Non–wellfounded sets

Philosophy Licenciate's thesis

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Abstract. This philosophy licenciate's thesis consists of two parts. First we study the non–wellfounded sets as fixed points of substitution. E.g. we show that ZFA implies that every function has a fixed point. As a corollary we determine for which functions f there is a function g such that $g = g \star f$. We also present a classification of non–wellfounded sets according to their branching structure.

In the second part we build a domain structure on a subclass of all non–wellfounded sets. We define a partial ordering $\sqsubseteq$ on the class of all non–wellfounded sets. That ordering is defined as a kind of end extension in the tree pictures of sets. Then the domain D is obtained by taking all inverse limits of $\sqsubseteq$–increasing sequences. This produces a subclass of the universe. We can show that all hereditarily finite sets belong to D.

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1 Introduction

Non–wellfounded sets have infinite descending membership sequences. This is somewhat counterintuitive since it would seem that they can not be formed by first forming their members which in turn have to be formed and so on. This common sensical idea of forming sets is behind the iterative hierarchy of sets, in which sets are arranged to stages according to how they are formed from earlier sets. Some philosophers hold that this is the only coherent conception of sets.

If mathematics and applications do not need non–wellfounded sets, then these sets may be left out of systematic study. As non–wellfounded sets have found applications in mathematical logic, computer science, and in philosophy, they have been increasingly investigated. Furthermore non–wellfounded set theory yields new problems, methods and also a rich theory for mathematical interest.

The concept of non–wellfounded sets is almost as old as axiomatic set theory itself. In 1917 Mirimanoff developed the concept of wellfounded and non–wellfounded sets. In 1920–1930 foundation axiom, that states that all sets are wellfounded, was added to the axioms of set theory by Zermelo, and proved relatively consistent from the other axioms by von Neumann.

Non–wellfounded sets have been studied by several authors during 1920–1980, but until Aczel's seminal work [1] no systematic study has taken place. Aczel develops the theory of non–wellfounded sets, formulates and proves consistent his *AFA* axiom, and studies the central concept of bisimulation. In [2] Barwise and Moss further develop the theory of non–wellfounded sets, coinduction, corecursion, and monotone operators.

Aczel's key notion is a graph picturing a set. His *AFA* states that every graph has a unique set that it pictures. In [2] the approach is via equation systems describing the elements of non–wellfounded sets. In this thesis we will use both these approaches and present a generalization of the latter.

This thesis consists of two parts. The first part will be published as a paper T. Hyttinen and M. Pauna, "On non–wellfounded sets as fixed points of substitutions", in Notre dame Journal of Formal Logic. In this part we present another method of defining non–wellfounded sets which is close to that of [2]. In this approach non–wellfounded sets are obtained as fixed points of substitutions. We also apply this theory to the theory of substitution of Barwise and Moss.

We also try to describe a bit the structure of non–wellfounded sets. First we define a certain game that describes the branching structure of the sets, and from this we can define the rank of a non–wellfounded set. We can produce a sequence of increasing classes of models for set theory in which there are not all solutions to equations, according to this rank.

In the second part we study the hereditarily finite sets HF_1 which also can be non–wellfounded, as a domain. Domains are approximation structures and here we are going to approximate the sets in HF_1 by their wellfounded counterparts. We will build the inverse limit of these approximations and the class of all inverse limits, $\widehat{HF}$ is a domain and also an ultra metric space.

All the necessary definitions are introduced at the place where they are needed. There is also a little introduction to the subject in the beginning of each of the two sections.

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2 On non–wellfounded sets as fixed points of substitutions

2.1 Introduction and definitions

In Aczel [1] and in Barwise and Moss [2] non–wellfounded sets and the antifoundation axiom (AFA) have been studied. The non–wellfounded sets are modelled by equations. In the equations we use urelements and the class of all urelements is denoted by $\mathcal{U}$. Urelements are not vital for the theory but often they are convenient, see e.g. [2], section 11. We recall the definitions of a flat system of equations and a solution to it from [2]:

Definition 1 .

(i) A flat system of equations is a triple (X, A, f) where X and A are sets of urelements, $X \cap A = \emptyset$, and $f : X \to \mathcal{P}(X \cup A)$ is a function.

(ii) A solution to a flat system of equations (X, A, f) is a function g such that $dom(g) = X$, and for all $x \in X$, $g(x) = \{g(y) \mid y \in f(x) \cap X\} \cup (f(x) \cap A)$.

The idea is that X is the set of indeterminates of the equations and A is the set of "constants". The equations are understood as $x = f(x)$, $x \in X$. For example, let $A = \{a\}$, $X = \{x\}$, and $f(x) = \{a, x\}$, then (X, A, f) is a flat system of equations. The solution to this system is a function g such that $g(x) = \{a, g(x)\}$. The antifoundation axiom, AFA, says that every flat system of equations has a unique solution.

Substitution operations $sub(s, b)$ are also studied in [2]. The operation $sub(s, b)$ means that in b all x are substituted by $s(x)$. We recall the definition of a substitution from [2]. If $A \subseteq \mathcal{U}$, then $V_{afa}[A]$ is the class of all sets x, such that support$(x) \subseteq A$, where support(x) is defined to be $TC(x) \cap \mathcal{U}$. So $V_{afa}[\mathcal{U}]$ is the class of all sets.

Definition 2 *Substitution is a function s such that $dom(s) \subset \mathcal{U}$. The substitution operation is the operation sub such that the domain of sub consists of a class of pairs $\langle s, b\rangle$ where s is a substitution and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$ such that the following conditions hold*

(i) If $x \in dom(s)$, then $sub(s, x) = s(x)$.

(ii) If $x \in \mathcal{U} - dom(s)$, then $sub(s, x) = x$.

(iii) For all sets b, $sub(s, b) = \{sub(s, p) \mid p \in b\}$.

In [2] it is shown that there is a unique substitution operation $sub(s, b)$ defined for all substitutions s and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. As a corollary to our theory of substitution fixed points, we obtain the same result, see corollary 18. Next we recall the definition of a composition of substitutions from [2].

Definition 3 *The substitution operation $sub(s,b)$ is also denoted by $b[s]$, and $[s]$ is the operation mapping each set or urelement b to $b[s]$. A substitution s is proper if for all $x \in dom(s)$, $s(x) \in V_{afa}[\mathcal{U}]$ whenever $s(x) \neq x$. If s and t are substitutions, then $t \star s$ is the substitution whose domain is $dom(s)$ and for every $x \in dom(s)$, $(t \star s)(x) = s(x)[t]$.*

It is shown in [2] that we can state the AFA axiom in terms of substitution. AFA is equivalent to the assertion that for every proper substitution e there is a unique proper substitution s such that $s = s \star e$.

We remark here first that if f is a substitution, not necessarily proper, then a substitution s such that $s = s \star f$ is not necessarily unique for the following reason: Let a and b be distinct urelements. Let $f(a) = b, f(b) = a$. Let $u \in \mathcal{U} - \{a, b\}$ and $s(a) = s(b) = u$. Then $s = s \star f$.

Second, if f is such that the domain of f contains sets, then s does not necessarily exist. For example, let a and b be distinct urelements, and let $f(a) = \{a\}$, $f(\{a\}) = b$. If s is such that $s = s \star f$, then $s(a) = (s \star f)(a) = f(a)[s] = sub(s, f(a)) = sub(s, \{a\}) = s(\{a\})$ by definition 3. Then by (ii) of definition 2, $s(\{a\}) = f(\{a\})[s] = sub(s, f(\{a\})) = sub(s, b) = b$. But by (iii) $s(a) = sub(s, \{a\}) = \{sub(s, a)\} = \{s(a)\}$, hence $s(a) = \Omega$, where Ω is the unique non–wellfounded set x such that $x = \{x\}$.

As a corollary of our theory of substitution fixed points, we will show that for every substitution f there is g such that $g = g \star f$, see lemma 16.

2.2 Fixed points approach

Next we study the non–wellfounded sets as fixed points of substitutions. Here we generalize the equation systems to arbitrary functions. The solutions are then defined in terms of the substitution. The fixed points are further generalizations of the solutions. This approach works well also in the situation without urelements. First we introduce some notation.

Definition 4 .

(i) With every function f we associate a class function f^* defined as follows. If $x \in dom(f)$, then $f^*(x) = f(x)$, otherwise $f^*(x) = x$.

(ii) If f and g are functions, then by $f[g]$ we mean a function such that $dom(f[g]) = dom(g)$ and $f[g](x)$ is defined as follows. If $f^*(x)$ is an urelement, then $f[g](x) = f^*(x)$ and otherwise $f[g](x) = \{g^*(y) \mid y \in f^*(x)\}$.

(iii) For all functions f and g, we say that g is a solution to f ($S(g,f)$), if $dom(g) = dom(f)$ and $g = f[g]$

From the above we see that in [2] a solution to a flat system (X, A, f) of equations is defined so that g is the solution to the system iff $S(g, f)$ holds.

So, in a sense, the solution to a flat system (X, A, f) of equations is obtained if in $f(x)$ all elements y from $f(x) \cap dom(f)$ are replaced by $f(y)$. Then all elements z in $f(y) \cap dom(f)$ are replaced by $f(z)$ and so on. So the solutions are some kind of restricted substitution–fixed points of the function from the system. In fact, in [2] it is shown that we get an equivalent theory if instead of equations we study substitution (cf. above). Because of the urelements, we define $\bigcup X = \bigcup\{x \mid x \in X$ and x is not an urelement$\}$.

Next we introduce the concept of a fixed point and the fixed point axiom.

Definition 5 .

(i) We say that g is a fixed point of f, $(FP(g, f))$, if $dom(f) \cup \bigcup f^*[dom(g)] \subseteq dom(g)$ and $g = f[g]$ (where $f^*[dom(g)] = \{f^*(y) \mid y \in dom(g)\}$).

(ii) We say that a function f is generating if for all $x \in dom(f)$ the following holds: if $f(x)$ is an urelement, then $x = f(x)$. We say that a generating f is basic if $dom(f) \subseteq \mathcal{U}$.

(iii) The fixed point axiom (FPA) is the following: every function has a fixed point.

Note that if $FP(g, f)$ holds and $f^*(x)$ is not an urelement, then $g(x) = \{g(y) \mid y \in f^*(x)\}$. The following example shows the difference between solutions and fixed points: Let x be an urelement, $dom(f) = \{x\}$ and $f(x) = (\emptyset, x)$. Then f itself is the solution to f but if g is a fixed point of f, then $g(x) = (\emptyset, g(x))$. Also following the notation from [2], if f is basic and $FP(g, f)$ holds, then for all $x \in dom(g)$, if $f(x)$ is not an urelement, then $g(x) = sub(g, f(x))$. Note that the basic functions are the same as the proper substitutions in [2].

Example 6 *Assume ZFC. Let X be a set and $f : X \to \mathcal{P}(X)$ be such that $f(x) = x \cap X$. Then f has a (unique) fixed point g (such that $dom(g) = dom(f)$), namely the Mostowski collapse of X.*

In [2] it is also shown that in the presence of the axiom AFA, bisimulation characterizes identity:

By $TC(x)$ we mean the transitive closure of x and in the case x is an urelement, $TC(x)$ is defined to be $\emptyset$.

Definition 7 .

