lim_{x\to 0} \big(p(x) - p(0)\big) \log \frac{1}{|x|}$$

exists and is finite. Denote by x_λ the largest value of x for which $p(x) = \lambda$. Then we conclude that the limit exists, specifically,

$$\lim_{\lambda\to 1} |K_\lambda|^{1-\lambda} = \lim_{\lambda\to 1} |2x_\lambda|^{1-p(x_\lambda)} = \lim_{x\to 0} x^{1-p(x)} = e^\alpha.$$

We next establish the behavior of the modular in the limit taking in our special case and show that the limit function is indeed a minimizer of the limit modular. These observations support the reasonability of our definitions.

Theorem 3.3. *Assume that p is a bounded even exponent on $[-1,1]$ such that p decreases on $[-1,0]$ and satisfies $p(0) = 1$. Assume further that $\alpha = \lim_{x\to 0} \big(p(x) - p(0)\big) \log \frac{1}{x} < \infty$. Let $u = \lim_{i\to\infty} u_{\lambda_i}$, where $\lambda_i \to 1$ is a sequence such that $c_{\lambda_i} \to c$. Then*

$$\varrho_{p_{\lambda_i}(\cdot)}(u'_{\lambda_i}) \to \varrho_{\mathrm{BV}^{p(\cdot)}}(u') := \int_{-1}^{1} |u'(x)|^{p(x)}\,dx + 2e^\alpha\Big(a - \int_{(0,1)} u'(x)\,dx\Big)$$

and u minimizes the energy $\varrho_{\mathrm{BV}^{p(\cdot)}}$ among functions which are absolutely continuous on $(-1,0)\cup(0,1)$ and attain the boundary values $\pm a$ at ± 1, respectively.

Proof. We may assume without loss of generality that u is non-decreasing. We denote by b (half of) the jump of u at the origin, $b = a - \int_{(0,1)} u'(x)\,dx$. For a Borel set $A \subset [-1,1]$ we define a measure μ by

$$\mu(A) = \int_{A\setminus\{0\}} u'(x)^{p(x)}\,dx + 2e^\alpha b\delta_0(A).$$

First we show that μ is a finite measure. From (3.2) and (3.4) we see that $u'(x)^{p(x)-1} = c/p(x) \leqslant c$ for $x \neq 0$. Thus

$$\int_{(-1,1)\setminus\{0\}} u'(x)^{p(x)}\,dx \leqslant c\int_{(-1,1)\setminus\{0\}} u'(x)\,dx = 2c\int_{(0,1)} u'(x)\,dx \leqslant 2ca.$$

The weight of the singular part is less than or equal to $2e^\alpha a$, thus μ is finite.

We denote $u_i := u_{\lambda_i}$, $p_i := p_{\lambda_i}$. Next, we approximate $\mu([-1,1])$ by the modulars $\varrho_{p_i(\cdot)}(u'_i)$. Since μ is a finite measure, we may fix $\varepsilon > 0$ and a neighborhood U of 0 such that $\mu(U\setminus\{0\}) < \varepsilon$. We are also free to choose the interval U so that the growth of u on U is less than $2b+\varepsilon$. Since $c_{\lambda_i} \to c$ by assumption, we see that $|u(z) - u_i(z)| \leqslant \varepsilon$ for all sufficiently large i, where $z \in \partial U$. Let v_i be the continuous function which is constant in each of the two components of $U \setminus K_{\lambda_i}$, linear in K_{λ_i}, and has boundary values u_i in ∂U. Since u_i is a minimizer we have $\varrho_{L^{p_i(\cdot)}(U)}(u'_i) \leqslant \varrho_{L^{p_i(\cdot)}(U)}(v'_i)$. As in the argument that was used to evaluate the contribution of the singularity (see the arguments after (3.5)) we have

$$\varrho_{L^{p_i(\cdot)}(U)}(u'_i) \leqslant \varrho_{L^{p_i(\cdot)}(K_{\lambda_i})}(v'_i) \leqslant \Big(\frac{3\varepsilon+2b}{|K_{\lambda_i}|}\Big)^{\lambda_i} |K_{\lambda_i}| \leqslant 4\varepsilon e^\alpha + \mu(0)$$

for all sufficiently large i. Here the growth of u_i is estimated by the growth of u. On the other hand, we already know by the same argument that $\varrho_{L^{p_i(\cdot)}(U)}(u_i') \geqslant \mu(0) - \varepsilon \geqslant \mu(U) - 2\varepsilon$ for all i large enough. Therefore

$$\varrho_{L^{p_i(\cdot)}(U)}(u_i') - 4\varepsilon e^{\alpha} \leqslant \mu(U) \leqslant \varrho_{L^{p_i(\cdot)}(U)}(u_i') + 2\varepsilon$$

for all sufficiently large i. We also know that u_i' is uniformly bounded in $[-1,1] \setminus U$, and so it follows by the dominated convergence that

$$\left|\varrho_{L^{p_i(\cdot)}([-1,1]\setminus U)}(u_i') - \mu([-1,1] \setminus U)\right| \leqslant \varepsilon$$

for all sufficiently large i. Combining these two estimates and letting $\varepsilon \to 0$ proves the claim regarding the convergence of modulars.

Finally we show that u is a minimizer of the the energy $\varrho_{\mathrm{BV}^{p(\cdot)}}$ among functions which are absolutely continuous on $(-1,0) \cup (0,1)$ and attain the boundary values $\pm a$ at ± 1, respectively. We already noticed after (3.4) that u attains the required boundary values. Assume to the contrary that v is an admissible function which has smaller $\varrho_{\mathrm{BV}^{p(\cdot)}}$-energy than u. Then we define a continuous function v_i by setting $v_i = v$ on $(-1,1) \setminus K_{\lambda_i}$ and requiring v_i to be linear on K_{λ_i}. For this sequence it is also true that

$$\varrho_{p_{\lambda_i}(\cdot)}(v_i') \to \varrho_{\mathrm{BV}^{p(\cdot)}}(v').$$

Since $\varrho_{\mathrm{BV}^{p(\cdot)}}(v') < \varrho_{\mathrm{BV}^{p(\cdot)}}(u')$ we obtain $\varrho_{p_{\lambda_i}(\cdot)}(v_i') < \varrho_{p_{\lambda_i}(\cdot)}(u_i')$ for some i, which is a contradiction since u_i was assumed to be the $p_{\lambda_i}(\cdot)$-minimizer. □

We also record the following facts:

Corollary 3.4. *Under the assumptions of Theorem 3.3 we have $c \leqslant e^{\alpha}$. If a singularity occurs in the derivative of u, then $c = e^{\alpha}$.*

Proof. First we show that $c \leqslant e^{\alpha}$. Assume on the contrary that $c > e^{\alpha}$. Then for any $m \in (e^{\alpha}, c)$ there exists i_0 such that $c_{\lambda_i} \geqslant m$ for every $i > i_0$. As before, we obtain

$$\begin{aligned}\lim_{i\to\infty} \int_0^1 \Big(\frac{c_{\lambda_i}}{p_{\lambda_i}(x)}\Big)^{\frac{p_{\lambda_i}(x)}{p_{\lambda_i}(x)-1}} dx &\geqslant \lim_{i\to\infty} |K_{\lambda_i}|\, m^{\frac{1}{\lambda_i - 1}} = \lim_{x\to 0^+} x\, m^{\frac{1}{p(x)-1}} \\ &= \lim_{x\to 0^+} \exp\Big(\frac{\log m - (p(x)-p(0))\log(1/x)}{p(x)-p(0)}\Big).\end{aligned}$$

Here the first inequality is based on the estimate $c_{\lambda_i}/p_{\lambda_i} \geqslant m$, which holds in K_{λ_i} for large values of i. The limit on the right-hand-side equals infinity since the numerator tends to $\log m - \alpha > 0$, whereas the denominator tends to zero. This contradicts (3.3) and therefore $c \leqslant e^{\alpha}$.

The second claim is verified by differentiating the energy $\varrho_{\mathrm{BV}^{p(\cdot)}}(u)$ with respect to c and using the minimization property of Theorem 3.3. The total $\mathrm{BV}^{p(\cdot)}$-energy as a function of c is given by

$$E(c) = 2\int_0^1 \Big(\frac{c}{p(x)}\Big)^{p(x)/(p(x)-1)} dx + 2e^{\alpha}\left(a - \int_0^1 \Big(\frac{c}{p(x)}\Big)^{1/(p(x)-1)} dx\right).$$

From this we easily see that $E'(c)$ equals

$$\begin{aligned}&2\int_0^1 \frac{1}{p(x)-1}\Big(\frac{c}{p(x)}\Big)^{1/(p(x)-1)} dx - 2e^{\alpha}\int_0^1 \frac{1}{p(x)-1}\frac{1}{c}\Big(\frac{c}{p(x)}\Big)^{1/(p(x)-1)} dx \\ &= 2\int_0^1 \frac{1}{p(x)-1}\Big(1 - \frac{e^{\alpha}}{c}\Big)\Big(\frac{c}{p(x)}\Big)^{1/(p(x)-1)} dx.\end{aligned}$$

Since $c \leqslant e^{\alpha}$, we see that the derivative of E is non-positive, which means that c should always be chosen as large as possible. In particular, if c can be chosen so large that the boundary values are attained without a jump, then this yields a minimizer. Otherwise, c should be chosen to equal e^{α}, and the rest of the growth of the minimizer is in the jump. □

4. Mixed BV-Sobolev norm

In this section we consider exponents p which satisfy the *strong log-Hölder continuity condition*: p is log-Hölder continuous in Ω and

$$\lim_{x\to y} \big|p(x)-1\big| \log \frac{1}{|x-y|} = 0$$

for every $y \in Y = \{\, y \in \Omega : p(y) = 1 \,\}$. The latter condition has been previously used by Acerbi and Mingione in connection to the regularity of minimizers, see [1,4]. Note that the strong log-Hölder continuity is slightly stronger than the condition required in Section 3, but it is still quite weak: every Hölder continuous exponent is strongly log-Hölder continuous. In Appendix A we will further study the case when p does not satisfy the strong log-Hölder condition.

