CRITICAL VARIABLE EXPONENT FUNCTIONALS IN IMAGE RESTORATION

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ABSTRACT. We study a variable exponent model for image restoration in the case that the exponent attains the critical value one. We prove existence and Γ-convergence. The results answer an open question by Li, Li and Pi [Variable exponent functionals in image restoration, *Appl. Math. Comput.* **216** (2010), no. 3, 870–882].

1. INTRODUCTION

To understand the role of the variable exponent in the image restoration problem we briefly recall the variational formulations of the isotropic and total variation smoothing. In the isotropic smoothing one minimizes the energy

$$\int_\Omega |\nabla u|^p + \lambda\, |u-f|^2\, dx, \tag{1.1}$$

with $p \equiv 2$, where $\lambda > 0$ is a parameter indicating the strength of the smoothing. In the total variation smoothing, introduced by Rudin, Osher & Fatemi [11], one minimizes the energy (1.1) with $p \equiv 1$. The first minimization problem is naturally solved in the Sobolev space $W^{1,2}(\Omega)$, whereas the second is solved in the space $\mathrm{BV}(\Omega)$ of functions of bounded variation. Since we would like to combine the strengths of these two approaches, it is natural to formulate the minimization problem (1.1) for an exponent $p = p(x)$ varying in the interval $[1, 2]$. This is the essence of the model proposed in [3] by Chen, Levine & Rao (see also[1, 2, 9]):

$$\int_\Omega |\nabla u|^{p(x)} + \lambda\, |u-f|^2\, dx. \tag{1.2}$$

For an overview on such variational problems with variable exponent see [8]. Recently Li, Li & Pi studied this model in the case $p^- := \inf p > 1$ in [10]. In the end of their paper they ask whether it is possible to extend their results to the case $p^- = 1$. In this paper we propose a solution to this problem and show that the our energy operator is a natural limit of (1.2).

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary, $s \geqslant 1$ a fixed constant, and $p : \Omega \to [1,\infty)$ be a bounded lower semicontinuous exponent.

We denote $Y := \{x \in \Omega : p(x) = 1\}$,

$$\mathrm{BV}^{p(\cdot)}(\Omega) := \{\, u \in \mathrm{BV}(\Omega) \cap W^{1,p(\cdot)}(\Omega \setminus Y)\,\},$$

and

$$\varrho_{\mathrm{BV}^{p(\cdot)}(A)}(u) := \|\nabla u\|(Y \cap A) + \int_{A\setminus Y} |\nabla u|^{p(x)}\, dx$$

Date: September 20, 2011.
2000 *Mathematics Subject Classification.* 94A08; 49J45,49N45,68U10.
Key words and phrases. Variable exponent; existence of minimizers; image restoration.

for every $A \subset \Omega$. Note that $\mathrm{BV}^{p(\cdot)}(\Omega) = W^{1,p(\cdot)}(\Omega)$ when $p^- > 1$; for more properties of $\mathrm{BV}^{p(\cdot)}$ see [7]. Our goal is to study the minimizing problem

$$\inf_u D_1(u) = \inf_u \Big(\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) + \lambda \int_\Omega |u-f|^s \, dx \Big)$$

in $\mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$, where λ is a fixed positive real number and $f \in L^s(\Omega)$ is the initial data. The following is our main result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open domain with Lipschitz boundary and let $p : \Omega \to [1,\infty)$ be lower semicontinuous. Then the minimizing problem*

$$\inf_{u \in \mathrm{BV}^{p(\cdot)}(\Omega)} \Big(\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) + \lambda \int_\Omega |u-f|^s \, dx \Big)$$

has a solution $u \in \mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$. Moreover, any minimizing sequence (u_i) has a subsequence such that $u_i \to u$ in $L^1(\Omega)$ and $\nabla u_i \rightharpoonup \nabla u$ weakly in $L^{p(\cdot)}(\Omega \setminus \overline{U})$ for every open $U \supset Y$.

In practice it is difficult to deal with the BV-part of the norm in $\mathrm{BV}^{p(\cdot)}(\Omega)$. Therefore we consider approximating functionals which are defined as follows. For $\delta \geqslant 1$ we set $p_\delta := \max\{p, \delta\}$ and define energies

$$D_\delta(u) := \varrho_{\mathrm{BV}^{p_\delta(\cdot)}(\Omega)}(u) + \lambda \int_\Omega |u-f|^s \, dx$$

in $\mathrm{BV}^{p_\delta(\cdot)}(\Omega) \cap L^s(\Omega)$. We extend operators D_δ to $L^1(\Omega)$ by setting $D_\delta(u) := \infty$ for $u \in L^1(\Omega) \setminus \big(\mathrm{BV}^{p_\delta(\cdot)}(\Omega) \cap L^s(\Omega)\big)$. We prove Γ-convergence of our auxiliary functionals D_δ to D_1; thus we give the following definition.