(i) We write $B(x, y)$ if there is $B \subseteq (\{x\} \cup TC(x)) \times (\{y\} \cup TC(y))$ such that

(a) $(x, y) \in B$,

(b) if $(a, b) \in B$ and $c \in a$, then there is $d \in b$ such that $(c, d) \in B$,

(c) if $(a, b) \in B$ and $d \in b$, then there is $c \in a$ such that $(c, d) \in B$,

(d) if $(a,b) \in B$, then a is an urelement iff b is an urelement and if they are urelements, then $a = b$.

We call this kind of a relation B a bisimulation relation between x and y.

(ii) We let the strong extensionality axiom (SEA) be the following axiom:

$$\forall x, y(B(x,y) \to x = y).$$

The axiom system ZFC^{-2} consists of pairing, union, power set, infinity, collection, separation, and choice, together with the axiom of urelements: $\forall p \forall q(\mathcal{U}(p) \to q \notin p)$, and the axiom of plenitude of urelements: for every set S there is an injective function $f : S \to \mathcal{U}$ whose image $f[S]$ is disjoint from S. So the list of the axioms of ZFC^{-2} is the same as that in page 28 of [2] excluding extensionality and replacing strong plenitude by plenitude of urelements. ZFA means $ZFC^{-2}+$ extensionality $+AFA$. By ZFC^{+} we mean $ZFC^{-2}+SEA+FPA$. So the difference between ZFA and ZFC^{+} is that we have replaced equations by substitution and uniqueness of the solutions by SEA.

We feel that our axiom system follows the lines of the axiom systems of [1] in that the axioms for the existence and the uniqueness of the solutions are separeted. Also we think that this approach is a bit more set theoretical in nature, since the fixed point axiom refers to functions instead of graphs or equation systems.

We start by showing that ZFA and ZFC^{+} are equivalent. Especially we show that ZFA implies that every function has a fixed point. Then we show that the fixed points of the basic functions are fixed points of themselves, thus the name fixed point. For all functions this does not hold. Finally, we study the following question: Do we need to assume the existence of all solutions to the flat systems of equations to get all fixed points? We show that the answer is (essentially) yes.

2.3 Equivalence of ZFA and ZFC^{+}

Item (ii) in the following Lemma is [2] Exercise 7.3 and item (i) is well–known.

Lemma 8 .

(i) $ZFC^{-2} \vdash \forall x, y \notin \mathcal{U}(\forall z(z \in x \leftrightarrow z \in y) \to B(x,y))$. Especially, $ZFC^{-2} \vdash \forall x, y(x = y \to B(x,y))$.

(ii) $ZFC \vdash \forall x, y(B(x,y) \to x = y)$.

Proof.

(i): Let B consist of (x,y) together with $(a,b) \in TC(x) \times TC(y)$, such that $a = b$. To show that B is a bisimulation between x and y, let $z \in x$. Then by the assumption, $z \in y$ also. By the definition of B, $(z,z) \in B$ and hence

B is a bisimulation between x and y. Especially, if $x = y$ and $x, y \notin \mathcal{U}$, then $\forall z(z \in x \leftrightarrow z \in y)$, and by the above, we can construct a bisimulation between x and y. If $x = y$ and $x, y \in \mathcal{U}$, then $\{(x, y)\}$ is a bisimulation between x and y.

(ii): By $\in$–induction: Assume the claim for all $x' \in x$ and that for some set y, $B(x, y)$ holds. This means that for every $x' \in x$ there is $y' \in y$ such that $B(x', y')$ holds. This is so because if B is the bisimulation between x and y, then $B \restriction ((TC(x') \cup \{x'\}) \times (TC(y') \cup \{y'\}))$ is a bisimulation between x' and y'. But this means that for every $x' \in x$ there is $y' \in y$ such that $x' = y'$, i.e. $x' \in y$. Similarly, if $y' \in y$, then there is $x' \in x$, such that $y' = x'$, i.e. $y' \in x$. By extensionality, $x = y$. $\square$

The following lemma is essentially proved in [2].

Lemma 9 *Assume $ZFC^{-2} + SEA$.*

(i) $\forall x, y \notin \mathcal{U}(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$ i.e. the extensionality axiom holds.

(ii) For all functions f, if $S(g, f)$ and $S(h, f)$ hold, then $g = h$.

(iii) For all functions f, if $FP(g, f)$ and $FP(h, f)$ hold and $A = dom(g) \cap dom(h)$, then $g \restriction A = h \restriction A$.

Notice that since the extensionality axiom holds in $ZFC^{-2} + SEA$, also the functions of the form $f[g]$ are well–defined and thus $S(g, f)$ and $FP(g, f)$ are well–defined.

Proof.

(i): Let B consist of (x, y) together with $(x, y) \in TC(x) \times TC(y)$, such that $x = y$. Then B is a bisimulation between x and y. This is so because if $a \in x$, then $a \in y$ also, and therefore $(a, a) \in B$. If $(a, a) \in B$, and $a \in \mathcal{U}$, then the condition (d) holds. If $a \notin \mathcal{U}$, and $c \in a$, then $(c, c) \in B$ and we are done.

(ii): Let B consist of $(g(a), h(a))$ together with $(x, y) \in TC(g(a)) \times TC(h(a))$ such that either

(1) $x = y$

or

(2) there is $z \in dom(f)$ such that $x = g(z)$ and $y = h(z)$.

To show that B is a bisimulation, let $z \in dom(f)$ and $(g(z), h(z)) \in B$. If $g(z)$ is an urelement, then $g(z) = f[g](z) = f^*(z) = f[h](z) = h(z)$.

Assume that $g(z)$ in not an urelement. Then $g(z) = f[g](z) = \{g^*(y) \mid y \in f^*(z)\}$. Let $c \in g(z)$. So $c = g^*(y)$ for some $y \in f^*(z)$. But because also $h(z) = \{h^*(y) \mid y \in f^*(z)\}$ and $g(z) = h(z)$, we have that $h^*(y) \in h(z)$. Now if $y \notin dom(f)$, then $y \notin dom(g) = dom(h)$. Therefore $g^*(y) = h^*(y) = y$ and $(y, y) \in B$. If $y \in dom(f)$, then $(g(y), h(y)) \in B$.

(iii): Let B consist of $(g(a), h(a))$ together with $(x, y) \in TC(g(a)) \times TC(h(a))$ for which there is $z \in \bigcup f^*[A]$ such that $x = g(z)$ and $y = h(z)$.

Assume that $(g(z), h(z)) \in B$ and $f^*(z)$ is not an urelement. So $g(z) = \{g(y) \mid y \in f^*(z)\}$. Now let $y \in \bigcup f^*[A]$, then $y \in A$ and therefore if $g(y) \in g(z)$, then also $h(y) \in h(z)$ and $(g(y), h(y)) \in B$. $\square$

Lemma 10 *Assume $ZFC^{-2} + SEA$.*

(i) Assume $FP(g, f)$ holds and let $x \in dom(g)$. If $TC(f^*(x)) \cap dom(f) = \emptyset$, then $g(x) = f^*(x)$.

(ii) If $FP(g, f)$ holds and for all $x \in dom(f)$, $TC(f(x) - dom(f)) \cap dom(f) = \emptyset$, then $S(g \restriction dom(f), f)$ holds.

Proof.
(i): We may assume that $f^*(x)$ is not an urelement, since if $f^*(x)$ is an urelement, then $g(x) = f[g](x) = f^*(x)$ by the assumption that $FP(g, f)$ holds. Let $A = TC(f^*(x)) \cap dom(g)$. Because for all $y \in A$, $y = f^*(y) \subseteq dom(g)$, A is transitively closed. Since $g(x) = \{g(y) \mid y \in f^*(x)\}$, it is enough to show that for all $y \in f^*(x)$, $g(y) = y$. Since $A \cap dom(f) = \emptyset$, $FP(g \restriction A, id_A)$ holds. Since A is transitively closed, $S(g \restriction A, id_A)$ holds. So by Lemma 9 (ii), it is enough to show that $S(id_A, id_A)$ holds, but this is clear.

(ii): Assume that $f(x)$ is not an urelement. By (i), $g(x) = \{g(y) \mid y \in f(x)\} = \{g(y) \mid y \in f(x) - dom(f)\} \cup \{g(y) \mid y \in f(x) \cap dom(f)\} = \{y \mid y \in f(x) - dom(f)\} \cup \{g(y) \mid y \in f(x) \cap dom(f)\} = f[g \restriction dom(f)](x)$. $\square$

Corollary 11 *Assume $dom(f) \subseteq \mathcal{U}$ and for all $x \in dom(f)$, $f(x) \subseteq \mathcal{U}$. If $FP(g, f)$ holds, then $S(g \restriction dom(f), f)$ holds.*

Proof. Let f be as in the assumption, then $TC(f(x) - dom(f)) \cap dom(f) = (f(x) - dom(f)) \cap dom(f) = \emptyset$. Hence, by lemma 10 (ii), $S(g \restriction dom(f), f)$ holds. $\square$

Lemma 12 *Assume ZFC^{-2}+ extensionality. Assume that f is a generating function, $A \subseteq dom(f)$, for all $x \in dom(f) - A$, $f(x)$ is an urelement, and $FP(g, f \restriction A)$ holds. If h is a function such that $dom(h) = dom(g) \cup dom(f)$, $h \restriction dom(g) = g$, and for all $x \in dom(f) - dom(g)$, $h(x) = f(x)$, then $FP(h, f)$ holds.*

Proof. If $x \in dom(f) - A$, then $f(x) = x \in \mathcal{U}$, because f is generating. Hence for all x, $f^*(x) = (f \restriction A)^*(x)$. Because $\bigcup f^*[dom(h)] = \bigcup f^*[dom(g) \cup dom(f)] = \bigcup f^*[dom(g) \cup (dom(f) - A)] = \bigcup f^*[dom(g)] = \bigcup (f \restriction A)^*[dom(g)] \subseteq dom(g) \subseteq dom(h)$, we have that $dom(f) \cup \bigcup f^*[dom(h)] \subseteq dom(h)$.

Assume $x \in dom(h)$ and $f^*(x) \notin \mathcal{U}$. Then $x \in dom(g)$ and $f[h](x) = \{h(y) \mid y \in f^*(x)\} = \{g(y) \mid y \in (f \restriction A)^*(x)\} = g(x) = h(x)$, because $f^*(x) = (f \restriction A)^*(x) \subseteq dom(g)$. Assume that $f^*(x) \in \mathcal{U}$. If $x \notin dom(g)$, then $h(x) = f(x) = x$. If $x \in dom(g)$, then $h(x) = g(x) = f^*(x)$. So we have shown that $h = f[h]$. $\square$.

The following lemma is proved for basic functions in [2] (cf. Theorem 8.5).

Lemma 13 *Assume ZFA. For all generating f there is g such that $FP(g, f)$ holds.*

Proof. Let f be a generating function. We show that f has a fixed point. By Lemma 12, we may assume that for all $x \in dom(f)$, $f(x)$ is not an urelement. Choose a transitively closed A so that $dom(f) \cup \bigcup rng(f) \subseteq A$. Then $\bigcup f^*[A] \subseteq A$. Choose a one-one function h so that $dom(h) = B = (A - \mathcal{U}) \cup dom(f)$, $rng(h) \subseteq \mathcal{U}$, $h(y) = y$ if $y \in dom(f) \cap \mathcal{U}$ and $rng(h) \cap A = dom(f) \cap \mathcal{U}$. Define f' so that $dom(f') = rng(h)$ and for all $x \in B$, $f'(h(x)) = \{h^*(y) \mid y \in f^*(x)\}$. Then $(rng(h), (A \cap \mathcal{U}) - rng(h), f')$ is a flat system of equations. Let g' be such that $S(g', f')$ holds and let g be such that $dom(g) = A$, $g \upharpoonright B = g' \circ h$ and $g \upharpoonright A - B = id_{A-B}$. We show that g is a fixed point of f. We have already shown that $dom(f) \cup f^*[dom(g)] \subseteq dom(g)$. So it is enough to show that for all $x \in A$, $g(x) = f[g](x)$. If $x \notin B$, then $g(x) = x$ and $x \notin dom(f)$. So $f^*(x) = x$ is an urelement and we have that $f[g](x) = f^*(x) = x = g(x)$.