Definition 4.1. A function $u \in L^1(\Omega)$ has *bounded variation*, denote $u \in \mathrm{BV}(\Omega)$, if

$$\sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ |\varphi| \leqslant 1 \right\} < \infty.$$

We denote $u \in \mathrm{BV}_{loc}(\Omega)$, if $u \in \mathrm{BV}(U)$ for every open set $U \subset\subset \Omega$.

It is well-known that for $u \in \mathrm{BV}_{loc}(\Omega)$ there is a Radon measure μ on Ω and a μ-measurable function $\sigma\colon \Omega \to \mathbb{R}^n$ such that $|\sigma| = 1$ μ-almost everywhere and

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu \tag{4.1}$$

for every $\varphi \in C_0^1(\Omega;\mathbb{R}^n)$, see [22, p. 167]. The measure μ is called the *total variation measure* and is denoted by $\|\nabla u\|$.

In what follows, we need a certain mixture of Sobolev and bounded variation norms. Hence we make the following definition.

Definition 4.2. Let p be a strongly log-Hölder continuous variable exponent on Ω. Then for $u \in \mathrm{BV}(\Omega) \cap W^{1,p(\cdot)}(\Omega \setminus Y)$ and a Borel set $E \subset \Omega$, we define

$$\varrho_{\mathrm{BV}^{p(\cdot)}(E)}(u) := \|\nabla u\|(Y \cap E) + \varrho_{L^{p(\cdot)}(E\setminus Y)}(\nabla u).$$

The norm in Ω is defined as usual by

$$\|u\|_{\mathrm{BV}^{p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \inf\big\{\lambda > 0 :\ \varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u/\lambda) \leqslant 1\big\}.$$

With this norm we define the space $\mathrm{BV}^{p(\cdot)}(\Omega)$ to consist of those measurable functions $u\colon \Omega \to \mathbb{R}$ for which $\|u\|_{\mathrm{BV}^{p(\cdot)}(\Omega)} < \infty$. We also denote $u \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)$, if $u \in \mathrm{BV}^{p(\cdot)}(U)$ for every open set $U \subset\subset \Omega$.

As a remark, we note some recent efforts to extend BV spaces to the framework of metric measure spaces, [8,9,45]. Like in those investigations, our first task is to prove some basic properties of our new BV-spaces.

Proposition 4.3. *The space* $\mathrm{BV}^{p(\cdot)}(\Omega)$ *equipped with the norm* $\|\cdot\|_{\mathrm{BV}^{p(\cdot)}(\Omega)}$ *is a Banach space.*

Proof. Let (u_i) be a Cauchy sequence in $\mathrm{BV}^{p(\cdot)}(\Omega)$. It is clear that $\mathrm{BV}^{p(\cdot)}(\Omega) \hookrightarrow \mathrm{BV}(\Omega)$, so $u_i \to u \in \mathrm{BV}(\Omega)$ for a suitable u. Also, (u_i) is a Cauchy sequence in $W^{1,p(\cdot)}(\Omega \setminus Y)$, so it converges to a Sobolev function u^* in $\Omega \setminus Y$. But $u^* = u$ almost everywhere in $\Omega \setminus Y$ by L^1-convergence, so that $u_i \to u$ in $\mathrm{BV}^{p(\cdot)}(\Omega)$. □

Recall that $u_\delta = \varphi_\delta * u$ is the standard mollification of u. We denote by M the Hardy–Littlewood maximal function,

$$Mu(x) = \sup_{r>0} \fint_{B(x,r)} u \, dx.$$

Note that we have the point-wise inequality

$$|u_\delta| \leqslant CMu \tag{4.2}$$

for every $u \in L^1_{loc}(\mathbb{R}^n)$.

In the next lemma we use a well-known technique due to Lars Diening to derive an estimate of the mollified function which is more useful than the usual, maximal function estimate in the case when $p \to 1$.

Lemma 4.4. *Let p be a bounded* log*-Hölder continuous exponent on an open set $\Omega \subset \mathbb{R}^n$. Then*

$$|u_\delta(x)|^{p(x)} \leqslant C\big(\varrho_{L^{p(\cdot)}(\Omega)}(u) + 1 + |\Omega|\big)^{p^+/p^-}\big((|u|^{p(\cdot)})_\delta(x) + 1\big)$$

for all $x \in \Omega$, $\delta \in (0,\infty)$, and $u \in L^{p(\cdot)}(\Omega)$.

Proof. Notice that we consider the zero-extension of u outside Ω. Let $u \in L^{p(\cdot)}(\Omega)$ and $x \in \Omega$. If $\delta > 1/2$, then we have (for a C depending on n and $\max|\varphi|$)

$$|u_\delta(x)| \leqslant C\int_\Omega |u|\,dy \leqslant C\big(\varrho_{L^{p(\cdot)}(\Omega)}(u) + |\Omega|\big),$$

and the claim is clear in this case. Assume now that $\delta \leqslant 1/2$ and denote $B = B(x,\delta)$. In the proof of [15, Lemma 3.3] it was shown that

$$|u_\delta(x)|^{p(x)} \leqslant C\delta^{-np(x)/p_B^-}\bigg(\int_B |u(y)|^{p(y)} + 1\,dy\bigg)^{p(x)/p_B^-}.$$

Combining this with the estimate

$$\bigg(\int_B |u(y)|^{p(y)} + 1\,dy\bigg)^{p(x)/p_B^- - 1} \leqslant \big(\varrho_{L^{p(\cdot)}(\Omega)}(u) + 1 + |B|\big)^{p^+/p^- - 1}$$

gives

$$|u_\delta(x)|^{p(x)} \leqslant C\big(\varrho_{L^{p(\cdot)}(\Omega)}(u) + 1 + |\Omega|\big)^{p^+/p^- - 1}\delta^{n(1-p(x)/p_B^-)}\bigg(\fint_B |u(y)|^{p(y)}\,dy + 1\bigg).$$

Now $\delta^{n(1-p(x)/p_B^-)} \leqslant C$ by log-Hölder continuity, so we are done. □

One central property of BV-spaces is that a BV-function can be approximated by smooth functions (cf. [22,54]). Usually this property is stated as a lim inf- and a lim sup-property. The former reads

$$\liminf_{\delta\to 0}\|\nabla u_\delta\|(U) \geqslant \|\nabla u\|(U),$$

where $u \in \mathrm{BV}(U)$ and $U \subset\subset \Omega$ is open. For the lim sup-property it is not possible to use convolution if we want to use open sets. Thus the property reads, for $u \in \mathrm{BV}(U)$ and $U \subset\subset \Omega$ open,

$$\limsup_{j\to\infty}\|\nabla u_j\|(U) \leqslant \|\nabla u\|(U),$$

where the functions $u_j \in \mathrm{BV}(U) \cap C^\infty(U)$ tend to u in L^1. An alternative lim sup-property is to use convolution, but give up the open set. For completeness we provide a proof below:

Lemma 4.5. *Let $u \in \mathrm{BV}(\Omega)$ and let $F \subset \Omega$ be closed. Then*

$$\limsup_{\delta\to 0}\|\nabla u_\delta\|(F) \leqslant \|\nabla u\|(F).$$

Proof. Fix $\varepsilon > 0$. Since the variation norm is a Radon measure, we may choose an open set U with $F \subset U \subset\subset \Omega$ so that $\|\nabla u\|(U) < \|\nabla u\|(F) + \varepsilon$. The claim is proved by slightly modifying the proof of Theorem 2, [22, Chapter 5.2.2]. Like in that book, we define

$$U_k = \Big\{x \in U : \ \mathrm{dist}(x,\partial U) > \frac{1}{m+k}\Big\}.$$

We choose m so large that $F \subset U_1$. As in [22], we form a sequence $u_j \in \mathrm{BV}(U) \cap C^\infty(U)$ which tends to u in L^1 so that

$$\limsup_{j\to\infty}\|\nabla u_j\|(U) \leqslant \|\nabla u\|(U),$$

Let us fix a sequence (δ_j) of positive real numbers tending to zero such that

$$\lim_{j\to\infty}\|\nabla u_{\delta_j}\|(F) = \limsup_{\delta\to 0}\|\nabla u_\delta\|(F).$$