Definition 1.4. The functionals $D_\delta : L^1(\Omega) \to \overline{\mathbb{R}}$, $\delta > 1$, are said to $\Gamma(L^1)$-converge to $D_1 : L^1(\Omega) \to \overline{\mathbb{R}}$ if the following holds for every sequence (δ_i) with $\delta_i > 1$ and $\lim_{i\to\infty} \delta_i = 1$:

(1) for every $u \in L^1(\Omega)$ and for every sequence (u_i) in $L^1(\Omega)$ converging to u in $L^1(\Omega)$, we have
$$D_1(u) \leqslant \liminf_{i\to\infty} D_{\delta_i}(u_i);$$
(2) for every $u \in L^1(\Omega)$ there exists a $L^1(\Omega)$-sequence (u_i) (called a recovery sequence) such that $u_i \to u$ in $L^1(\Omega)$ and
$$D_1(u) \geqslant \limsup_{i\to\infty} D_{\delta_i}(u_i).$$

For Γ-convergence we require a stronger assumption on the exponent, namely so-called strong $\log$-Hölder continuity:

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$ and

$$\lim_{x\to y} \big|p(x) - 1\big| \log \frac{1}{|x-y|} = 0$$

for every $y \in Y$.

Theorem 1.5. *Let Ω be an open rectangle and let p be strongly $\log$-Hölder continuous. Then D_δ $\Gamma(L^1)$-converges to D_1.*

It can be seen from the proof that in fact we also have $\Gamma(\text{w-}L^1)$ convergence with respect to weak-L^1.

2. Existence and lower semicontinuity

We first give a lower semicontinuity result which will be used both for existence and Γ-convergence. Here $\rightharpoonup$ denotes the weak convergence.

Theorem 2.1. *Suppose that $u_i \rightharpoonup u$ in $L^1(\Omega)$ and either $\delta_i = 1$ for all i or $\delta_i > 1$ and $\lim_{i\to\infty} \delta_i = 1$. Then*

$$D_1(u) \leqslant \liminf_{i\to\infty} D_{\delta_i}(u_i).$$

In particular, if the limit inferior is finite, then $u \in \mathrm{BV}^{p(\cdot)}(\Omega) \cap L^s(\Omega)$.

Proof. Let (u_i) be a sequence in $L^1(\Omega)$ converging weakly to u in $L^1(\Omega)$. By picking a subsequence, if necessary, we may assume that (u_i) gives the limit inferior (and thus so does its every subsequence). Denote $p_i := p_{\delta_i}$. To estimate the derivatives we are free to assume that $\alpha := \liminf_{i\to\infty} D_{\delta_i}(u_i) < \infty$ and $D_{\delta_i}(u_i) < \infty$ for every i.

Then $u_i - f$ is bounded in $L^s(\Omega)$, so it converges weakly to some function (by reflexivity if $s > 1$, and by assumption when $s = 1$); uniqueness of the limit implies that this function is $u - f$. Hence the weak lower semicontinuity of the integral yields

$$\lambda \int_\Omega |u - f|^s \, dx \leqslant \liminf_{i\to\infty} \lambda \int_\Omega |u_i - f|^s \, dx. \tag{2.2}$$

Denote $\Omega_k := \{p > 1 + \frac{1}{k}\}$; since p is lower semicontinuous, Ω_k is open. Then

$$\int_{\Omega_k} |\nabla u_i|^{p(x)} \, dx \leqslant \int_{\Omega_k} |\nabla u_i|^{p_i(x)} + 1 \, dx \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(\nabla u_i) + |\Omega| \leqslant 2\alpha + |\Omega|,$$

when i is large enough. Hence (∇u_i) is a bounded sequence in the reflexive space $L^{p(\cdot)}(\Omega_k)$. By reflexivity, $\nabla u_i \rightharpoonup g$ (up to a subsequence) in $L^{p(\cdot)}(\Omega_k)$ for some function g. Since $u_i \rightharpoonup u$ in $L^1(\Omega)$, we see that if $\varphi \in C_0^\infty(\Omega_k)$, then

$$\int_{\Omega_k} g \cdot \varphi \, dx = \lim_{i\to\infty} \int_{\Omega_k} \nabla u_i \cdot \varphi \, dx = -\lim_{i\to\infty} \int_{\Omega_k} u_i \operatorname{div} \varphi \, dx = \int_{\Omega_k} u \operatorname{div} \varphi \, dx,$$

so actually $g = \nabla u$ in Ω_k.