Assume that $x \in B$. Then $f'(h(x))$ is not an urelement and so $g(x) = g'(h(x)) = \{g'(y) \mid y \in f'(h(x))\} =$

$$\{g'(y) \mid y \in f'(h(x)) - rng(h)\} \cup \{g'(y) \mid y \in f'(h(x)) \cap rng(h)\} \qquad (1)$$

Now $f'(h(x)) - rng(h) = \{h^*(z) \mid z \in f^*(x)\} - rng(h) = \{z \mid z \in f^*(x) - B\} = f^*(x) - B$. If $y \in f^*(x) - B = f'(h(x)) - rng(h)$, then $y \notin dom(f')$, since $rng(h) = dom(f')$. Because $f^*(x) \subseteq A$, we have that $f^*(x) - B \subseteq \mathcal{U}$, so $y \in \mathcal{U}$. Hence $g'(y) = f'[g'](y) = (f')^*(y) = y$. On the other hand $f'(h(x)) \cap rng(h) = \{h^*(y) \mid y \in f^*(x)\} \cap rng(h) = \{h(y) \mid y \in f^*(x) \cap B\}$. Thus we have that (1) is equal to $\{y \mid y \in f^*(x) - B\} \cup \{g'(y) \mid y \in \{h(z) \mid z \in f^*(x) \cap B\}\} = \{g(y) \mid y \in f^*(x) - B\} \cup \{g'(h(z)) \mid z \in f^*(x) \cap B\} = \{g(y) \mid y \in f^*(x)\}$. □

Lemma 14 *Assume $ZFC^{-2} + SEA$. If every generating function has a fixed point, then every function has a fixed point.*

Proof. Let f be a function. Let A be a transitively closed set such that $dom(f) \cup rng(f) \subseteq A$. Let B be the set of those $x \in A$ such that $f^*(x) \neq x$ is an urelement and let C be the set of those $x \in A$ such that $f^*(x) = x$ is an urelement. Let h be a one-one function such that $dom(h) = B$ and $rng(h) \subseteq \mathcal{U} - A$. Define f' so that $dom(f') = A - B$ and for all $x \in dom(f')$, if $x \in C$, then $f'(x) = x$ and otherwise $f'(x) = \{h^*(y) \mid y \in f^*(x)\}$. Then f' is generating and so by Lemma 13, it has a fixed point g'. Let $D = g'[dom(f')] - \mathcal{U}$ and define f'' so that $dom(f'') = D$ and for all $x \in dom(f'')$, $f''(x) = \{h'(y) \mid y \in x\}$, where $h'(y) = y$, if $y \notin rng(h)$ and otherwise $h'(y) = f(h^{-1}(y))$. Then f'' is generating and let g'' be a fixed point of f''. We define g so that $dom(g) = A$, for all $x \in dom(g)$, if $f^*(x)$ is an urelement, then $g(x) = f^*(x)$ and otherwise, $g(x) = g''(g'(x))$. We show that g is a fixed point of f.

Since $rng(f) \subseteq A$ and A is transitively closed, $\bigcup f^*[A] \subseteq A$. Also $dom(f) \subseteq A$. So it is enough to prove that for all $x \in A$, $g(x) = f[g](x)$. If $x \in B \cup C$, the claim is clear. So assume $x \in A - (B \cup C)$. Then $g(x) = g''(g'(x)) = \{g''(y) \mid y \in f''(g'(x))\} = \{g''(h'(y)) \mid y \in g'(x)\} = \{g''(h'(g'(y))) \mid y \in f'(x)\} = \{g''(h'(g'(h^*(y)))) \mid y \in f^*(x)\}$. We have several cases:

1. $y \in C$: Then $h^*(y) = y$, $g'(y) = (f')^*(y) = y$, $h'(y) = y$ and $g''(y) = (f'')^*(y) = y$. Also $g(y) = y$ and so $g''(h'(g'(h^*(y)))) = g(y)$.

2. $y \in B$: Then $(f')^*(h^*(y)) = h^*(y)$ and so $g'(h^*(y)) = h^*(y)$. Furthermore $h'(h^*(y)) = f(y)$ and since $f(y) \notin dom(f'')$, $g''(f(y)) = f(y)$. So $g''(h'(g'(h^*(y)))) = f(y) = g(y)$.

3. $y \in A - (B \cup C)$: Clearly $h^*(y) = y$ and $g'(y)$ is a set. So $h'(g'(y)) = g'(y)$. Then $g''(h'(g'(h^*(y)))) = g''(g'(y)) = g(y)$.

By 1-3, $g(x) = \{g''(h'(g'(h^*(y)))) \mid y \in f^*(x)\} = \{g(y) \mid y \in f^*(x)\} = f[g](x)$. □

Corollary 15 *ZFC^+ is equivalent to ZFA.*

Proof. Assume ZFC^+ and that (X, A, f) is a flat system of equations. Then $f(x) \subseteq \mathcal{U}$ for every $x \in dom(f)$. Let g be such that $FP(g, f)$ holds. Then by corollary 11, $S(g \restriction dom(f), f)$ holds. Thus AFA holds. The extensionality axiom follows from lemma 9 (i) and hence ZFA holds.

Assume ZFA. Then by lemmata 13 and 14, every function has a fixed point. So FPA holds. By theorem 7.3 of [2], the strong extensionality axiom holds in ZFA. Hence ZFC^+ holds. □

Lemma 16 *If f is a substitution, then there is a function g such that $g = g \star f$.*

Proof. Let $u \in \mathcal{U}$ be such that $u \notin dom(f)$. We define f' as follows. Let $dom(f') = dom(f)$. If $f(x)$ is not an urelement, or $f(x) = x$, or $f(x) \in \mathcal{U} - dom(f)$, then let $f'(x) = f(x)$. If there is $n < \omega$ such that $\forall m < n : f^m(x) \in \mathcal{U}$ and $f^m(x) \neq f^{m-1}(x)$ but $f^n(x) \notin \mathcal{U}$, or $f^n(x) = f^{n-1}(x)$, or $f^n(x) \in \mathcal{U} - dom(f)$, then let $f'(x) = f^n(x)$. Otherwise let $f'(x) = u$.

Let g' be such that $FP(g', f')$ holds and let $g = g' \restriction dom(f)$. We show that $g = g \star f$.

If $f(x) \in dom(f)$ is an urelement, then by the definition of f', we have that $f'(f(x)) = f'(x)$. Now if $f'(x) \in \mathcal{U}$, $g(x) = f'(x) = f'(f(x)) = g(f(x)) = sub(g, f(x))$. If $f'(x) \notin \mathcal{U}$, $g(x) = \{g'(y) \mid y \in f'(x)\} = \{g'(y) \mid y \in f'(f(x))\} = g(f(x)) = sub(g, f(x))$. If $f(x) \in \mathcal{U} - dom(f)$, then $g^*(x) = f'(x) = f(x) = sub(g, f(x))$. So we have shown that if $f(x)$ is an urelement, then $g(x) = g(f(x))$, hence we need to show that for all $x \in dom(f)$, if $f(x) \notin \mathcal{U}$, then $g'(x) = sub(g, f(x))$. For this, we define a bisimulation B so that $(a, b) \in B$ iff there is a $y \in dom(g')$ such that $a = g'(y)$ and $b = sub(g, f^*(y))$, or $a = b \in \mathcal{U} \cap dom(g')$, or $a = g'(y) = b$.

To show that B is bisimulation, let $(g'(y), sub(g, f^*(y))) \in B$ for some $y \in dom(g')$. We have several cases:

1. $y \in \mathcal{U} - dom(f)$: Then $(f')^*(y) = f^*(y) = y$ so $g'(y) = y = sub(g, f^*(y))$.

2. $y \in dom(f)$ and $f(y) \in \mathcal{U}$: As above we have that $g'(y) = g(f(y)) = sub(g, f^*(y))$.

3. $y \in dom(f)$, $f(y) \notin \mathcal{U}$: Because $f(y) \notin \mathcal{U}$, we have that $f(y) = f'(y)$, so

$$\begin{aligned} g'(y) &= \{g'(z) \mid z \in f'(y)\} \\ sub(g, f^*(y)) &= \{sub(g, z) \mid z \in f(y)\}. \end{aligned}$$

Assume $z \in f(y)$, so $z \in dom(g')$. If $z \notin dom(f)$, then $(g'(z), sub(g, f^*(z))) \in B$. If $z \in dom(f)$, then $sub(g, z) = g(z) = g'(z)$ and $(g'(z), g'(z)) \in B$. □

For a class function F, $FP(G, F)$ is defined as for the set functions. We show that under ZFC^+ also the class functions have fixed points.

Lemma 17 *Assume ZFC^+. Let $F : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$ be a definable class function. Then there exists a unique definable class function $G : V_{afa}[\mathcal{U}] \cup \mathcal{U} \to V_{afa}[\mathcal{U}] \cup \mathcal{U}$ such that $FP(G, F)$ holds.*

Proof. Let $x \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. If $F(x) \in \mathcal{U}$, then let $G(x) = F(x)$. Otherwise we define $G(x)$ as follows. Let $A_0 = TC(\{x\})$, $A_{n+1} = A_n \cup TC(F[A_n])$, and $A(x) = \bigcup_{n<\omega} A_n$. Then $A(x)$ is transitively closed and $F[A(x)] \subseteq A(x)$. Now let g be a function such that $FP(g, F \restriction A(x))$ holds and define $G(x) = g(x)$.

We show that $g(y)$ does not depend on the choice of $A(x)$ as long as $y \in A(x)$ and $F[A(x)] \subseteq A(x)$. Let A and A' be transitively closed sets such that $y \in A$, $F[A] \subseteq A$, and $F[A'] \subseteq A'$. Let g and g' be such that $FP(g, F \restriction A)$ and $FP(g', F \restriction A')$ hold. Let $C = A \cap A'$, then C is transitively closed, $y \in C$, and $F[C] \subseteq C$. We show that $g \restriction C = g' \restriction C$ from which the claim follows. So let $z \in C$ and let B consist of $(g(z), g'(z))$ together with $(a, b) \in TC(g(z)) \times TC(g'(z))$ such that either $a = b$ or there is $c \in C$ such that $a = g(c)$ and $b = g'(c)$. To show that B is a bisimulation between $g(z)$ and $g'(z)$, let $c \in C$ and $(g(c), g'(c)) \in B$. Assume that $F(c)$ is not an urelement. Then $g(c) = \{g(d) \mid d \in (F \restriction C)(c)\}$. Now if $d \in (F \restriction C)(c)$, then $d \in C$ and so $(g(d), g'(d)) \in B$. So B is a bisimulation and hence for all $z \in C$, $g(z) = g'(z)$.