A look at the proof of the BV-lim sup-property, [22, 5.2.2. Theorem 2], shows that $u_j|_F = u_{\delta_j}|_F$, when j is so large that $\delta_j < d(F, \Omega \setminus U_1)$. Thus

$$\limsup_{\delta\to 0} \|\nabla u_\delta\|(F) \leqslant \limsup_{j\to\infty} \|\nabla u_j\|(U) \leqslant \|\nabla u\|(U) < \|\nabla u\|(F) + \varepsilon.$$

The claim follows from this as $\varepsilon \to 0$. □

As the main result of this section we prove the analogue of the previous lemma in the BV-$W^{1,p(\cdot)}$ space. It is one in the line of results indicating that the definition adopted is sensible, at least for our class of exponents.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded and let p be a bounded, strongly* log*-Hölder continuous exponent in Ω. If $u \in \mathrm{BV}^{p(\cdot)}(\Omega)$ and $F \subset \Omega$ is closed, then*

$$\limsup_{\delta\to 0} \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u_\delta) \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u).$$

Proof. As in the proof of Theorem 3.3, we split the modular into three parts: in Y, away from Y and near ∂Y. For the first two we can directly investigate the limit behavior. The third case is handled by restricting to so small a neighborhood of the boundary that the term becomes insignificant. We move on to the details.

Since the function u_δ is smooth in F, we have

$$\varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u_\delta) = \varrho_{L^{p(\cdot)}(F)}(\nabla u_\delta),$$

so we commence to work with the right hand side of this equation. Let $E_{\delta'} = \{ x \in \mathbb{R}^n : \mathrm{dist}(x, E) < \delta' \}$ denote the open δ'-neighborhood of $E \subset \mathbb{R}^n$ for $\delta' > 0$. We write $Y' = \overline{(Y \cap F)_{\delta'}} \cap F$ for $\delta' > 0$ and start by dividing the $p(\cdot)$-energy integral of u_δ into two parts:

$$\int_F |\nabla u_\delta|^{p(x)}\, dx = \int_{F\setminus Y'} |\nabla u_\delta|^{p(x)}\, dx + \int_{Y'} |\nabla u_\delta|^{p(x)}\, dx.$$

Assume that $0 < \delta \leqslant \frac{1}{2}\delta'$ and $x \in F \setminus Y'$. We define $p_0 = p^-_{F_{\delta'}\setminus Y_{\delta'/2}}$ and note that $p_0 > 1$. Using the differentiation rule $D_i(u_\delta)(x) = (D_i u)_\delta(x)$, $i = 1, \dots, n$, the estimate (4.2), and Lemma 4.4 we have

$$|\nabla u_\delta(x)|^{p(x)/p_0} \leqslant CM\big(|\nabla u|^{p(\cdot)/p_0}\big)(x) + C.$$

Note that the constant C depends on $\varrho_{L^{p(\cdot)}(\Omega\setminus Y)}(\nabla u)$. Since $p_0 > 1$ we conclude by the L^{p_0}-maximal function theorem that

$$C\big(M|\nabla u|^{p(\cdot)/p_0}\big)^{p_0} + C$$

is a point-wise L^1 majorant of $|\nabla u_\delta|^{p(\cdot)}$ in $F \setminus Y'$. Since $u \in W^{1,1}(F \setminus Y)$, it is clear that $|\nabla u_\delta| \to |\nabla u|$ almost everywhere in $F \setminus Y$ as $\delta \to 0$. Hence the Dominated Convergence Theorem implies that

$$\lim_{\delta\to 0} \int_{F\setminus Y'} |\nabla u_\delta|^{p(x)}\, dx = \int_{F\setminus Y'} |\nabla u|^{p(x)}\, dx \leqslant \int_{F\setminus Y} |\nabla u|^{p(x)}\, dx$$

for any $\delta' > 0$.

Thus we have shown that

$$\limsup_{\delta\to 0} \int_F |\nabla u_\delta|^{p(x)}\, dx \leqslant \int_{F\setminus Y} |\nabla u|^{p(x)}\, dx + \limsup_{\delta\to 0} \int_{Y'} |\nabla u_\delta|^{p(x)}\, dx. \tag{4.3}$$

This inequality holds for every $\delta' > 0$. Now we fix $\varepsilon > 0$ and restrict our attention to so small values of δ' that $|Y' \setminus Y| < \varepsilon$ and

$$\int_{Y'\setminus Y} |\nabla u|^{p(x)}\, dx < \varepsilon.$$

We estimate the remaining integral from (4.3) by another split:

$$\int_{Y'} |\nabla u_\delta|^{p(x)}\, dx = \int_{Y'\setminus Y_\delta} |\nabla u_\delta|^{p(x)}\, dx + \int_{Y_\delta\cap Y'} |\nabla u_\delta|^{p(x)}\, dx.$$

For the first integral we use almost the same method as before, except that we do not use the maximal function as an intermediary (because p is not now bounded away from 1 as $\delta \to 0$). We have

$$|\nabla u_\delta(x)|^{p(x)} \leqslant C\big(|\nabla u|^{p(\cdot)}\big)_\delta(x) + C$$

for $x \in Y' \setminus Y_\delta$ by Lemma 4.4. Therefore

$$\int_{Y'\setminus Y_\delta} |\nabla u_\delta|^{p(x)}\,dx \leqslant C \int_{Y'\setminus Y_\delta} \big(|\nabla u|^{p(\cdot)}\big)_\delta\,dx + C|Y' \setminus Y|.$$

Swapping the order of integration on the right hand side give us the upper bound $C\varepsilon$ for the whole right hand side, in view of our previous assumption of the smallness of $Y' \setminus Y$. Combining this with (4.3), we have

$$\limsup_{\delta\to 0} \int_F |\nabla u_\delta|^{p(x)}\,dx \leqslant \int_{F\setminus Y} |\nabla u|^{p(x)}\,dx + \limsup_{\delta\to 0} \int_{Y_\delta\cap Y'} |\nabla u_\delta|^{p(x)}\,dx + C\varepsilon. \tag{4.4}$$

We therefore proceed to estimate the last remaining integral of ∇u_δ.

Since $|\sigma| = 1$ for $\|\nabla u\|$-almost every point, we infer from (4.1) that

$$|\nabla u_\delta(x)| = \left|\int_{\mathbb{R}^n} u(y)\,\nabla\varphi_\delta(x-y)\,dy\right| \leqslant C\int_{\mathbb{R}^n} |\varphi_\delta(x-y)|\,d\|\nabla u\|(y) \leqslant C\delta^{-n}\max|\varphi|\,\|\nabla u\|(\Omega) =: M\delta^{-n}.$$

This holds for $x \in Y_\delta$ whenever $B(x,\delta) \subset \Omega$. Let $\delta_0 > 0$ be so small that the condition in the previous sentence holds for all $\delta \in (0,\delta_0)$, and additionally $\max\{1, M\}^{p^+_{Y_\delta}-1} < 1+\varepsilon$ for the same range. Then we obtain the estimate

$$|\nabla u_\delta(x)|^{p(x)-1} \leqslant \max\{1, M\}^{p^+_{Y_\delta}-1}\delta^{-n(p(x)-1)} \leqslant (1+\varepsilon)\delta^{-n(p(x)-1)}$$

for all $x \in Y_\delta \cap F$ and $\delta \in (0,\delta_0)$.

Let us denote by $Y(\delta)$, $\delta > 0$, the set of all points $y \in Y$ for which

$$|z-y|^{-n(p(z)-1)} < 1+\varepsilon$$

for every $z \in \overline{B}(y,\delta)$. Note that the strong log-Hölder assumption implies that there exists $\delta_y > 0$ for every $y \in Y$ so that $y \in Y(\delta)$ if and only if $\delta \in (0,\delta_y)$. Further, the continuity of the function $y \mapsto \sup_{z\in B(y,\delta)} |z-y|^{-n(p(z)-1)}$ implies that $Y(\delta)$ is open in Y. Next we denote by $\Omega(\delta)$ the set of those points x in Y_δ which have the unique point $y \in Y$ satisfying $|x-y| = d(x,Y)$ and $y \in Y(\delta)$. We note that $\Omega(\delta)$ is open, $Y_\delta \setminus \Omega(\delta)$ is increasing in δ, and that $\lim_{\delta\to 0^+} Y_\delta \setminus \Omega(\delta) = \emptyset$. Since $\|\nabla u\|$ is a Radon measure we may choose $\delta_1 \in (0,\delta_0)$ so that

$$\|\nabla u\|\big(Y_{\delta_1} \setminus \Omega(\delta_1)\big) < \varepsilon \quad\text{and}\quad \|\nabla u\|(Y_{2\delta_1} \cap F_{\delta_1}) \leqslant \|\nabla u\|(Y\cap F) + \varepsilon. \tag{4.5}$$

If $x \in \Omega(\delta)$, then clearly $\delta^{-n(p(x)-1)} \leqslant 1+\varepsilon$. We conclude that

$$\begin{aligned}\int_{Y_\delta\cap F\cap\Omega(\delta)} |\nabla u_\delta(x)|^{p(x)}\,dx &\leqslant \int_{Y_\delta\cap F\cap\Omega(\delta)} |\nabla u_\delta(x)|(1+\varepsilon)^2\,dx \leqslant (1+\varepsilon)^2 \int_{Y_{\delta_1}\cap F} |\nabla u_\delta(x)|\,dx \\ &\leqslant (1+\varepsilon)^3\|\nabla u\|(Y_{2\delta_1}\cap F_{\delta_1}) \leqslant (1+\varepsilon)^3(\|\nabla u\|(Y\cap F)+\varepsilon)\end{aligned}$$

for all sufficiently small δ. Here we used [54, Theorem 5.3.1] for the third inequality.