By Young's inequality, $a^{p(\cdot)} \leqslant a^{p_i(\cdot)} + (p_i - p) \leqslant a^{p_i(\cdot)} + \delta_i - 1$. Hence by the weak lower semicontinuity of the modular, we have

$$\begin{aligned}\int_{\Omega_k} |\nabla u|^{p(x)} \, dx &\leqslant \liminf_{i\to\infty} \int_{\Omega_k} |\nabla u_i|^{p(x)} \, dx = \liminf_{i\to\infty} \int_{\Omega_k} |\nabla u_i|^{p_i(x)} + (\delta_i - 1) \, dx \\ &= \liminf_{i\to\infty} \int_{\Omega_k} |\nabla u_i|^{p_i(x)} \, dx \leqslant \liminf_{i\to\infty} \int_{\Omega\setminus Y} |\nabla u_i|^{p_i(x)} \, dx\end{aligned}$$

for every k. Letting $k \to \infty$ we obtain by the monotone convergence that $|\nabla u| \in L^{p(\cdot)}(\Omega \setminus Y)$.

To finish the proof, we choose for every $\varepsilon > 0$ an open neighborhood $U \subset \Omega$ of Y such that $|U \setminus Y| < \varepsilon$, $\int_{U\setminus Y} |\nabla u|^{p(x)} \, dx < \varepsilon$ and $|\partial U| = 0$. Since $u_i \rightharpoonup u$ in $L^1(\Omega)$ we obtain by [6, Theorem 1, p.172] that $u \in \mathrm{BV}(\Omega)$ and $\|\nabla u\|(U) \leqslant \liminf_{i\to\infty} \|\nabla u_i\|(U)$. By the argument in the previous part of the proof,

$$\int_{\Omega\setminus\overline{U}} |\nabla u|^{p(x)} \, dx \leqslant \liminf_{i\to\infty} \int_{\Omega\setminus\overline{U}} |\nabla u_i|^{p_i(x)} \, dx.$$

Hence by the pointwise inequality $|t|^{p(x)} \leqslant |t|^{p_i(t)} + \delta_i - 1$, we conclude that

$$\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) \leqslant \|\nabla u\|(U) + \int_{\Omega\setminus\overline{U}} |\nabla u|^{p(x)}\,dx + \int_{U\setminus Y} |\nabla u|^{p(x)}\,dx$$
$$\leqslant \liminf_{i\to\infty} \|\nabla u_i\|(U) + \liminf_{i\to\infty} \int_{\Omega\setminus\overline{U}} |\nabla u_i|^{p_i(x)}\,dx + \varepsilon.$$

We consider the sequences $\delta_i = 1$ and $\delta_i > 1$ separately. In the former case, since $|t| \leqslant |t|^{p_i(t)} + 1$, we find that

$$\|\nabla u_i\|(U) = \|\nabla u_i\|(Y) + \int_{U\setminus Y} |\nabla u_i|\,dx \leqslant \|\nabla u_i\|(Y) + \int_{U\setminus Y} |\nabla u_i|^{p_i(x)}\,dx + |U\setminus Y|,$$

while in the latter case, since $\nabla u_i \in L^{p_i(\cdot)}(\Omega)$, we obtain

$$\|\nabla u_i\|(U) = \int_U |\nabla u_i|\,dx \leqslant \int_U |\nabla u_i|^{p_i(x)}\,dx + |U\setminus Y| + (\delta_i - 1)|Y|$$

by using $|t| \leqslant |t|^{p_i(t)} + p_i(t) - 1$ in Y. In both cases we thus have

$$\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) \leqslant \liminf_{i\to\infty} \varrho_{\mathrm{BV}^{p_i(\cdot)}(\Omega)}(u_i) + 2\epsilon.$$

As $\varepsilon \to 0$, the claim follows from this and (2.2). □

We can then prove the existence of minimizers.