We show that G is a fixed point of F. Let $x \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$. If $F(x) \in \mathcal{U}$, then $G(x) = F[G](x) = F(x)$. If $F(x) \notin \mathcal{U}$, let $A(x)$ be the transitively closed set such that $x \in A(x)$ and $F[A(x)] \subseteq A(x)$. Let g be such that $FP(g, F \restriction A(x))$ holds. Then $G(x) = g(x) = \{g(y) \mid y \in (F \restriction A(x))^*(x)\} = \{G(y) \mid y \in F(x)\}$, because $A(x)$ is transitively closed and $F[A(x)] \subseteq A(x)$. □

As a corollary we have the Theorem 8.1 of [2].

Corollary 18 *There is a unique operation $sub(s, b)$ as in definition 2 defined for all substitutions s and sets b.*

Proof. Assume s is a substitution. Define a class function F by $F(x) = x$ if $x \notin dom(s)$ and $F(x) = s(x)$ otherwise. By the above lemma, let G be such that $FP(G, F)$ holds. We claim that $G(x) = sub(s, x)$ for all x.

If x is an urelement, then $G(x) = F(x) = sub(s, x)$. Let x be a set. Define the relation B by $(a, b) \in B$ iff $a = G(y)$ and $b = sub(s, y)$ for some $y \in \{x\} \cup TC(x)$. We show that B is a bisimulation between $G(x)$ and $sub(s, x)$. The case for urelements is as in the above, so let $(G(y), sub(s, y)) \in B$ where $G(y)$ is a set. Then $G(y) = \{G(z) \mid z \in F(y)\}$ and $sub(s, y) = \{sub(s, z) \mid z \in y\}$. Because $y \notin dom(s)$, $F(y) = y$ and we see that the bisimulation can be continued. □

We finish this section by showing that a fixed point of a fixed point of a basic f is a fixed point of f. Thus the name fixed point.

Lemma 19 *Assume ZFC^+. For every function f there is a function g such that $FP(g, f)$ holds and $rng(g) \cup \bigcup rng(g) \subseteq dom(g)$.*

Proof. Let f be a function and g' such that $FP(g', f)$. We define inductively functions f_n and g_n for $n < \omega$ as follows. Let $f_0 = f$ and $g_0 = g'$.

Let $A_n = dom(g_n) \cup rng(g_n) \cup \bigcup rng(g_n)$ and $dom(f_{n+1}) = dom(f_n) \cup A_n$. Define $f_{n+1}(x) = f_n(x)$, if $x \in dom(f_n)$, and otherwise $f_{n+1}(x) = x$. Let g_{n+1} be such that $FP(g_{n+1}, f_{n+1})$ holds.

Because for every n, $dom(f_n) \subseteq dom(g_{n+1})$, we have that

$$rng(g_n) \cup \bigcup rng(g_n) \subseteq dom(g_{n+1}). \tag{2}$$

From the definition of f_n if follows that for all n, $f_n \subseteq f_{n+1}$ and also $f \subseteq f_n$. Clearly if $x \notin dom(f)$, then $f_n(x) = x$, hence for every x and n, $f_n^*(x) = f^*(x)$.

It is clear that $dom(f) \subseteq dom(g_n)$. Also $f^*[dom(g_n)] = f[dom(g_n) \cap dom(f)] \cup (dom(g_n) - dom(f)) = f_n^*[dom(g_n)]$. Hence $\bigcup f^*[dom(g_n)] \subseteq \bigcup f_n^*[dom(g_n)] \subseteq dom(g_n)$. If $x \in dom(g_n)$ and $f^*(x)$ is an urelement, then $g_n(x) = f_n^*(x) = f^*(x)$. If $f^*(x)$ is not an urelement, then $g_n(x) = \{g_n(y) \mid y \in f_n^*(x)\} = \{g_n(y) \mid y \in f^*(x)\}$. Thus $FP(g_n, f)$.

Now because for every $n < \omega$, $FP(g_n, f)$, $FP(g_{n+1}, f)$, and $dom(g_n) \subseteq dom(g_{n+1})$, we have by lemma 9 (iii), that $g_n \subseteq g_{n+1}$. So we can define $g = \bigcup_{n<\omega} g_n$. By (2) we have that $rng(g) \cup \bigcup rng(g) \subseteq dom(g)$.

Finally, we show that $FP(g, f)$ holds. Because for all n, $FP(g_n, f)$, we have that $dom(f) \cup \bigcup f^*[dom(g)] \subseteq dom(g)$. Let $x \in dom(g)$. Then for some n, $x \in dom(g_n)$. If $f^*(x) \in \mathcal{U}$, then $g(x) = g_n(x) = f^*(x)$. Otherwise $g(x) = g_n(x) = \{g_n(y) \mid y \in f_n^*(x)\} = \{g(y) \mid y \in f^*(x)\} = f[g](x)$. □

Lemma 20 *Assume $ZFC^{-2} + SEA$, f is basic, $FP(g, f)$ holds and $rng(g) \cup \bigcup rng(g) \subseteq dom(g)$. Then for all $x \in dom(g)$, $g(g(x)) = g(x)$ and if $g(x) \notin \mathcal{U}$, then $g(x) = \{g(y) \mid y \in g(x)\}$. Especially, $FP(g, g)$ and $S(g, g)$ hold.*

Proof. Let $a \in dom(g)$. Let B consist of $(g(a), g(g(a)))$ together with $(x, y) \in TC(g(a)) \times TC(g(g(a)))$ such that either $x = y$ or there is $z \in dom(g)$ such that $x = g(z)$ and $y = g(g(z))$.

We show that B is a bisimulation between $g(a)$ and $g(g(a))$. Assume that $z \in dom(g)$, $f^*(z) \notin \mathcal{U}$, and $(g(z), g(g(z))) \in B$.

Let $y \in g(z) = \{g(w) \mid w \in f^*(z)\}$. So $y = g(w)$ for some $w \in f^*(z)$. Because $g(w) \in rng(g) \subseteq dom(g)$, $g(g(w))$ is defined. So $(g(w), g(g(w))) \in B$.

Let $y \in g(g(z)) = \{g(w) \mid w \in f^*(g(z))\}$. So $y = g(w)$ for some $w \in f^*(g(z)) \subseteq dom(g)$. Thus $(g(w), g(g(w))) \in B$.

For the second claim, assume that $g(x) \notin \mathcal{U}$. So $g(x) \notin dom(f)$ and thus $g(x) = g(g(x)) = \{g(y) \mid y \in g(x)\} = g[g](x)$. Thus $S(g, g)$ holds. Because $dom(g) \cup \bigcup g^*[dom(g)] \subseteq dom(g) \cup rng(g) \cup \bigcup rng(g) \subseteq dom(g)$, we have that $FP(g, g)$ holds. $\Box$

The assumption that f is basic is needed in Lemma 20:

Example 21 *Assume ZFC^+ (the first example works also in ZFC).*

(i) We define sets e^n, $n < \omega$, so that $e^0 = \emptyset$ and $e^{n+1} = \{e^n\}$. Let f be such that $f(e^3) = e^2$, $f(e^2) = \{e^0, e^1\}$ and for $n < 2$, $f(e^n) = e^n$. Then $FP(f, f)$ holds, but $f(f(e^3)) = \{e^0, e^1\} \neq f(e^3)$

(ii) As in [2], let Ω, be such that $\Omega = \{\Omega\}$. Define f so that $f(\emptyset) = \{\emptyset\}$ and $f(\Omega) = \emptyset$. Let g be such that $FP(g, f)$ holds: Then since $g(x) = \{g(y) \mid y \in f^*(x)\}$, $g(\emptyset) = \Omega$ and $g(\Omega) = \emptyset$. But $\{g(y) \mid y \in g(\emptyset)\} = \{\emptyset\} \neq g(\emptyset)$, so it is not the case that $FP(g, g)$.

2.4 A model in which not all equations have solutions

We now turn to the question: Do we need to assume that all flat systems of equations have solutions in order to get all fixed points. First we show how to construct a model of set theory from a given transitive class of non–wellfounded sets.

Definition 22 *Let $C \subset V_{afa}[\mathcal{U}] \cup \mathcal{U}$ be a transitive class.*

$$\mathbf{C}(C) = \{x \in V_{afa}[\mathcal{U}] \cup \mathcal{U} \mid \text{there is no sequence } x_i, i < \omega \text{ such that } x_0 \in TC(x) \text{ and } \forall i, x_{i+1} \in x_i, x_i \notin C\}.$$

Intuitively this class is the same as the following class V'': Let $V'_0 = C \cup \mathcal{U}$, $V'_{\alpha+1} = \{x \mid x \subseteq V'_\alpha\}$, $V'_\beta = \bigcup_{\alpha<\beta} V'_\alpha$, when β is a limit ordinal, and let $V'' = \bigcup\{V'_\alpha \mid \alpha \text{ is an ordinal}\}$.

Lemma 23 *Assume that $C \subset V_{afa}[\mathcal{U}] \cup \mathcal{U}$ is a transitive class and $V' = \mathbf{C}(C)$, then $V' \models ZFC^{-2} + SEA$.*

Proof. Now V' is a transitive class, so the axioms of extensionality and strong extensionality hold in V'. If x is a subset of V', then clearly $x \in V'$. Hence the power set axiom holds in V'.

The axiom of urelements, $\forall p \forall q(\mathcal{U}(p) \to \neg(q \in p))$ holds in V'. The pairing and union axioms also hold in V'. Because $\omega \in V'$, $\emptyset$ and the successor operation are absolute for V', V' satisfies the axiom of infinity.

For the collection it is enough to show that for each formula $\phi(x, y, A, w_1, \ldots, w_n)$ and each $A, w_1, \ldots, w_n \in V'$, if $\forall x \in A \exists! y \in V' \phi^{V'}(x, y, A, w_1, \ldots, w_n)$, then $\exists Y \in V'(\{y \mid \exists x \in A, \phi^{V'}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y)$. So let $Y = \{y \in V' \mid \exists x \in A, \phi^{V'}(x, y, A, w_1, \ldots, w_n)\}$. Then $Y \subset V'$ and hence $Y \in V'$.

Since for every $z \in V'$, $P(z) \subseteq V'$, we have that V' satisfies the separation axiom. For the axiom of choice, we can show that if $x \in V'$ and x can be well–ordered, then $(x \text{ can be well–ordered})^{V'}$: If $R \subseteq x \times x$ well–orders x, then since $x \times x \in V'$ we have that $R \in V'$. The formula "R totally orders x" is absolute for V'. For well–ordering we have to check that $(\forall y \phi(y, x, R))^{V'}$, where $\phi(y, x, R)$ is

$$y \subseteq x \wedge y \neq \emptyset \to \exists z \in y \forall w \in y((w, z) \notin R).$$

Now ϕ is absolute for V' so it is enough to show that $\forall y \in V' \phi(y, x, R)$, which follows since R well–orders x. Thus the axiom of choice holds in V'.