For $x \notin \Omega(\delta)$ we still have $\delta^{-n(p(y)-1)} \leqslant C$ with a constant depending on the log-Hölder continuity constant of p and the dimension n. Therefore

$$\int_{(Y_\delta\cap F)\setminus\Omega(\delta)} |\nabla u_\delta(x)|^{p(x)}\,dx \leqslant C\int_{Y_\delta\setminus\Omega(\delta)} |\nabla u_\delta(x)|\,dx \leqslant C\|\nabla u_\delta\|\big(\overline{Y_{\delta_1}} \setminus \Omega(\delta_1)\big)$$

for all $\delta \in (0,\delta_1]$. Now we use Lemma 4.5 on the closed set $\overline{Y_{\delta_1}} \setminus \Omega(\delta_1)$ to conclude that

$$\limsup_{\delta\to 0} \int_{(Y_\delta\cap F)\setminus\Omega(\delta)} |\nabla u_\delta(x)|^{p(x)}\,dx \leqslant C\|\nabla u\|\big(\overline{Y_{\delta_1}} \setminus \Omega(\delta_1)\big) < C\varepsilon.$$

Here the second inequality follows from (4.5).

We then combine the conclusions from the previous two paragraphs to conclude that

$$\limsup_{\delta\to 0} \int_{Y_\delta \cap F} |\nabla u_\delta(x)|^{p(x)}\, dx \leqslant (1+\varepsilon)^3 \|\nabla u\|(Y\cap F) + C\varepsilon.$$

Combining this with (4.4) gives

$$\limsup_{\delta\to 0} \int_{F} |\nabla u_\delta|^{p(x)}\, dx \leqslant \int_{F\setminus Y} |\nabla u|^{p(x)}\, dx + (1+\varepsilon)^3 \|\nabla u\|(Y\cap F) + C\varepsilon,$$

from which the claim follows by letting $\varepsilon \to 0$. □

If p is not strongly log-Hölder continuous, then even basic approximation results appear to be problematic (see Example A.1). In view of our examples in Appendix A it is fair to conclude that it will be difficult to build a satisfactory theory for the spaces $\mathrm{BV}^{p(\cdot)}$ in this case. Therefore we confine our attention to the case of strongly log-Hölder continuous exponents in our main theorems about $\mathrm{BV}^{p(\cdot)}$ minimizers.

5. Caccioppoli-type estimates

In this section we begin our investigation of higher-dimensional minimizers. We start by recalling the following basic definition, which is known to be equivalent with the $p(\cdot)$-minimization condition if $p^- > 1$ [37, Theorem 5.7].

Definition 5.1. Let $1 < p^- \leqslant p^+ < \infty$. We say that a function $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a *(weak) $p(\cdot)$-solution* in Ω, if

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx = 0 \tag{5.1}$$

for every test function $\varphi \in W^{1,p(\cdot)}(\Omega)$ with compact support in Ω.

If we know that C^∞-functions are dense in $W^{1,p(\cdot)}(\Omega)$, then it suffices to test with functions that belong to $C_0^\infty(\Omega)$. Density of smooth functions is investigated in [24,40,50,53], and a basic result is that density holds at least if the exponent is log-Hölder continuous.

The existence of $p(\cdot)$-solutions has been discussed in [29,37,39]. In what follows, we consider the Dirichlet problem with bounded Sobolev boundary values in a bounded open set $\Omega \subset \mathbb{R}^n$. Let p be a variable exponent on Ω such that $1 < p^- < p^+ < \infty$ and let $f \in L^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ be the boundary value function. Then there is a unique $p(\cdot)$-solution $u \in W^{1,p(\cdot)}(\Omega)$ satisfying $u - f \in W_0^{1,p(\cdot)}(\Omega)$, see [39, Theorem 2]. The function u is called the solution of Dirichlet problem with boundary values f.

Remark 5.2. We could instead consider a bounded open set $\Omega \subset \mathbb{R}^n$ where the Dirichlet problem is solvable for any continuous boundary function f, see [7] and Remark 7.2.

In Sections 6 and 7 we will get back to studying limits of sequences of solutions corresponding to truncated exponents as in Section 3. Before that we record some basic lemmas regarding solutions. Notice that we assume $p^- > 1$ for these lemmas.

Lemma 5.3 (Caccioppoli-type estimate)**.** *Let $1 < p^- \leqslant p^+ < \infty$ and $\eta \in C_0^\infty(\Omega)$ with $0 \leqslant \eta \leqslant 1$. If u is a $p(\cdot)$-solution in Ω, then*

$$\int_\Omega |\nabla u|^{p(x)}\eta^{p^+}\, dx \leqslant (2p^+)^{2p^+} \int_\Omega |u|^{p(x)}|\nabla\eta|^{p(x)} dx.$$

Proof. Using $u\eta^{p^+} \in W^{1,p(\cdot)}(\Omega)$ as a test function (i.e., setting $\varphi = u\eta^{p^+}$ in (5.1)) we obtain

$$\begin{aligned} 0 &= \int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u\eta^{p^+})\, dx \\ &= \int_\Omega p(x)|\nabla u|^{p(x)}\eta^{p^+}\, dx + \int_\Omega p(x)|\nabla u|^{p(x)-2} u p^+ \eta^{p^+-1}\nabla u\cdot\nabla\eta\, dx. \end{aligned}$$

Since the first integral in the right hand side is non-negative this implies that

$$p^- \int_\Omega |\nabla u|^{p(x)}\eta^{p^+}\, dx \leqslant p^+ \int_\Omega p(x)|\nabla u|^{p(x)-1}|u|\ \eta^{p^+-1}|\nabla\eta|\ dx. \tag{5.2}$$

To the right hand side we apply Young's inequality,

$$ab \leqslant \left(\frac{1}{\varepsilon}\right)^{p(x)-1} \frac{a^{p(x)}}{p(x)} + \varepsilon \frac{b^{p'(x)}}{p'(x)}$$

(for $0 < \varepsilon \leqslant 1$), with $a = |u|\,|\nabla\eta|\,\eta^{p^+ - \frac{p^+}{p'(x)} - 1}$ and $b = |\nabla u|^{p(x)-1}\eta^{\frac{p^+}{p'(x)}}$. This gives

$$\begin{aligned} p^- \int_\Omega |\nabla u|^{p(x)}\eta^{p^+}\,dx \leqslant p^+ \left(\frac{1}{\varepsilon}\right)^{p^+-1} &\int_\Omega |u|^{p(x)}|\nabla\eta|^{p(x)}\eta^{p^+-p(x)}dx \\ &+ p^+(p^+-1)\varepsilon \int_\Omega |\nabla u|^{p(x)}\eta^{p^+}\,dx. \end{aligned} \tag{5.3}$$

By choosing

$$\varepsilon = \min\left\{1, \frac{p^-}{2p^+(p^+-1)}\right\}$$

we can absorb the last term in (5.3) to the left hand side, obtaining

$$\int_\Omega |\nabla u|^{p(x)}\eta^{p^+}\,dx \leqslant p^+ \left(\frac{2p^+(p^+-1)}{p^-} + 1\right)^{p^+-1} \frac{2}{p^-} \int_\Omega |u|^{p(x)}|\nabla\eta|^{p(x)}dx. \qquad \square$$

For the next version of the Caccioppoli estimate, we need a Poincaré inequality for the modulars. In the variable exponent setting the Poincaré inequality is typically stated in a norm-form, but the proof can be easily modified. Here and elsewhere u_B denotes the integral average of u in B.

Lemma 5.4. *Let p be a* log-*Hölder continuous exponent with $1 < p^- \leqslant p^+ < \infty$, and let $B \subset \Omega$ be a ball. Then the variable exponent, modular Poincaré inequality*

$$\varrho_{L^{p(\cdot)}(B)}\big((u-u_B)\operatorname{diam}(B)^{-1}\big) \leqslant C\varrho_{L^{p(\cdot)}(B)}(\nabla u) + C|B|$$

holds for all $u \in W^{1,p(\cdot)}(B)$ with $\varrho_{L^{p(\cdot)}(B)}(\nabla u) \leqslant 1$.