Proof of Theorem 1.3. Let

$$E := \inf_{u\in\mathrm{BV}^{p(\cdot)}(\Omega)} \Big(\varrho_{\mathrm{BV}^{p(\cdot)}(\Omega)}(u) + \int_\Omega \lambda|u-f|^s\,dx\Big);$$

since 0 is an admissible test function, $E \leqslant \lambda\,\varrho_{L^s(\Omega)}(f) < \infty$. Let (u_i) be a minimizing sequence. Then $\|u_i\|_{BV(\Omega)} \leqslant c < \infty$ so by [6, Theorem 4, p.176] we may choose a subsequence which converges in $L^1(\Omega)$. By Theorem 2.1 with $\delta_i = 1$, $D_1(\lim_{i\to\infty} u_i) \leqslant \liminf_{i\to\infty} D_1(u_i) = E$. Hence $\lim_{i\to\infty} u_i$ is the desired minimizer. The additional claim holds by the proof of Theorem 2.1. □

3. The recovery sequence

We construct the recovery sequence using a suitable convolution. For this we need that p is strongly log-Hölder continuous, as defined in the introduction.

We denote $\Omega_{a,b} := \{x \in \Omega : a < \operatorname{dist}(x, \partial\Omega) < b\}$ and set $\Omega_a := \Omega_{a,\infty}$. For brevity, we also write

$$\varrho_{LBV^{p(\cdot)}(E)}(u) := \varrho_{L^{p(\cdot)}(E)}(u) + \varrho_{BV^{p(\cdot)}(E)}(u)$$

for $u \in \mathrm{BV}^{p(\cdot)}(\Omega)$ and

$$\varrho_{W^{1,p(\cdot)}(E)}(u) := \varrho_{L^{p(\cdot)}(E)}(u) + \varrho_{L^{p(\cdot)}(E)}(|\nabla u|)$$

for a Sobolev function $u \in W^{1,p(\cdot)}(\Omega)$ whenever $E \subset \Omega$ is measurable. Clearly

$$\varrho_{LBV^{p(\cdot)}(E)}(u) = \varrho_{W^{1,p(\cdot)}(E)}(u) \quad \text{if } u \in W^{1,p(\cdot)}(\Omega).$$

We need the following extension result:

Theorem 3.1. *Let Q be an open rectangle, and let $a > 0$ be less than half the length of its shorter side. If $u \in W^{1,p(\cdot)}(Q_a)$, then there exists an extension $Eu \in W^{1,p(\cdot)}(Q)$ such that*

$$\varrho_{W^{1,p(\cdot)}(Q_{0,a})}(Eu) \leqslant c\,\varrho_{W^{1,p(\cdot)}(Q_{a,2a})}(u).$$

Proof. Existence of an extension is proved e.g. in [4, Theorem 8.5.12]. The inequality can be seen by analyzing the proof. An easier proof for this case is in [5]. □

Let u_δ be the standard mollification of u.

Proposition 3.2 (Theorem 4.6, [7]). *Let p be strongly $\log$-Hölder continuous with $p^+ < \infty$. Assume that $u \in \mathrm{BV}^{p(\cdot)}(\Omega)$ and $F \subset \Omega$ is closed. Then $u_\delta \to u$ in $L^{p(\cdot)}(\Omega)$ as $\delta \to 0^+$ and*

$$\limsup_{\delta\to 0^+} \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u_\delta) \leqslant \varrho_{\mathrm{BV}^{p(\cdot)}(F)}(u).$$

Lemma 3.3. *Let p be strongly $\log$-Hölder continuous with $p^+ < \infty$. Let Q be a rectangle. Then for every $u \in \mathrm{BV}^{p(\cdot)}(Q)$ and $\epsilon > 0$ there exists $\lambda_0 > 1$ and $\tilde u \in W^{1,p_\lambda(\cdot)}(Q)$ such that*

$$\varrho_{W^{1,p_\lambda(\cdot)}(Q)}(\tilde u) < \varrho_{LBV^{p(\cdot)}(Q)}(u) + \epsilon \quad \text{and} \quad \varrho_{L^{p(\cdot)}(Q)}(u-\tilde u) < \epsilon$$

for every $\lambda \in (1, \lambda_0)$.