For the axiom of plenitude, which is: $\forall S \notin \mathcal{U}(\exists f : S \to \mathcal{U}$ such that f is injective and $f[S] \cap S = \emptyset)$, let S be a set in V'. Let $g : S \to \mathcal{U}$ be an injection in $V_{afa}[\mathcal{U}]$. Then also $g \in V'$, because V' is closed under the power set operation. We have shown that $V' \models ZFC^{-2} + SEA$. □

Lemma 24 *Assume that $C \subset V_{afa}[\mathcal{U}]$ is a transitive class and there exist x_i, $i < \omega$ such that $x_{i+1} \in TC(x_i)$ and $x_i \notin C$. If $V' = \mathbf{C}(C)$, then $V' \models ZFC^{-2} + SEA$ and $V' \not\models AFA$.*

Proof. We define the canonical flat system of equations for x_0 as follows. Let h be an injection such that $dom(h) = TC(x_0)$, $rng(h) \subseteq \mathcal{U}$ and if $a \in TC(x_0) \cap \mathcal{U}$, then $h(a) = a$. Let $A = TC(x_0) \cap \mathcal{U}$, $a_0 \in \mathcal{U} - rng(h)$, $X = (rng(h) \cup \{a_0\}) - A$. Define f in X such that $f(a_0) = \{h(y) \mid y \in x_0\}$ and $f(h(z)) = \{h(y) \mid y \in z\}$ for $z \in dom(h)$. So f is a system of equations which belongs to V' and it was constructed so that for the solution g to f, we have that $g(a_0) = x_0$.

Now x_0 can not be in V', because of the definition of V'. Because being a solution to a flat system of equations is an absolute property for V', we have that f has not a solution in V'. □

Next we introduce the notion of a flat $\mathcal{P}$–coalgebra from [2], which corresponds to a flat system of equations with no atoms.

Definition 25 *A flat $\mathcal{P}$–coalgebra is a pair (X, f) such that $X \subseteq \mathcal{U}$ is a set of urelements and $f : X \to \mathcal{P}(X)$. A function g is a substitution solution to the flat $\mathcal{P}$–coalgebra if $FP(g, f)$ holds.*

It is shown in [2] that if g is a substitution solution to a flat $\mathcal{P}$–coalgebra, then $rng(g) \subseteq \mathcal{P}^*$, where $\mathcal{P}^*$ is the greatest fixed point of the operator $\mathcal{P}$ and it is equal to the class of all pure sets $V_{afa}[\emptyset]$. (In case thera are no urelements, then $\mathcal{P}^*$ is the whole universe and the example below does not hold anymore, but see the next section.)

Example 26 *Assume $ZFC^{-2}+SEA$. The following does not imply AFA: Every flat $\mathcal{P}$–coalgebra has a substitution solution.*

Proof. Let $C = V_{afa}[\emptyset]$. If (X, f) is a flat $\mathcal{P}$–coalgebra and g its solution, then $rng(g) \subseteq C$. If we let $V' = \mathbf{C}(C)$, then $dom(g) \in V'$ and hence $g \in V'$, because V' is closed under the power set operation. Because being a flat $\mathcal{P}$–coalgebra and a solution to it are absolute properties for V', we have that in V' every flat $\mathcal{P}$–coalgebra has a solution.

Let x be an urelement and $f(x) = \{a, x\}$, where $a \in \mathcal{U}$ and $a \neq x$. Then f is an equation system and it has a solution g in $V_{afa}[\mathcal{U}]$. Then $g(x) = \{a, g(x)\} \notin C$, and we get the x_i's as required in lemma 24, by setting $x_i = g(x)$. Hence by lemma 24, AFA does not hold in V'. □

Γ–coalgebras can be seen as systems of equations (see [2] section 16) and if we restrict our interest to flat Γ–coalgebras, then the class of solutions can be seen as the final Γ–coalgebra. One may wonder if the same can be done (e.g. by fixed points) for a larger class of Γ–coalgebras than the flat ones. This does not seem to be the case or at least a much deeper understanding of non–wellfounded sets is needed. The crusial property of the flat Γ–coalgebras (X, e) is that X is new for Γ (i.e. for all substitutions t and sets a, $\Gamma(a[t]) = \Gamma(a)[t]$), this forces X to be flat in the usual sense of the word. And without something like this the theory does not work. E.g. the crusial lemma 16.1 in [2] fails:

Let $\Gamma(X) = \mathcal{P}(\mathcal{P}(X) - \{\emptyset\})$. This is a monotone and proper operator, i.e. if $X \subseteq Y$, then $\Gamma(X) \subseteq \Gamma(Y)$, and for all sets a, $\Gamma(a) \subseteq V_{afa}[\mathcal{U}]$. Let $X = \{\emptyset, \{\emptyset\}\}$, and let $e(\emptyset) = \{\{\emptyset\}\}$ and $e(\{\emptyset\}) = \emptyset$. Then (X, e) is a Γ–coalgebra that is not flat. If s is a solution to e, then $s(\emptyset) = \{\emptyset\}$ and $s(\{\emptyset\}) = \emptyset$ but $\{\emptyset\} \notin \Gamma^*$. Hence s is not a Γ–morphism of (X, e) into (Γ^*, id).

2.5 Classifying non–wellfounded sets

Here we have a fixed class of urelements, $\mathcal{U}$, and all sets can contain urelements as their members. So from now on we denote by V_{afa} the class $V_{afa}[\mathcal{U}] \cup \mathcal{U}$ of [BM]. But as in the above, urelements are not vital in here. We regard arbitrary functions as equation systems and when we speak of the indeterminates of the equation systems, we mean the elements of their domain.

We define a series of classes of equation systems E_α, $\alpha \in \mathbf{ON}$ in increasing complexity. From these equation systems we obtain a series of classes of non–

wellfounded sets,

$$V_{afa}^0 \subset V_{afa}^1 \subset \cdots \subset V_{afa}^\alpha \subset \cdots ,$$

so that $V_{afa}^{\alpha+1} \not\subset V_{afa}^{\alpha}$. We also define the *rank* of a non–wellfounded set x as the least α such that $x \in V_{afa}^\alpha$.

The non–wellfounded sets become more complicated in the series $V_{afa}^0 \subset V_{afa}^1 \subset \cdots$ according to the branching structure of the non–wellfounded sets. V_{afa}^0 is the class of wellfounded sets. In V_{afa}^1 there are sets, which can be described as either Ω and sets that can be obtained from it by standard set theoretical operations or sets which have a non-wellfounded $\in$–sequence of length ω such that going down this sequence one has ω chances to branch out of that sequence. But in V_{afa}^1 after branching the sets are wellfounded. In V_{afa}^2 there are sets in which there are ω chances to branch to sets in which there are again ω chances to branch into sets in which there are only finite number of possibilities to branch. So the rank tells how many times it is possible to branch arbitrarily deep.

A non–wellfounded set of rank ω has elements of arbitrarily high rank below ω. In a non–wellfounded set of rank $\omega + 1$ one can find a non–wellfounded $\in$–sequence in which there are ω chances to branch into sets of rank ω. And so on in the higher degrees.

There is also the possibility that this branching process goes on arbitrarily long. In this case we say that the rank is ∞. First we need to characterize the different classes of equation systems. This is done in game theoretic terms.

Definition 27 *Let f be a system of equations. A sequence $\vec{x} = \langle x_i \mid i < \omega \rangle$, where $x_i \in dom(f), i < \omega$, of indeterminates of f is called descending if for all $i < \omega$, $x_{i+1} \in f(x_i)$.*

We describe a game $G_\alpha(\mathcal{E})$, where $\alpha \in \mathbf{ON}$, that is played on a given system of equations f as follows. There are two players, black and white. First the black player chooses a descending sequence $\vec{x}$ of indeterminates. Then white chooses an ordinal $\alpha_0 < \alpha$ and a natural number $n < \omega$. Black must respond with a descending sequence of indeterminates $\vec{y}$ such that for some $m \geq n$, $y_0 \in f(x_m)$, and $y_0 \neq x_{m+1}$. So $\vec{y}$ branches out of $\vec{x}$. Then again white chooses an ordinal $\alpha_2 < \alpha_1$ and a natural number and so on.

The length of this game is the number of pairs of moves by black and white. This length is finite, since there are no infinite descending sequences of ordinals.

We say that black has a winning strategy in the game $G_\alpha(f)$ if she is able to respond to white's moves until white has no more moves. White wins otherwise, that is if black is not able to respond with a descending sequence of indeterminates to white's move.

There is also a game of infinite length. In $G_\infty(f)$ the white player does not choose ordinals, only indices. Hence the length of this game is ω.

More formally, we say that a move of the white player is a pair (α, n), where α is an ordinal and $n < \omega$. We use the projection function $\pi_2(\alpha, n) = n$ to get the second coordinate of the pair (α, n). In $G_\alpha(f)$ we say that a sequence $\vec{w}$ is a *legal* sequence of white's moves of length k if

$$\vec{w} = \langle(\alpha_i, n_i) \mid i < k\rangle,\ \forall i \in \omega(n_i < \omega),\ \text{and } \alpha > \alpha_0 > \alpha_1 > \cdots > \alpha_{k-1}$$

We define the winning strategy σ for black as follows.

Definition 28 *Let f be a system of equations and α an ordinal. A winning strategy for the black player in the game $G_\alpha(f)$ is a function σ of two arguments, a natural number k and a legal sequence $\vec{w}$ of white's moves of length k, that satisfies the following conditions:*

(i) $\sigma(0, \emptyset) = \vec{x}$, *where $\vec{x}$ is a descending sequence of the indeterminates of f*

(ii) $\sigma(k+1, \vec{w}) = \vec{y}$, *where $\vec{y}$ is a descending sequence of indeterminates such that the following holds. Denote by $\vec{x}$ the previous move of black, i.e. $\sigma(k, \vec{w} \restriction k)$ and denote by n white's last move, i.e. $\pi_2(w_k)$. We require from $\vec{y}$ that $\exists m \geq n(y_0 \in f(x_m)$ and $y_0 \neq x_{m+1})$.*

We say that the black player *wins* a game if she has a winning strategy. The white player wins, if the black player does not win.

We may also define a similar game played on non–wellfounded sets, $G_\alpha(x)$. We say that a sequence $\langle x_i \mid i < \omega\rangle$ is a non–wellfounded sequence, if for all $i < \omega$, $x_{i+1} \in x_i$. If we replace in the above definitions the system of equations f by a set x and the descending sequences of indeterminates by non–wellfounded sequences, then we have the corresponding definition for sets. For sets we also require that $\sigma(0, \emptyset)$ is a non–wellfounded sequence starting from x.

If white wins the game $G_0(x)$, then x is well–founded. If white wins $G_1(x)$, and black wins $G_0(x)$, then in $TC(x)$ there are non–wellfounded sets but no sets in which we can branch two times as described above. Also, if black wins G_α, then black wins G_β for all $\alpha \leq \beta$ and if white wins G_α, then white wins G_β for all $\beta \geq \alpha$.

Definition 29 .

(i) $E_\alpha = \{f \mid \textit{white wins } G_\alpha(f)\}$

(ii) $E_\infty = \{f \mid \textit{black wins } G_\infty(f)\}$,

(iii) $V_{afa}^\alpha = \{x \mid \textit{white wins } G_\alpha(x)\}$

(iv) $V_{afa}^\infty = \{x \mid \textit{black wins } G_\infty(x)\}$

(v) *AFA_α is the statement that all the systems of equations in E_α have solutions.*

From the preceding definition it follows that if x is a set and f is its canonical system of equations, then $x \in V_{afa}^{\alpha}$ iff $f \in E_{\alpha}$. Also black player's winning strategy in the game on sets can be straightforwardly converted into a winning strategy in the game on equation systems. Thus all the solutions to the equation systems from E_{α} are in V_{afa}^{α}.