Proof. We have by [32, Lemma 4] that

$$\frac{|u(x)-u(y)|}{\operatorname{diam}(B)} \leqslant C(M|\nabla u|(x) + M|\nabla u|(y))$$

for almost all $x, y \in B$ (where M is the Hardy–Littlewood maximal operator). By the estimate

$$|u(x) - u_B| \leqslant \fint_B |u(x)-u(y)|\,dy,$$

we arrive at

$$\frac{|u(x)-u_B|}{\operatorname{diam}(B)} \leqslant C\left(M|\nabla u|(x) + \fint_B M|\nabla u|(y)\,dy\right).$$

Next, we use the constant exponent Hölder inequality for the exponent p_B^- and the maximal inequality to derive

$$\fint_B M|\nabla u|(y)\,dy \leqslant \left(\fint_B (M|\nabla u|(y))^{p_B^-}\,dy\right)^{1/p_B^-} \leqslant C\left(\fint_B |\nabla u(y)|^{p_B^-}\,dy\right)^{1/p_B^-}.$$

Thus we have

$$\frac{|u(x)-u_B|}{\operatorname{diam}(B)} \leqslant CM|\nabla u|(x) + C\left(\fint_B |\nabla u(y)|^{p(y)} + 1\,dy\right)^{1/p_B^-}.$$

We raise both sides of this inequality to the power $p(x)$ and integrate over $x \in B$ to obtain

$$\int_B \left(\frac{|u(x)-u_B|}{\operatorname{diam}(B)}\right)^{p(x)} dx \leqslant C\int_B \left[\big(M|\nabla u|(x)\big)^{p(x)} + \left(\fint_B |\nabla u(y)|^{p(y)}\,dy + 1\right)^{p_B^+/p_B^-}\right] dx.$$

The log-Hölder continuity of the exponent implies that $|B|^{-p_B^+/p_B^-} \leqslant C\,|B|^{-1}$, and [15, Lemma 3.3] implies that

$$\varrho_{L^{p(\cdot)}(B)}(M|\nabla u|) \leqslant C\varrho_{L^{p(\cdot)}(B)}(\nabla u) + C|B|.$$

Using this in the previous inequality gives

$$\int_B \left(\frac{|u(x)-u_B|}{\operatorname{diam}(B)}\right)^{p(x)} dx \leqslant C\int_B |\nabla u(x)|^{p(x)}\,dx + C\biggl(\int_B |\nabla u(y)|^{p(y)}\,dy\biggr)^{p_B^+/p_B^-} + C|B|.$$

The claim follows after we replace the exponent p_B^+/p_B^- by 1, which is permissible by the assumption $\varrho_{L^{p(\cdot)}(B)}(\nabla u) \leqslant 1$. □

We state one more auxiliary result by combining the Caccioppoli estimate with the Poincaré inequality. This result shows, in particular, that the the energy-integrals of the sequence (u_j) in next corollary have a kind of equi-uniform continuity with respect to the integration domain.

Corollary 5.5. *Let p be log-Hölder continuous in Ω with $1 < p^- \leqslant p^+ < \infty$. Assume that (u_j) is a sequence of $p(\cdot)$-solutions in Ω which converges in $L^{p(\cdot)}_{loc}(\Omega)$ to a function $u \in W^{1,p(\cdot)}_{loc}(\Omega)$. Fix a set $D \subset\subset \Omega$ with $\varrho_{L^{p(\cdot)}(D)}(\nabla u) \leqslant 1$. Then*

$$\limsup_{j\to\infty} \int_B |\nabla u_j|^{p(x)}\,dx \leqslant C\int_{2B} |\nabla u|^{p(x)}\,dx + C|B|$$

for every ball $2B \subset D$.

Proof. Fix a ball B with $2B \subset D$ and note that $u_j - (u_j)_{2B}$ is also a $p(\cdot)$-solution in $2B$. We choose $\eta \in C_0^\infty(2B)$ with $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in B, and $|\nabla\eta| \leqslant C/\operatorname{diam}(B)$. Then the Caccioppoli estimate, Lemma 5.3, gives

$$\int_B |\nabla u_j|^{p(x)}\,dx \leqslant C\int_{2B} |u_j - (u_j)_{2B}|^{p(x)} \operatorname{diam}(B)^{-p(x)}\,dx$$

for a constant C depending only on p^+. By assumption, it is clear that $u_j - (u_j)_{2B} \to u - u_{2B}$ in $L^{p(\cdot)}(2B)$. To conclude, we apply the Poincaré inequality, Lemma 5.4, to u:

$$\varrho_{L^{p(\cdot)}(2B)}\bigl((u-u_{2B})\operatorname{diam}(B)^{-1}\bigr) \leqslant C\varrho_{L^{p(\cdot)}(2B)}(\nabla u) + C|B|.$$

Combining the previous two inequalities and the convergence of the functions gives the result. □

6. Energy minimizers: auxiliary results

We now return to the study of a variable exponent p which attains the value 1. Recall that $\Omega \subset \mathbb{R}^n$ is bounded, $Y = \{\, x \in \Omega : p(x) = 1 \,\}$, and $p_\lambda = \max(p,\lambda)$ for $\lambda > 1$.

Assume that $f \in L^\infty(\Omega) \cap W^{1,p_\delta(\cdot)}(\Omega)$ for some $\delta > 1$. Then for any $1 < \lambda \leqslant \delta$ there exists a $p_\lambda(\cdot)$-solution with boundary values f. This unique solution is denoted by u_λ. As has been mentioned before, we will construct the solution of the $p(\cdot)$-minimization problem as a limit of the functions u_λ. In this section we will derive some basic results regarding the limit function. Note that in these results we do not need to assume that the exponent is strongly log-Hölder continuous.

In what follows, we use the notation $\rightharpoonup$ for weak convergence, and $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$ for bounded and compact embeddings, respectively.

Proposition 6.1. *Let p be a bounded continuous exponent and let (λ_j) be a sequence decreasing to 1. Suppose that (u_{λ_j}) is a sequence of $p_{\lambda_j}(\cdot)$-solutions in Ω with boundary values $f \in W^{1,p_\delta(\cdot)}(\Omega) \cap L^\infty(\Omega)$, $\delta > 1$.*

Then there exists a subsequence (λ_j) and $u \in L^\infty(\Omega)$ such that

(i) *$u_{\lambda_j} \to u$ in $L^{p_\delta(\cdot)}(\Omega)$ for every $\delta \in [1, \frac{n}{n-1})$;*
(ii) *$u_{\lambda_j} \rightharpoonup u$ in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$; and*
(iii) *u is a $p(\cdot)$-solution in $\Omega \setminus Y$.*

Proof. We write $u_{\lambda_j} = u_j$ and $p_{\lambda_j} = p_j$. Fix open sets $U \subset\subset V \subset\subset \Omega$ and choose a function $\eta \in C_0^\infty(V)$ with $\chi_U \leqslant \eta \leqslant 1$. Then it follows from Lemma 5.3 that

$$\begin{aligned}\int_U |\nabla u_j(x)|^{p_j(x)}\,dx &\leqslant C\int_V |u_j(x)|^{p_j(x)}|\nabla\eta(x)|^{p_j(x)}\,dx\\ &\leqslant C(1+\operatorname{ess\,sup}|f|)^{p^+}(1+\sup|\nabla\eta|)^{p^+}|\Omega|,\end{aligned}$$

as it is clear that $|u_j| \leqslant \operatorname{ess\,sup}|f|$ almost everywhere. Since

$$\int_U |\nabla u_j(x)|^{p(x)}\,dx \leqslant \int_U |\nabla u_j(x)|^{p_j(x)}\,dx + |U|,$$

we conclude that the sequence (u_j) is bounded in $W^{1,p(\cdot)}(U)$.

As u and u_j are bounded by a constant independent of j, we see that it suffices to prove (i) in $L^{p_\delta(\cdot)}_{loc}(\Omega)$. Since the exponent is uniformly continuous in V there exists $r > 0$ such that the inequality $p_B^+ \leqslant (p_B^-)^* - \varepsilon$ holds for every ball $B \subset V$ with radius less than r. Here the asterisk denotes the Sobolev conjugate exponent and $\varepsilon > 0$. For $0 < \varepsilon < \frac{n}{n-1} - \delta$ we have $(p_B^-)^* - \varepsilon \geqslant p_\delta(x)$ when $x \in B$ since $(p_B^-)^* \geqslant \frac{n}{n-1}$. Then for this range of ε and for every ball $B \subset V$ with radius less than r we have the chain of embeddings

$$W^{1,p(\cdot)}(B) \hookrightarrow W^{1,p_B^-}(B) \hookrightarrow\hookrightarrow L^{(p_B^-)^*-\varepsilon}(B) \hookrightarrow L^{p_\delta}(B).$$

The set U can be covered by finitely many balls $B_i \subset V$ with radius less than r. Combining these embeddings we get a compact embedding $W^{1,p(\cdot)}(\cup B_i) \hookrightarrow\hookrightarrow L^{p_\delta(\cdot)}(\cup B_i)$. Since we concluded that (u_j) is bounded in $W^{1,p(\cdot)}_{loc}(\Omega)$, the compact embedding implies that we may choose a subsequence which converges in $L^{p_\delta(\cdot)}(U)$. This gives Claim (6.1).