Proof. Fix $u \in \mathrm{BV}^{p(\cdot)}(Q)$ and $\epsilon \in (0,1)$. Let us choose first $a > 0$ such that $\varrho_{LBV^{p(\cdot)}(Q_{0,3a})}(u) < \epsilon$. Using Proposition 3.2, we then choose $\delta \in (0, \frac{a}{2})$ such that

$$\varrho_{LBV^{p(\cdot)}(Q_{a,2a})}(u_\delta) < \epsilon, \quad \varrho_{L^{p(\cdot)}(Q_a)}(u-u_\delta) < \epsilon,$$

and

$$\varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u_\delta) < \varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u) + \epsilon.$$

Since $u_\delta \in C^\infty(Q)$, the dominated convergence theorem implies that

$$\lim_{\lambda\to 1^+} \varrho_{LBV^{p_\lambda(\cdot)}(E)}(u_\delta) = \varrho_{LBV^{p(\cdot)}(E)}(u_\delta)$$

for any measurable set E with $\overline{E} \subset Q$. Hence we can choose $\lambda_0 > 1$ such that

$$\varrho_{LBV^{p_\lambda(\cdot)}(Q_{a,2a})}(u_\delta) < \epsilon \quad \text{and} \quad \varrho_{LBV^{p_\lambda(\cdot)}(\overline{Q_a})}(u_\delta) < \varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u) + \epsilon$$

for every $\lambda \in (1, \lambda_0)$.

Suppose that $\lambda \in (1, \lambda_0)$. By Theorem 3.1, we extend $u_\delta|_{Q_a}$ to a function $\tilde u \in W^{1,p_\lambda(\cdot)}(Q)$. Then

$$\begin{aligned}
\varrho_{LBV^{p_\lambda(\cdot)}(Q)}(\tilde u) &= \varrho_{LBV^{p_\lambda(\cdot)}(\overline{Q_a})}(u_\delta) + \varrho_{LBV^{p_\lambda(\cdot)}(Q_{0,a})}(\tilde u)\\
&\leqslant \varrho_{LBV^{p(\cdot)}(\overline{Q_a})}(u) + \epsilon + c\varrho_{LBV^{p_\lambda(\cdot)}(Q_{a,2a})}(u_\delta)\\
&\leqslant \varrho_{LBV^{p(\cdot)}(Q)}(u) + c\epsilon.
\end{aligned}$$

Note that the extension is independent of λ, i.e. the same function $\tilde u$ can be used for every $\lambda \in (1, \lambda_0)$. Hence the first inequality is proved.

To prove the $L^{p(\cdot)}$-inequality, we estimate

$$\begin{aligned}
\varrho_{L^{p(\cdot)}(Q)}(u-\tilde u) &\leqslant \varrho_{L^{p(\cdot)}(Q_a)}(u-u_\delta) + 2^{p^+}\Big(\varrho_{L^{p(\cdot)}(Q_{0,a})}(u) + \varrho_{L^{p(\cdot)}(Q_{0,a})}(\tilde u)\Big)\\
&\leqslant \epsilon + 2^{p^+}(\epsilon + c\epsilon).
\end{aligned}$$

Here again the latter inequality follows from Theorem 3.1 and the choice of a. □

We are now ready to prove the Γ-convergence.

Proof of Theorem 1.5. Condition (1) of Γ-convergence was established in Theorem 2.1. Here we prove Condition (2). So let $\delta_i \to 1^+$ and $u \in L^1(Q)$. If $\varrho_{BV^{p(\cdot)}(Q)}(u) = \infty$, there is nothing to prove. So we assume that $u \in BV^{p(\cdot)}(Q)$.

Let $\tilde{u}_j$ be the function $\tilde{u}$ from Lemma 3.3 corresponding to $\epsilon = \frac{1}{j}$ and let $\lambda_j > 1$ be less than the corresponding λ_0. We are free to assume that (λ_j) decreases to 1. Fix $\epsilon > 0$. For each i, let $j(i)$ be the largest index j for which $\lambda_j > \delta_i$. Since $\delta_i \to 1^+$, we have $j(i) \to \infty$. Now we choose $u_i = \tilde{u}_{j(i)}$ as the sequence in Condition (2). By Lemma 3.3,

$$\limsup_{i\to\infty} \varrho_{W^{1,p_{\delta_i}(\cdot)}(Q)}(u_i) \leqslant \varrho_{LBV^{p(\cdot)}(Q)}(u) \quad \text{and} \quad \lim_{i\to\infty} \varrho_{L^{p(\cdot)}(Q)}(u_i) = \varrho_{L^{p(\cdot)}(Q)}(u).$$

Therefore

$$\limsup_{i\to\infty} \varrho_{BV^{p_{\delta_i}(\cdot)}(Q)}(u_i) \leqslant \varrho_{BV^{p(\cdot)}(Q)}(u).$$

and $u_i \to u$ in $L^1(Q)$ by Hölder's inequality. Hence Condition (2) holds. □

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