The solution set x to an equation system f does not always have the same rank as f. For example define an equation system f such that $dom(f) = \{u_{\eta} \in \mathcal{U} \mid \eta \in 2^{<\omega}\}$, where u_{η}'s are distinct, by $f(u_{\eta}) = \{u_{\eta^\frown\{0\}}, u_{\eta^\frown\{1\}}\}$. Then $f \notin E_{\alpha}$ for all α but the solution set of f is Ω, by exercise 7.1 of [2], and $\Omega \in V_{afa}^{1}$.

Definition 30 *The non–wellfoundedness rank of a set x, denoted by* nwfrank(x) *is the least α such that $x \in V_{afa}^{\alpha}$, if there is such and ∞ otherwise.*

Note that for x such that $\text{nwfrank}(x) \in \mathbf{ON}$ we have that $\text{nwfrank}(x) = \min\{\alpha \mid \text{white wins } G_{\alpha}(x)\} = \sup\{\alpha + 1 \mid \text{black wins } G_{\alpha}(x)\}$. We also have that if $x \in y$, then $\text{nwfrank}(x) \leq \text{nwfrank}(y)$ but not necessarily $\text{nwfrank}(x) < \text{nwfrank}(y)$. In fact, Marshall and Schwarze [6] have shown that it is not possible to define a rank function r such that if $x \in y$, then $r(x) < r(y)$, in set theory without the foundation axiom. Another notion of rank for non–wellfounded sets, defined using modal logic, appears in [2], section 11.

Lemma 31 *Black wins $G_{\alpha}(x)$ iff there is a non–wellfounded sequence $\vec{x}$ starting from x such that for all $\beta < \alpha$ the set*

$$A_{\beta} = \{i < \omega \mid \exists y \in x_i (y \neq x_{i+1} \text{ and black wins } G_{\beta}(y))\}$$

is an unbounded subset of ω.

Proof. Assume that black has a winning strategy σ in the game $G_{\alpha}(x)$. Let $\vec{x} = \sigma(0, \emptyset)$. Let $n < \omega$ and $\beta < \alpha$. If we let (β, n) be the first move of the white player, then $\sigma(1, (\beta, n))$ is a non–wellfounded sequence $\vec{y}$ such that $\exists i \geq n (y_0 \in x_i$ and $y_0 \neq x_{i+1})$ by the definition of a winning strategy. The winning strategy σ' for black in the game $G_{\beta}(y_0)$ is defined by the following equations

$$\begin{aligned} \sigma'(0, \emptyset) &= \sigma(1, (\beta, n)) \\ \sigma'(m+1, \vec{w}) &= \sigma(m+2, (\beta, n)^\frown\vec{w}). \end{aligned}$$

Hence A_{β} is unbounded in ω.

Assume on the other hand that there is a non–wellfounded sequence $\vec{x}$ starting from x and satisfying the condition. We prove that black has a winning strategy σ in the game $G_{\alpha}(x)$. Let $\sigma(0, \emptyset) = \vec{x}$. Let $(\beta, n) \in \alpha \times \omega$ be white's first move. Since A_{β} is unbounded in ω there is $i \geq n$ such that $\exists y \in x_i (y \neq x_{i+1}$ and black wins $G_{\beta}(y)$ with winning strategy $\sigma')$. Then let

$$\begin{aligned} \sigma(1, (\beta, n)) &= \sigma'(0, \emptyset) \\ \sigma(m+2, (\beta, n)^\frown\vec{w}) &= \sigma'(m+1, \vec{w}). \end{aligned}$$

□

Corollary 32 nwfrank$(x) \geq \alpha$ *iff for all* $\beta < \alpha$ *there is a non–wellfounded sequence* $\vec{x}$ *starting from* x *such that*

$$\{i < \omega \mid \exists y \in x_i(y \neq x_{i+1} \text{ and } \mathrm{nwfrank}(y) \geq \beta)\}$$

is an unbounded subset of ω.

By corollary 32 we could have also defined the classes V^α_{afa} via the concept of nwfrank, by letting wellfounded sets have rank 0. So we have that $V^\alpha_{afa} = \{x \mid \mathrm{nwfrank}(x) \leq \alpha\}$.

Theorem 33 $V^\alpha_{afa} \models ZFC^{-2} + SEA + AFA_\alpha$.

Proof. Let $V' = \mathbf{C}(V^\alpha_{afa})$. We claim that $V^\alpha_{afa} = V'$ from which the conclusion follows. Since all the solutions to the equation systems from E_α are in V^α_{afa}, $V^\alpha_{afa} \models AFA_\alpha$.

By the definition of V', we have that $V^\alpha_{afa} \subseteq V'$. We show that $V' \subseteq V^\alpha_{afa}$ by showing that if nwfrank$(x) > \alpha$, then $x \notin V'$.

Assume towards a contradiction that there is $x \in V'$ for which nwfrank$(x) > \alpha$. So the white player does not have a winning strategy in $G_\alpha(x)$ and this means that the black player has. From this it follows by lemma 31, that there is a non–wellfounded sequence $\vec{x}$ starting from x such that for all $\beta < \alpha$ the set

$$A_\beta = \{i < \omega \mid \exists y \in x_i(y \neq x_{i+1} \text{ and black wins } G_\beta(y))\}$$

is an unbounded subset of ω.

Let $i < \omega$. Black wins $G_\beta(x_i)$ for all $\beta < \alpha$, because $\vec{x} \restriction [i, \omega]$ is now a non–wellfounded sequence where black wins. So by lemma 31, black wins $G_\alpha(x_i)$. Hence nwfrank$(x_i) > \alpha$, and so $x_i \notin V^\alpha_{afa}$ which violates the definition of $\mathbf{C}(V^\alpha_{afa})$.
□

From the preceding proof we can extract the following corollaries:

Corollary 34 *If* $\alpha < \gamma$, *then* $V^\alpha_{afa} \subsetneq V^\gamma_{afa}$.

Proof. If nwfrank$(x) = \gamma > \alpha$, then by the proof of the theorem, we have that $x \notin V^\alpha_{afa}$.

We construct an example of a set x for which nwfrank$(x) = \gamma$ as follows.

Let $X = \{u_\alpha \mid \alpha \leq \gamma\}$ be a set of distinct urelements. Let f be such that $dom(f) = X$ and for $\alpha \leq \gamma$, let

$$f(u_\alpha) = \begin{cases} \{u_\beta \mid \beta < \alpha\} & \text{if } \alpha \text{ is a limit,} \\ \{u_\alpha, u_{\alpha-1}\} & \text{if } \alpha \text{ is a successor,} \\ \emptyset & \text{if } \alpha = 0. \end{cases}$$

Let g be the solution to f and let $x = g(u_\gamma)$. We show by induction that $\text{nwfrank}(g(u_\alpha)) = \alpha$ for $\alpha \leq \gamma$. It is clear that $\text{nwfrank}(g(u_0)) = 0$.

Assume the claim for α. By the construction, there is a non–wellfounded sequence $\vec{x} = \langle g(u_{\alpha+1}), g(u_{\alpha+1}), \ldots \rangle$ starting from $g(u_{\alpha+1})$. Let $i < \omega$. Then $g(u_\alpha) \in x_i$ and $\text{nwfrank}(g(u_\alpha)) = \alpha$, hence by corollary 32, $\text{nwfrank}(g(u_{\alpha+1})) \geq \alpha+1$. We show that white wins $G_{\alpha+1}(g(u_{\alpha+1}))$, whence $\text{nwfrank}(g(u_{\alpha+1})) = \alpha+1$. For the first move black has to choose $\vec{x}$ (other choices would be worse). But black can not win $G_{\alpha+1}(g(u_{\alpha+1}))$ in $\vec{x}$ by lemma 31, because for all $i < \omega$, there is no $y \in x_i$ such that black wins $G_\alpha(y)$.

Assume the claim for $\beta < \alpha$. By the construction, $g(u_\alpha) = \{g(u_\beta) \mid \beta < \alpha\}$. For every $\beta < \alpha$, $\text{nwfrank}(g(u_\beta)) = \beta$ so $\text{nwfrank}(g(u_\alpha)) \geq \alpha$. We show that white wins $G_\alpha(g(u_\alpha))$. Black has to choose some non–wellfounded sequence starting from $g(u_\alpha)$, say $\vec{x} = \langle g(u_\beta), g(u_\beta), \ldots \rangle$. Then white chooses some ordinal γ such that $\beta < \gamma < \alpha$. Now black cannot win $G_\gamma(g(u_\beta))$, because $\text{nwfrank}(g(u_\beta)) = \beta < \gamma$. □

Corollary 35 *If $\alpha < \gamma$, then $V^\alpha_{afa} \not\models AFA_\gamma$.*

Proof. If we let f be the canonical equation system for a set x such that $\text{nwfrank}(x) = \gamma$, then $f \in E_\gamma$. But f does not have a solution in V^α_{afa}. □

Next we show that all the AFA_α axioms together with AFA_∞ imply AFA. But note that $\forall\alpha AFA_\alpha \not\vdash AFA$.

Lemma 36 $\vdash AFA \leftrightarrow (AFA_\infty \wedge \forall\alpha\, AFA_\alpha)$.

Proof. Let f be an arbitrary system of equations, and assume that the white player does not win $G_\alpha(f)$ for any α. We show that then black wins $G_\infty(f)$. For a descending sequence of indeterminates $\vec{u}$ of f, let

$$r(\vec{u}) = \sup\{\alpha \mid \text{black wins } G_\alpha(f) \text{ where the first move of black is } \vec{u}\}.$$

There is an ordinal α such that if $\vec{u}$ is a descending sequence of indeterminates of f and $r(\vec{u}) \geq \alpha$, then $r(\vec{u}) = \infty$. This is so because otherwise the set $\{\vec{u} \mid \vec{u}$ is a descending sequence of indeterminates of $f\}$ and hence f would be a proper class.

We describe a winning strategy for black in the game $G_\infty(f)$ as follows. There is a descending sequence of indeterminates $\vec{u}_0$ of f such that $r(\vec{u}_0) \geq \alpha$ since otherwise we could take

$$\gamma = \sup\{r(\vec{u}) \mid \vec{u} \text{ from } f \text{ such that } r(\vec{u}) \neq \infty\}$$

and white would win $G_{\gamma+1}(f)$. Let $\vec{u}_0$ be the first move of black. Let n be the first move of white. Because $r(\vec{u}_0) \geq \alpha$, then by the above, $r(\vec{u}_0) = \infty$. So there is a descending sequence $\vec{u}_1$ such that it branches out of $\vec{u}_0$ below n and $r(\vec{u}_1) \geq \alpha$. So this way we can continue the game arbitrarily long. □

By the previous lemma, we also see that $V_{afa}^{\infty} \cup \{x \mid \exists\alpha(x \in V_{afa}^{\alpha})\} = V_{afa}$.

3 A domain order over non–wellfounded sets

In this section we construct a domain structure on a subclass of $V_{afa}[\mathcal{U}]$. The approximation order $\sqsubseteq$ is based on a certain kind of tree extension on canonical tree pictures of sets.

3.1 Introduction

Domain theory can be seen as a theory of approximation and it is used widely in computer science to model various programming languages and abstract data types. On the other hand, non–wellfounded sets, studied in Aczel [1] and in Barwise and Moss [2], have found applications in computer science. There are connections between these two theories. In an appendix to his book [1], Peter Aczel writes:

> A natural way to try to understand non–well–founded sets is to view them as limits, in some sense, of their well–founded approximations. This approach is inspired by Scott's theory of domains, but it cannot be done in any simple minded way, as I found out.