Suppose now that additionally $\overline{V} \cap Y = \emptyset$. Then $1 < p_U^- \leqslant p_U^+ < \infty$ and hence the space $W^{1,p(\cdot)}(U)$ is reflexive [43]. Since (u_j) is a bounded sequence in this space, we find that there exists a weakly converging subsequence. Thus (6.1) is proven.

It follows that $u \in W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$, and hence to prove (6.1) we need to check that (5.1) is satisfied for every test function $\varphi \in W^{1,p(\cdot)}(\Omega \setminus Y)$ with compact support in $\Omega \setminus Y$. With U and V as above we fix such a φ and with $\operatorname{spt}\varphi \subset U$. Then we choose $\eta \in C_0^\infty(V)$ with $\chi_U \leqslant \eta$. Since $p_V^- > 1$, we have $p_j = p$ in V for all large j. Let (u_j) be a subsequence that converges weakly to $u \in W^{1,p(\cdot)}(V)$. Using $\eta(u - u_j)$ as a test function for the solution u_j implies that

$$\int_V p(x)|\nabla u_j|^{p(x)-2}(u-u_j)\nabla u_j\cdot\nabla\eta\,dx + \int_V p(x)|\nabla u_j|^{p(x)-2}\eta\nabla u_j\cdot(\nabla u - \nabla u_j)\,dx = 0.$$

From this we infer as in the fixed exponent case that there exists a subsequence (u_j) for which

$$|\nabla u_j|^{p(x)-2}\nabla u_j \rightharpoonup |\nabla u|^{p(x)-2}\nabla u$$

in $L^{p'(\cdot)}(U)$. We skip here the details which are exactly the same as in [36, Theorem 11]. By the weak convergence in $L^{p'(\cdot)}(U)$ we obtain

$$\begin{aligned}0 = \int_V p(x)|\nabla u_j|^{p(x)-2}\nabla u_j\cdot\nabla\varphi\,dx &= \int_U p(x)|\nabla u_j|^{p(x)-2}\nabla u_j\cdot\nabla\varphi\,dx\\ \to \int_U p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx &= \int_V p(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx,\end{aligned}$$

which completes the proof. □

Remark 6.2. Under the assumptions of the previous lemma, u also has a minimization property inside Y. More precisely, for every open $U \subset Y$, every compact $K \subset U$ and every $v \in \mathrm{BV}_{loc}(U)$ with $v = u$ in $U \setminus K$ we have

$$\|\nabla u\|(K) \leqslant \|\nabla v\|(K).$$

To derive this we first notice that we have $p_j(x) = \lambda_j$ for every $x \in U$. Thus the functions u_j are as λ_j-solutions continuous and the claim follows from Proposition 6.1 and [42, Proposition 4.5].

Proposition 6.3. *Let p be a bounded continuous exponent and (λ_j) be a sequence decreasing to 1. Let (u_{λ_j}) be a sequence of $p_{\lambda_j}(\cdot)$-solutions in Ω with boundary values $f \in W^{1,p_\delta(\cdot)}(\Omega) \cap L^\infty(\Omega)$, $\delta > 1$, so that $u_{\lambda_j} \to u$ in $L^{p(\cdot)}(\Omega)$ and $u_{\lambda_j} \rightharpoonup u$ in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$. Then*

(i) *$u \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)$; and*

(ii) *$\varrho_{\mathrm{BV}^{p(\cdot)}(E)}(u) \leqslant \liminf_{j\to\infty} \varrho_{L^{p_j(\cdot)}(E)}(\nabla u_{\lambda_j})$ for every Borel set $E \subset\subset \Omega$.*

Note that $p_j \geqslant p$ on the right hand side of the inequality in (ii). This means that the claim is stronger than the usual lower semicontinuity.

Proof. Again, we denote $u_{\lambda_j} = u_j$ and $p_{\lambda_j} = p_j$. By the Caccioppoli estimate, Lemma 5.3, $(|\nabla u_j|)$ is bounded in $L^1_{loc}(\Omega)$ and thus by [22, Theorem 1, p. 172] we have $u \in \mathrm{BV}_{loc}(\Omega) \cap L^\infty(\Omega)$. Therefore, to prove (i) we need only verify that $|\nabla u| \in L^{p(\cdot)}(U \setminus Y)$ for every open $U \subset\subset \Omega$. We write $Y_k = \{x \in \Omega\colon p(x) \leqslant 1 + 1/k\}$ for $k \in \mathbb{N}$. The functions u_j converge weakly to u in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$, so by Lemma 2.1 and $p \leqslant p_j$ we find that

$$\begin{aligned}\int_{U\setminus Y_k} |\nabla u|^{p(x)} dx &\leqslant \liminf_{j\to\infty} \int_{U\setminus Y_k} |\nabla u_j|^{p(x)} dx \\ &\leqslant |U \setminus Y_k| + \liminf_{j\to\infty} \int_{U\setminus Y_k} |\nabla u_j|^{p_j(x)} dx.\end{aligned}$$

Fix an open set V with $U \subset\subset V \subset\subset \Omega$ and $\eta \in C^\infty_0(V)$ with $\eta = 1$ in U. Lemma 5.3 implies that

$$\begin{aligned}\int_{U\setminus Y_k} |\nabla u_j|^{p_j(x)} dx &\leqslant C \int_V |u_j|^{p_j(x)} |\nabla \eta|^{p_j(x)} dx \\ &\leqslant C(1 + \operatorname{ess\,sup} |f|)^{p^+} (1 + \sup |\nabla \eta|)^{p^+} |V| < \infty.\end{aligned}$$

By the monotone convergence theorem,

$$\int_{U\setminus Y} |\nabla u|^{p(x)} dx = \lim_{k\to\infty} \int_U |\nabla u|^{p(x)} \chi_{Y_k} dx = \lim_{k\to\infty} \int_{U\setminus Y_k} |\nabla u|^{p(x)} dx.$$

Combining this with the previous inequalities completes the proof of (i).

To prove (ii), let $\varepsilon \in (0,1)$ and fix a Borel set $E \subset\subset \Omega$. Choose an open neighborhood U of $E \cap Y$ which is so small that $\varrho_{\mathrm{BV}^{p(\cdot)}(U\setminus Y)}(u) < \varepsilon$ and $|U \setminus Y| < \varepsilon$. By the lower semicontinuity of the total variation norm, Hölder's inequality, and the pointwise estimate $|\nabla u_j|^{p_j^-} \leqslant |\nabla u_j|^{p_j(x)} + \chi_{U\setminus Y}$ we find that

$$\begin{aligned}\|\nabla u\|(U) &\leqslant \liminf_{j\to\infty} \int_U |\nabla u_j|\, dx \leqslant \liminf_{j\to\infty} |U|^{1-1/p_j^-} \left(\int_U |\nabla u_j|^{p_j^-}\, dx \right)^{1/p_j^-} \\ &\leqslant \liminf_{j\to\infty} \int_U |\nabla u_j|^{p_j(x)}\, dx + |U \setminus Y|.\end{aligned}$$

Using this estimate and the assumptions from the beginning of the paragraph we conclude that

$$\varrho_{\mathrm{BV}^{p(\cdot)}(U)}(u) \leqslant \liminf_{j\to\infty} \int_U |\nabla u_j|^{p_j(x)}\, dx + 2\varepsilon.$$

Since u_j converges to u weakly in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$ we obtain by Lemma 2.1 that

$$\int_{E\setminus U} |\nabla u(x)|^{p(x)}\, dx \leqslant \liminf_{j\to\infty} \int_{E\setminus U} |\nabla u_j(x)|^{p_j(x)}\, dx.$$

In conclusion, we have

$$\varrho_{\mathrm{BV}^{p(\cdot)}(E)}(u) = \varrho_{\mathrm{BV}^{p(\cdot)}(E\cap U)}(u) + \varrho_{\mathrm{BV}^{p(\cdot)}(E\setminus U)}(u) \leqslant \liminf_{j\to\infty} \int_E |\nabla u_j|^{p_j(x)} dx + 2\varepsilon,$$

from which the claim follows as $\varepsilon \to 0$. □

7. Energy minimizers: main result

In the previous section we investigated the limit behavior of a sequence of solutions in Sobolev space. We also found that the limit is well behaved as long as we are separated from ∂Y, either outside Y (Proposition 6.1) or inside (Remark 6.2).

In this section we prove that we do get a global minimization property, as long as we stay away from the boundaries of the domain. This claim is as strong as in the fixed exponent case, cf. [42, Proposition 4.5].