In [7], Mislove, Moss, and Oles show how to construct a domain structure on HF_1, the set of all hereditarily finite sets, which may be non–wellfounded. Their construction involves initial continuous algebras. Here we try to define a domain ordering directly giving a set–theoretic definition in terms of certain kinds of tree extensions.

In [3], Boffa describes a way to approximate non–wellfounded sets by their wellfounded components by an inverse limit construction. The approximating sets are taken from the class of all wellfounded and hereditarily finite sets, HF_0. We show that this actually defines a domain ordering in the appropriate domain completion, $\widehat{HF}_0$.

This approach differs from the one in [7] in that it does not use protosets. Lindström [5] has also studied non–wellfounded sets as the inverse limits of wellfounded sets in constructive set theory.

3.2 Domains

We give the basic definitions of domains. As a reference to Domain Theory we use Stoltenberg–Hansen, Lindström, and Griffor [8]. Let $(D, \sqsubseteq)$ be a partially ordered set. A set $A \subseteq D$ is called directed, if $A \neq \emptyset$ and for all $a_1, a_2 \in A$ there is $a \in A$ such that $a_1 \sqsubseteq a$ and $a_2 \sqsubseteq a$. We denote by $\bigsqcup A$ the least upper bound of A, if it exists.

Definition 37 *Let $D = (D, \sqsubseteq, \bot)$ be a partially ordered set with a least element $\bot$. Then D is called a complete partial order if $\bigsqcup A$ exists for all directed $A \subseteq D$.*

In domains we have an abstract notion of finiteness, or compactness as it is called, by the following definition.

Definition 38 *Let D be a complete partial order. An element $a \in D$ is said to be compact, if whenever $A \subseteq D$ is a directed set and $a \sqsubseteq \bigsqcup A$, then there is some $x \in A$ such that $a \sqsubseteq x$. We say that D is algebraic if for each $x \in D$, the set* $\mathrm{approx}(x) = \{a \sqsubseteq x \mid a \text{ is compact}\}$ *is directed and* $x = \bigsqcup \mathrm{approx}(x)$.

We add one more condition to obtain the definition of Scott–Ershov domains. Two elements a and b of D are called consistent if there is c such that $a \sqsubseteq c$ and $b \sqsubseteq c$.

Definition 39 *An algebraic complete partial order D is called a domain if for any two consistent compact elements a and b, $a \sqcup b$ exists.*

Domains can be obtained also by so called ideal completions of conditional upper semi lattices.

Definition 40 *A partial order $P = (P, \sqsubseteq, \bot)$ with least element $\bot$ is a conditional upper semi lattice with least element (abbreviated cusl) if whenever $\{a, b\}$ is consistent in P, then $a \sqcup b$ exists in P.*

We cite the following well–known representation theorem for domains from [8]. The notion of an ideal can be defined for domains in the usual way. A principal ideal generated by an element a of the domain is denoted by $[a] = \{b \mid b \sqsubseteq a\}$.

Theorem 41 *Let P be a cusl and let $\overline{P} = \{I \subseteq P \mid I \text{ an ideal}\}$. Then the structure $\overline{P} = (\overline{P}, \subseteq, [\bot])$ is a domain. Furthermore, the compact elements of $\overline{P}$, denoted by $\overline{P}_c$ are precisely the principal ideals. Finally, the map $\iota : P \to \overline{P}_c$ defined by $\iota(a) = [a]$ is order–preserving.*

3.3 Non–wellfounded sets

Here we follow the lines of [1] in formulating the *AFA* axiom. But we use urelements as in [2]. Although they are not vital to the theory, they will simplify a few things in some applications. We denote the class of all urelements by $\mathcal{U}$. (Officially there is a predicate $\mathcal{U}(x)$ in the language and axioms, $\forall x \forall y(\mathcal{U}(x) \to y \notin x)$, and an axiom stating that for any set there is an equipollent set of urelements. We denote $\mathcal{U}(x)$ also by $x \in \mathcal{U}$. The axiom of extensionality is also adjusted to hold only on sets.)

We work in the set theory ZFC^- which is like ordinary Zermelo–Fraenkel set theory, but the foundation axiom is excluded. Non–wellfounded sets are modeled by directed graphs, which consist of nodes and edges between the nodes. For a graph G we denote the set of its edges by G_E and its vertices by G_V.

We require that the graphs are pointed, i.e. there is a distinguished point, and accessible, i.e. every node is accessible from the point. We abbreviate accessible pointed graph by apg. We may assign sets of urelements to the nodes of an apg with a labelling function f such that $dom(f) = G_E$ and $f(x) \subset \mathcal{U}$ for all $x \in dom(f)$.

Definition 42 *Let G be an apg and f its labelling. A decoration of G is a function d such that $d(n) = \{d(n') \mid (n, n') \in G_V\} \cup f(n)$.*

The *AFA* axiom now says that every apg has a unique decoration. So to an apg G we may assign a set x such that $x = d(n)$, where d is the unique decoration of G and n is the distinguished point of G. We also say then that G is a picture of the set x. In the next section we show how to assign to a set a canonical tree picture of it.

The uniqueness condition in the *AFA* axiom is equivalent to the strong extensionality axiom which states that bisimulation characterizes identity. We recall the definition of bisimulation.

Definition 43 *We write $B(x, y)$ if there is a relation $B \subseteq (\{x\} \cup TC(x)) \times (\{y\} \cup TC(y))$ such that*

(i) $(x, y) \in B$,

(ii) if $(a, b) \in B$ and $c \in a$, then there is $d \in b$ such that $(c, d) \in B$,

(iii) if $(a, b) \in B$ and $d \in b$, then there is $c \in a$ such that $(c, d) \in B$,

(iv) if $(a, b) \in B$, then a is an urelement iff b is an urelement and if they are urelements, then $a = b$.

We call this kind of a relation B a bisimulation relation between x and y.

3.4 A partial order of non–wellfounded sets

We are going to define a domain ordering on a subclass of non–wellfounded sets. Here we first define a partial order $\sqsubseteq$ on all non–wellfounded sets.

When trying to see non–wellfounded sets as limits of wellfounded sets one idea could be to see them as some kind of substitution limits, or fixed points as in Hyttinen and Pauna [4]. But then the ordering does not come out as a domain ordering.

In order to construct the partial order, we see sets as trees, called canonical tree pictures. For a set a, we can define the canonical tree picture of a, denoted by $T(a)$ as in [1]. The graph relation $(n, n') \in G_V$ is from now on to be denoted by $n \to n'$.

Definition 44 *Let a be a set. Then $T(a)$ consists of all finite sequences $\langle a_i : i < k\rangle$ such that $a_0 = a$, and for all $i < k-1$, $a_{i+1} \in a_i$. For t and t' in $T(a)$, we let $t \to t'$ (i.e. $(t,t') \in T(a)_V$) iff t' is obtained from t by adding one element to t.*

Here we have implicitly the labelling f of a tree picture T by urelements, so that if the last element of a node t is an urelement u, then we require that f assigns to the predecessor of t the urelement u. Formally, if $t = t^{'\frown} u \in T$, then $u \in f(t')$. If we drop the condition that $a_{i+1} \in a_i$, then we only say that T is a tree picture. By *AFA*, understood as in [1], every tree picture is a picture of a unique set.

Definition 45 *Let T be a tree picture and $t \in T$.*

(i) $\to^*$ *is the transitive and reflexive closure of* $\to$*,*

(ii) $\ln(t) = |\{t' \mid t' \to^* t\}|$ *is the length of* t*,*

(iii) t_n*, where* $n \leq \ln(t)$*, is the nth element of* t*,*

(iv) $\mathrm{last}(t) = t_{\ln(t)-1}$ *is the last element of* t*,*

(v) $t_T = \{t' \in T \mid t \to t'\}$ *is the set of immediate successors of* t*,*

(vi) $Tt = \{t' \in T \mid t \to^* t'\}$ *is the subtree of* T *whose root is* t*.*

We define an ordering on tree pictures as a certain kind of end extension. We say that a node t is a leaf node when there is no $t' \leftarrow t$. So either the last element of $t \in T$ is the empty set or an urelement when T is a canonical tree picture.

Definition 46 *Let a and b be sets. Then $a \sqsubseteq b$ if there is a partial epimorphism $f : T(b) \to T(a)$ such that for all $t \in dom(f)$*

(i) *for* $t, t' \in dom(f)$, $t \to t'$ *iff* $f(t) \to f(t')$*,*

(ii) $f(\langle b\rangle) = \langle a\rangle$*,*

(iii) *if* $\mathrm{last}(f(t)) \in \mathcal{U}$*, then* $\mathrm{last}(t) = \mathrm{last}(f(t))$*,*

(iv) *if* $t', t'' \leftarrow t$ *and* $t' \in dom(f)$*, then* $t'' \in dom(f)$*,*

(v) *if* $\mathrm{last}(f(t)) = \emptyset$ *and there is* $t' \in dom(f)$ *such that* $\ln(t') > \ln(t)$*, then* $\mathrm{last}(t) = \emptyset$*.*

In what follows, we mean by an *epimorphism* a function that satisfies the previous definition. For a set a we define its height $\mathrm{ht}(a) = \mathrm{ht}(T(a)) = \sup\{\ln(t) \mid t \in T(a)\}$.

Lemma 47 *Assume $a \sqsubseteq b$ and $f : T(b) \to T(a)$ is an epimorphism witnessing this. Then* *(i)* $\mathrm{ht}(dom(f)) = \mathrm{ht}(a) \leq \mathrm{ht}(b)$,

(ii) $dom(f) = T(b) \upharpoonright \mathrm{ht}(a)$,

(iii) *if* $\mathrm{ht}(a) = \mathrm{ht}(b)$, *then* $a = b$.

Proof. Here we understand $dom(f)$ as the subtree of $T(b)$ where f is defined.

(i) This follows from the fact that f is surjective and for all $t \in dom(f)$, $\ln(t) = \ln(f(t))$.

(ii) We show the claim by induction on $t \in T(b)$, where $\ln(t) \leq \mathrm{ht}(T(a))$. First, the root, $b \in dom(f)$. Assume $t \in dom(f)$ and $t' \leftarrow t$. If there is $h' \leftarrow f(t)$, then $\ln(t) < \mathrm{ht}(T(a))$ and for some $t'' \leftarrow t$, $f(t'') = h'$, because f is surjective and respects $\rightarrow$. Now by the condition (iv) of definition 46, also $t' \in dom(f)$.

Assume then that $f(t)$ is a leaf node. If $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(t) \in \mathcal{U}$. Assume $\mathrm{last}(f(t)) = \emptyset$. Now if for some $h \in T(a)$, $\ln(h) > \ln(f(t))$, then there is, by (i), some $t'' \in dom(f)$ such that $\ln(h) = \ln(t'')$. But then, by (v) of definition 46, $\mathrm{last}(t) = \emptyset$. If for all $h \in T(a)$, $\ln(h) \leq \ln(f(t))$, then $f(t)$ is maximal in $T(a)$, so $\ln(f(t)) = \ln(t) = \mathrm{ht}(T(a))$. Hence $\ln(t') > \mathrm{ht}(T(a))$ and so $t' \notin dom(f)$.