Theorem 7.1. *Let p be strongly* log*-Hölder continuous and bounded. Let $f \in W^{1,p_\delta(\cdot)}(\Omega)\cap L^\infty(\Omega)$ for some $\delta > 1$ and let (u_{λ_j}) be a sequence of $p_{\lambda_j}(\cdot)$-solutions in Ω with boundary values f where (λ_j) decreases to 1. Then there exists a function $u \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)\cap L^\infty(\Omega)$ so that*

- (i) *there is a subsequence (λ_j) such that $u_{\lambda_j} \to u$ in $L^{p(\cdot)}(\Omega)$;*
- (ii) *u is a $p(\cdot)$-solution in $\Omega \setminus Y$;*
- (iii) *for every $D \subset\subset \Omega$ the function u belongs to $W^{1,p(\cdot)}(D\setminus Y)$; and*
- (iv) *u is a minimizer of the energy $\varrho_{\mathrm{BV}^{p(\cdot)}}$, i.e. for every compact $F \subset \Omega$ and every $v \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)$ with $v = u$ in $\Omega \setminus F$ we have*

$$\varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u) \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(v).$$

Proof. We choose a subsequence λ_j so that Propositions 6.1 and 6.3 hold. Then Claims (i), (ii) and (iii) follow.

To prove Claim (iv), let $F \subset \Omega$ be compact and denote $p_j = p_{\lambda_j}$, $u_j = u_{\lambda_j}$. Fix a function $v \in \mathrm{BV}^{p(\cdot)}_{loc}(\Omega)$ with $v = u$ in $\Omega \setminus F$. By Proposition 6.1, (u_j) converges to u weakly in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$. By Proposition 6.3 we also know that

$$\varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u) \leqslant \liminf_{j\to\infty} \varrho_{L^{p_j(\cdot)}(F)}(\nabla u_j).$$

To conclude the proof, we will show that

$$\int_F |\nabla u_j|^{p_j(x)} dx \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(v) + C\varepsilon,$$

for every $\varepsilon \in (0,1)$, when j is large enough. Fix open sets U, V and Ω' with $F \subset U \subset\subset V \subset\subset \Omega' \subset\subset \Omega$, such that

$$|\Omega' \setminus F| < \varepsilon \quad \text{and} \quad \varrho_{\mathrm{BV}^{p(\cdot)}(\Omega'\setminus F)}(u) = \varrho_{\mathrm{BV}^{p(\cdot)}(\Omega'\setminus F)}(v) < \varepsilon.$$

Let $\eta \in C^\infty_0(V)$ be a cutoff function with $\chi_U \leqslant \eta \leqslant 1$. We consider mollifications u_δ and v_δ with $\delta > 0$ so small that $v_\delta = u_\delta$ outside U. Since $v_\delta \in C^\infty(\overline{V})$ we may test u_j with $\eta(v_\delta - u_j)$. We use the product formula $\nabla[\eta(v_\delta - u_j)] = (v_\delta - u_j)\nabla\eta + \eta\nabla v_\delta - \eta\nabla u_j$ and obtain

$$\begin{aligned}\int_V & p_j(x)|\nabla u_j|^{p_j(x)-2}(v_\delta - u_j)\nabla u_j\cdot\nabla\eta\,dx \\ &+ \int_V p_j(x)|\nabla u_j|^{p_j(x)-2}\eta\nabla u_j\cdot\nabla v_\delta\,dx - \int_V p_j(x)|\nabla u_j|^{p_j(x)}\eta\,dx = 0.\end{aligned}$$

Next we arrange the terms suitably and use the triangle inequality to derive

$$\int_V p_j(x)|\nabla u_j|^{p_j(x)}\eta\,dx \leqslant p^+ \underbrace{\int_V |\nabla u_j|^{p_j(x)-1}|v_\delta - u_j|\,|\nabla\eta|\,dx}_{=:\,\mathtt{I}} + \underbrace{\int_V p_j(x)|\nabla u_j|^{p_j(x)-1}|\nabla v_\delta|\,dx}_{=:\,\mathtt{II}}. \tag{7.1}$$

We proceed to estimate the terms $\mathtt{I}$ and $\mathtt{II}$ in the previous inequality using Young's inequality. Thus

$$\mathtt{I} \leqslant \varepsilon\int_{V\setminus U}|\nabla u_j|^{p_j(x)}\,dx + \left(\frac{1}{\varepsilon}\right)^{p^+}\int_{V\setminus U}|v_\delta - u_j|^{p_j(x)}|\nabla\eta|^{p_j(x)}\,dx.$$

The first integral is bounded by the Caccioppoli estimate, Lemma 5.3, independent of j. For the second integral we obtain

$$\int_{V\setminus U} |v_\delta - u_j|^{p_j(x)} |\nabla\eta|^{p_j(x)}\,dx \leqslant (1+\sup|\nabla\eta|)^{p^+} \int_{V\setminus U} |u_\delta - u_j|^{p_j(x)}\,dx$$
$$\leqslant C2^{p^+-1}\left[\int_{V\setminus U} |u_\delta - u|^{p_j(x)}\,dx + \int_{V\setminus U} |u-u_j|^{p_j(x)}\,dx\right].$$

Here we used the fact that $u_\delta = v_\delta$ in $V\setminus U$. Fix any number $\kappa \in (1, n/(n-1))$ and assume that $\lambda_j < \kappa$. Since u is bounded, we obtain $u_\delta \to u$ in $L^{p_\kappa(\cdot)}(V\setminus U)$. The estimate

$$\int_{V\setminus U} |u_\delta - u|^{p_j(x)} dx \leqslant \int_{V\setminus U} |u_\delta - u|\,dx + \int_{V\setminus U} |u_\delta - u|^{p_\kappa(x)} dx$$

implies that the first integral is smaller than $C\varepsilon^{p^+ + 1}$ if δ is small enough. A similar argument based on Proposition 6.1(6.1) implies that $\int_{V\setminus U} |u-u_j|^{p_j(x)}\,dx \to 0$ as $j\to\infty$. Thus we may assume that $\mathtt{I} < C\varepsilon$, by choosing j large enough and $\delta>0$ small enough.

For the term $\mathbb{I}$ we also use Young's inequality:

$$\mathbb{I} \leqslant \int_V p_j(x)\big[|\nabla u_j|^{p_j(x)}/p_j'(x) + |\nabla v_\delta|^{p_j(x)}/p_j(x)\big]\,dx$$
$$= \int_V (p_j(x)-1)|\nabla u_j|^{p_j(x)}\,dx + \int_V |\nabla v_\delta|^{p_j(x)}\,dx.$$

We next use the estimates for $\mathtt{I}$ and $\mathbb{I}$ in (7.1):

$$\int_V p_j(x)|\nabla u_j|^{p_j(x)}\eta\,dx \leqslant C\varepsilon + \int_V (p_j(x)-1)|\nabla u_j|^{p_j(x)}\,dx + \int_V |\nabla v_\delta|^{p_j(x)}\,dx.$$

We next subtract $\int_U (p_j(x)-1)|\nabla u_j|^{p_j(x)}\,dx$ from both sides and use the fact $\eta = 1$ in U. This gives

$$\int_U |\nabla u_j|^{p_j(x)}\,dx \leqslant \int_V |\nabla v_\delta|^{p_j(x)}\,dx + C\varepsilon + \underbrace{\int_{V\setminus U} (p_j(x)-1)\,|\nabla u_j|^{p_j(x)}\,dx}_{=:\,\mathbb{III}}. \tag{7.2}$$

Below we will show that $\mathbb{III} \leqslant C\varepsilon$. Before that we see how the estimate then gives the claim. By the dominated convergence theorem with constant majorant, we obtain that

$$\int_V |\nabla v_\delta|^{p_j(x)} dx \to \int_V |\nabla v_\delta|^{p(x)} dx$$

for a fixed $\delta>0$ as $j\to\infty$. Applying this in (7.2) gives

$$\int_U |\nabla u_j|^{p_j(x)}\,dx \leqslant \int_V |\nabla v_\delta|^{p(x)}\,dx + C\varepsilon$$

for any fixed small $\delta>0$ if j is large enough. Hence by Theorem 4.6 and the choice of V, we have

$$\int_U |\nabla u_j|^{p_j(x)}\,dx \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(V)}(v) + C\varepsilon \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(v) + C\varepsilon$$

for all sufficiently large j. As was stated in the second paragraph of the proof, this estimate gives the claim.

Finally we estimate Term $\mathbb{III}$. We have

$$\mathbb{III} \leqslant (p^+-1)\int_{V\setminus(U\cup Y_\delta)} |\nabla u_j|^{p_j(x)}\,dx + (p^+_{Y_\delta}-1)\int_{V\cap Y_\delta} |\nabla u_j|^{p_j(x)}\,dx,$$

for $\delta>0$, where $Y_\delta := \{x\in\Omega:\ \operatorname{dist}(x,Y)<\delta\}$ is as before. We fix δ so small that $p^+_{Y_\delta} - 1 < \varepsilon$. The second integral on the right hand side is uniformly bounded by the Caccioppoli estimate, Lemma 5.3, so we have an estimate of the type $C\varepsilon$ for this term.

We then consider the first integral on the right hand side. We denote

$$r := \min\{\operatorname{dist}(U,\partial\Omega'),\delta\} \quad\text{and}\quad B_x := B(x,\tfrac{1}{20}r).$$

Then the family $\{B_x\}_{x\in\overline{V}\setminus(U\cup Y_\delta)}$ is an open cover of $\overline{V}\setminus(U\cup Y_\delta)$. By compactness and Vitali's covering theorem [22, Theorem 1, p. 27] we find a finite subcollection $(B_{x_i})_{i=1}^N$ of pairwise disjoint balls such the collection $(5\overline{B}_{x_i})_{i=1}^N$ covers $\overline{V} \setminus (U \cup Y_\delta)$. For convenience, denote $B_i := 5\overline{B}_{x_i}$. Since the balls B_i have the same radius, we conclude that the collection of balls $(2B_i)_{i=1}^N$ has overlap bounded by a constant $c(n)$. We note that $D := \cup_{i=1}^N (2B_i)$ is a set with $p_D^- > 1$, and that u_j is a sequence of solutions converging in $L^{p(\cdot)}(D)$ (since $p_j = p$ in D, starting from some index). By the initial smallness assumptions on $\Omega' \setminus F$ and Corollary 5.5 we find that

$$\begin{aligned}\int_{V\setminus(U\cup Y_\delta)} |\nabla u_j|^{p_j(x)}\,dx &\leqslant \sum_{i=1}^N \int_{B_i} |\nabla u_j|^{p_j(x)}\,dx \leqslant \sum_{i=1}^N C\int_{2B_i} |\nabla u|^{p(x)}\,dx + C\,|B_i| \\ &\leqslant C\int_{\Omega'\setminus F} |\nabla u|^{p(x)}\,dx + C\,|\Omega'\setminus F| \leqslant C\varepsilon.\end{aligned}$$

Notice that this upper bound holds for every $\delta > 0$. This completes the estimate of Ⅲ, and the whole proof. □

Remark 7.2. Assuming in Theorem 7.1 that "$\{u_{\lambda_j}\}$ is a locally bounded sequence" instead of assuming that functions have bounded Sobolev boundary values f gives similar results. This is so because assumption on "boundedness" is only needed when Lemma 5.3 is applied to get a uniform upper bound for local Dirichlet integrals of u_{λ_j}. In particular, one could consider an open set Ω for which the Dirichlet problem is solvable in the classical sense. Such domains are characterized in [7] by means of capacity.

Appendix A. Exponents which are not strongly log-Hölder continuous: pathological examples

In this appendix we give one-dimensional examples which suggest that our theory breaks down without the strong log-Hölder assumption on the exponent p.

Example A.1. Consider the case when p is defined in $\mathbb{R}$ by

$$p(x) = 1 + x\chi_{\{x\geqslant 0\}} + \frac{b}{\log(1/|x|)}\chi_{\{x<0\}},$$

($k > 0$). This exponent has an interesting feature: let $u \in \mathrm{BV}^{p(\cdot)}([-1,1])$ have a jump of height $b > 0$ at the origin. We define two smoothenings of u;

$$u_\delta^-(x) = \fint_{x-\delta}^{x} u(y)\,dy \quad \text{and} \quad u_\delta^+(x) = \fint_{x}^{x+\delta} u(y)\,dy.$$

Both of these functions lie in $W^{1,p(\cdot)}([-1/2,1/2])$, but

$$\lim_{\delta\to 0} \varrho_{p(\cdot)}\big((u_\delta^-)'\big) - \lim_{\delta\to 0} \varrho_{p(\cdot)}\big((u_\delta^+)'\big) = (e^k - 1)b \neq 0,$$

since the first limit corresponds to $\mathrm{BV}^{p(\cdot)}$-modular (cf. Theorem 3.3) with weight e^k and the origin, whereas the second one corresponds to a weight of 1.

It is easily seen that Theorem 4.6 holds also for the convolutions u_δ^+ and u_δ^-. Therefore, this example implies that if we want the theorem to hold for exponents which are not strongly log-Hölder continuous, then we need to use a singular part of the form $\|\nabla u\, w^+\|$, where

$$w^+(x) = \limsup_{y\to x} |p(y) - 1|\,\log\frac{1}{|x-y|},$$

for $x \in Y$. On the other hand, if we want to show that

$$\varrho_{\mathrm{BV}^{p(\cdot)}}(u) \leqslant \liminf_{\delta\to 0} \varrho_{\mathrm{BV}^{p(\cdot)}}(u_\delta)$$

(compare with Proposition 6.3(ii)) then we are led to use the weight w^-, which is defined like w^+, but with a lim inf.

Example A.2. We concluded previously (cf. (3.5)) that the contribution of the interval $K_\lambda = \{x\colon p(x) \leqslant \lambda\}$ to the appropriate energy is given by

$$\int_{K_\lambda} |u'(x)|^{p_\lambda(x)}\,dx = \Big(\frac{b+\epsilon(\lambda)}{|K_\lambda|}\Big)^\lambda |K_\lambda|.$$

Since $\epsilon(\lambda) \to 0$, it plays no role in the further considerations. Let us next consider an exponent for which

$$\limsup_{\lambda\to 1} |K_\lambda|^{1-\lambda} > \liminf_{\lambda\to 1} |K_\lambda|^{1-\lambda}.$$

It is easily constructed. We fix functions $f_1(x) := 1 + |x|$ and $f_2(x) := 1 + k/\log(1/|x|)$. We define the exponent as follows:

(i) set $x_1 = -1$;

(ii) if we have chosen $x_1 < x_1' < \ldots < x_i < 0$, then we choose $x_i' \in (x_1, 0)$ so that $f_1(x_1) > f_2(x_i')$. (This is possible, since $f_1(x_i) > 1$ and $f_2 \searrow 1$.)

(iii) if we have chosen $x_1 < x_1' < \ldots < x_i' < 0$, then we choose $x_{i+1} \in (x_i'/2, 0)$ arbitrarily, and return to step (2).

The previous procedure gives us an increasing sequence $x_1, x_1', x_2, x_2', \ldots$ which tends to 0. The exponent p is defined by the graph which consists of the segments joining $(x_i, f_1(x_i))$ with $(x_i', f_2(x_i'))$ and $(x_i', f_2(x_i'))$ with $(x_{i+1}, f_1(x_{i+1}))$. The function p is extended to $(0, 1]$ as an even function, with $p(0) = 1$. This construction gives us a log-Hölder continuous exponent p. We define $\lambda_i = f_1(x_i)$ and $\lambda_i' = f_2(x_i')$. Then clearly

$$\limsup_{\lambda\to 1} |K_\lambda|^{1-\lambda} \geqslant \lim_{i\to\infty} |K_{\lambda_i'}|^{1-\lambda_i'} = e^k > 1 = \lim_{i\to\infty} |K_{\lambda_i}|^{1-\lambda_i} = \liminf_{\lambda\to 1} |K_\lambda|^{1-\lambda}.$$

Although we were not able to prove a definite result to this effect, it seems that the existence of multiple accumulation points of the sequence c_i is of no importance for the minimizer: in each case the limiting constant should be chosen as large as possible (cf. Corollary 3.4).

However, if the interval contains two or more points where $p = 1$, then this turns into a significant consideration.

Example A.3. For concreteness, consider exponents on $[-2, 2]$ such that p is continuous and decreasing on $[-2, -1]$ and $[0, 1]$, with $p(-1) = 1$ and $p(1) = 1$, and is symmetrically extended: $p(\pm 1 + x) = p(\pm 1 - x)$ for $x \in [0, 1]$. As above we may construct the exponent so that K_λ consists of two components $K_\lambda^\pm$ about ± 1 with

$$\lim_{i\to\infty} |K^+_{\lambda_i'}|^{1-\lambda_i'} = e^k > 1 = \lim_{i\to\infty} |K^+_{\lambda_i}|^{1-\lambda_i}$$

and

$$\lim_{i\to\infty} |K^-_{\lambda_i'}|^{1-\lambda_i'} = 1 < e^k = \lim_{i\to\infty} |K^-_{\lambda_i}|^{1-\lambda_i}.$$

Here (λ_i) and (λ_i') are some suitable sequences tending to 1.

Now we can argue as in Theorem 3.3 and find that the limit corresponding to the sequence λ_i minimizes

$$\varrho_{L^{p(\cdot)}((-2,-1)\cup(-1,1)\cup(1,2))}(u') + u'(\{-1\}) + e^k u'(\{+1\})$$

(where u' is regarded as a measure, and $u'(\{-1\})$ denotes the singular part at -1) but on the other hand the limit corresponding to the sequence λ_i' minimizes

$$\varrho_{L^{p(\cdot)}((-2,-1)\cup(-1,1)\cup(1,2))}(u') + e^k u'(\{-1\}) + u'(\{+1\})$$

This means that if the boundary values are such that the derivative of the minimizer contains a singular part, then this singular part will be supported at -1 in the first case and at 1 in the second. In particular, we see that the there are two different limits of the sequence u_i, so the "minimizer" is not unique in this case!

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