(iii) Let $B = \{(\mathrm{last}(f(t)), \mathrm{last}(t)) \mid t \in T(b)\}$. We show that B is a bisimulation between a and b. First, we have that $(a, b) \in B$. Assume $(\mathrm{last}(f(t)), \mathrm{last}(t)) \in B$. Let $x \in \mathrm{last}(f(t))$, then there is $h' \in T(a)$ such that $\mathrm{last}(h') = x$ and $h' \leftarrow f(t)$. By (ii), there is some $t' \in T(b)$ such that $f(t') = h'$. Because $f(t') = h' \leftarrow f(t)$ we have that $t' \leftarrow t$. Hence $\mathrm{last}(t') \in \mathrm{last}(t)$ and $(\mathrm{last}(f(t'), \mathrm{last}(t'))) \in B$.

Let $y \in \mathrm{last}(t)$, so $y = \mathrm{last}(t')$ for some $t' \leftarrow t$. Now by (ii), $t' \in dom(f)$. So $f(t) \rightarrow f(t')$ and $(\mathrm{last}(f(t')), \mathrm{last}(t')) \in B$. Assume $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(t) = \mathrm{last}(f(t))$. If $\mathrm{last}(t) \in \mathcal{U}$, then also $\mathrm{last}(f(t)) \in \mathcal{U}$ and hence $\mathrm{last}(f(t)) = \mathrm{last}(t)$. □

By the previous lemma, we see that all strictly $\sqsubseteq$–increasing sequences are at most of length ω.

Lemma 48 *For all sets* a, b, *and* c *we have*

(i) $a \sqsubseteq a$,

(ii) *if* $a \sqsubseteq b$ *and* $b \sqsubseteq a$, *then* $a = b$,

(iii) *if* $a \sqsubseteq b$ *and* $b \sqsubseteq c$, *then* $a \sqsubseteq c$.

Proof. (i) The identity $id_{T(a)} : T(a) \rightarrow T(a)$ is an epimorphism.

(ii) If $f : T(b) \rightarrow T(a)$ and $g : T(a) \rightarrow T(b)$, then $\mathrm{ht}(a) \leq \mathrm{ht}(b)$ and $\mathrm{ht}(b) \leq \mathrm{ht}(a)$, hence $\mathrm{ht}(a) = \mathrm{ht}(b)$. So by (iii) of lemma 47, $a = b$.

(iii) Assume $f : T(b) \rightarrow T(a)$ and $g : T(c) \rightarrow T(b)$ are epimorphisms. Define $h : T(c) \rightarrow T(a)$ by $h(t) = f(g(t))$, if $g(t)$ and $f(g(t))$ are defined. We show that h is an epimorphism. First we have that $h(c) = a$ and that it respects

$\rightarrow$. Assume $t' \in T(a)$. Then for some $t'' \in T(b)$, $f(t'') = t'$, but also for some $t \in T(c)$, $g(t) = t''$, so $h(t) = f(g(t)) = t'$.

If $h(t) = f(g(t)) \in \mathcal{U}$, then $\mathrm{last}(f(g(t))) = \mathrm{last}(g(t)) = \mathrm{last}(t)$. Assume that $\mathrm{last}(f(g(t))) = \emptyset$ and there is some $t' \in dom(h)$ such that $\ln(t') > \ln(t)$. So $t' \in dom(g)$ and $g(t') \in dom(f)$. Now $\ln(g(t')) > \ln(g(t))$, so $\mathrm{last}(g(t)) = \emptyset$ and hence $\mathrm{last}(t) = \emptyset$. □

Lemma 49 *Assume $f : T(b) \rightarrow T(a)$ is a partial epimorphism. If $n \leq \mathrm{ht}(b)$, then $f \restriction (T(b) \restriction n) : T(b) \restriction n \rightarrow T(a) \restriction n$ is also a partial epimorphism. Especially, if $n \leq \mathrm{ht}(a)$, then $T(b) \restriction n$ and $T(a) \restriction n$ picture the same sets.*

Proof. If $n \geq \mathrm{ht}(a)$, then $f \restriction (T(b) \restriction n) = f$. If $n \leq \mathrm{ht}(a)$, then we see by the definition of a partial epimorphism, that $f \restriction (T(b) \restriction n)$ is a partial epimorphism. The last remark follows from (iii) of lemma 47. □

Corollary 50 *The following are equivalent.*

(i) $a \sqsubseteq b$,

(ii) $T(a)$ *and* $T(b) \restriction \mathrm{ht}(a)$ *are pictures of the same sets,*

(iii) $T(b) \restriction \mathrm{ht}(a)$ *is a picture of* a.

□

We have now obtained that epimorphisms are unique.

Corollary 51 *If f and f' are partial epimorphisms from $T(b)$ to $T(a)$, then $f = f'$.*

Proof. By lemma 47 (ii), $dom(f) = T(b) \restriction \mathrm{ht}(a) = dom(f')$. Let $B = \{(f(t), f'(t)) \mid t \in dom(f)\}$. We show that B is a bisimulation relation on $T(a)$ from which the claim follows. If $\mathrm{last}(f(t)) \in \mathcal{U}$, then $\mathrm{last}(f(t)) = \mathrm{last}(t) = \mathrm{last}(f(t'))$. Let $f(t) \rightarrow s$. Because f is surjective, there is some $t' \in dom(f)$ such that $f(t') = s \leftarrow f(t)$. Hence $t \rightarrow t'$, so $f'(t) \rightarrow f'(t')$ and $(f(t'), f'(t')) \in B$. So B is a bisimulation on $T(a)$. □

Lemma 52 *Assume $a \sqsubseteq c$ and $b \sqsubseteq c$. Then*

(i) if $\mathrm{ht}(a) = \mathrm{ht}(b)$, *then* $a = b$,

(ii) if $\mathrm{ht}(a) < \mathrm{ht}(b)$, *then* $a \sqsubseteq b$.

Proof. Let $f_a : T(c) \rightarrow T(a)$ and $f_b : T(c) \rightarrow T(b)$ be the partial epimorphisms.

(i) Since $\mathrm{ht}(a) = \mathrm{ht}(b)$, we have that $dom(f_a) = T \restriction \mathrm{ht}(a) = dom(f_b)$ by (iv) of lemma 47. Define a bisimulation B between a and b as

$$B = \{(\mathrm{last}(f_a(t)), \mathrm{last}(f_b(t))) \mid t \in dom(f_a)\}.$$

First, $(a,b) \in B$. Assume $(\text{last}(f_a(t)), \text{last}(f_b(t))) \in B$ and $x \in \text{last}(f_a(t))$, i.e. $s \leftarrow f_a(t)$ for some s such that $\text{last}(s) = x$. There is some $t' \in dom(f_a)$ such that $f_a(t') = s$. Then $t \to t'$ and $f_a(t) \to f_a(t')$. Also, $t' \in dom(f_b)$ and $f_b(t) \to f_b(t')$. So we have that $\text{last}(f_b(t')) \in \text{last}(f_b(t))$ and $(\text{last}(f_a(t')), \text{last}(f_b(t'))) \in B$, hence B is a bisimulation between a and b.

(ii) Now $T(c) \upharpoonright \text{ht}(a)$ and $T(a)$ are pictures of the set a, by lemma 49. But then $T(b) \upharpoonright \text{ht}(a)$ is also a picture of a. So by corollary 50, $a \sqsubseteq b$. □

Corollary 53 *If a and b are consistent, then $a \sqcup b$ exists.*

Proof. Actually, the predecessors of a set are linearly ordered, by lemma 52, so either $a \sqsubseteq b$ or $b \sqsubseteq a$, or equivalently $a \sqcup b = b$ or $a \sqcup b = a$. □

The previous corollary states the property of conditionally closedness for domains in the class of all non–wellfounded sets. Below we are going to show how to restrict the class to obtain a domain.

3.5 Inverse limits of projective sequences

Here we show how to obtain limits of $\sqsubseteq$–increasing sequences of wellfounded sets. We need first a lemma stating that epimorphisms can always be composed to yield a unique epimorphism.

Lemma 54 *If $f_{cb} : T(c) \to T(b)$ and $f_{ba} : T(b) \to T(a)$ are partial epimorphisms, then $dom(f_{ba}) \subseteq ran(f_{cb})$ and $f_{ca} = f_{ba} \circ f_{cb}$ is the unique epimorphism from $T(c)$ to $T(a)$.*

Proof. Now $\text{ht}(c) \geq \text{ht}(b) \geq \text{ht}(a)$ and $dom(f_{ba}) = T(b) \upharpoonright \text{ht}(a) \subseteq T(b) = ran(f_{cb})$. So $f_{ba}(f_{cb}(t))$ is defined iff $t \in T(c) \upharpoonright \text{ht}(a)$. Since $f_{ba} \circ f_{cb}$ is a partial epimorphism, cf. proof of lemma 48 (iii), it is unique by corollary 51. □.

Assume that $\vec{a} = \langle a_i \mid i < \omega \rangle$ is a strictly $\sqsubseteq$-increasing sequence of sets. So for all $i < \omega$, $\text{ht}(a_i) < \omega$, and there is an epimorphism $f_{i+1,i} : T(a_{i+1}) \to T(a_i)$. We show next how to construct an upper bound for $\vec{a}$. The idea is to construct the inverse limit of the projective sequence $(T(a_{i+1}), f_{i+1,i})_{0 \leq i < \omega}$.

Let

$$T = \{(t_n) \in \prod_{n=k}^{\omega} T(a_n) \mid k < \omega, f_{n+1,n}(t_{n+1}) = t_n \text{ and } (t_k \notin dom(f_{k,k-1}) \text{ or } k = 0)\}.$$

Let $(t_n)_{k \leq n < \omega} \to (t'_n)_{k' \leq n < \omega}$ iff $k' \geq k$ and $t_n \to t'_n$ for all $n \geq k'$. Note that if $t_k \notin dom(f_{k,k-1})$, then $t_k \notin dom(f_{k,k'})$ for any k' such that $0 \leq k' < k$, since in that case $f_{k,k'} = f_{k-1,k'} \circ f_{k,k-1}$.

We show first that if $(t_n)_{k \leq n < \omega} \to (t'_n)_{k' \leq n < \omega}$, then $k' = k$ or $k' = k + 1$. First assume that t_k is a leaf node in $T(a_k)$, so $k' > k$. We show that in this case, we have that $k' = k+1$. Assume $k' > k+1$ and k'' is such that $k < k'' < k'$. Then $t_{k''}$ is a leaf node in $T(a_{k''})$ because otherwise $t_{k''} \to f_{k',k''}(t'_{k'})$, since $t_{k'} \to t'_{k'}$ and $f_{k',k''}(t_{k'}) = t_{k''}$, hence $t'_{k'} \in dom(f_{k',k''})$, a contradiction. But because $\vec{a}$ was strictly increasing, there is a node $t \in T(a_{k'})$ such that $\ln(t) > \ln(t_{k''})$. Since $t_{k''}$ is a leaf, $\text{last}(f_{k',k''}t_{k'}) = \emptyset$, and hence also $\text{last}(t_{k'}) = \emptyset$. But on the other hand, $t_{k'} \to t'_{k'}$, a contradiction.

Assume then that t_k is not a leaf node. So some $t'' \leftarrow t_k$ in $T(a_k)$. Now if $k' > k$, then $f_{k',k}(t_{k'}) = t_k$ and $t_{k'} \to t'_{k'}$ imply that $f_{k',k}(t'_{k'}) = t''$, which is a contradiction, since $t'_{k'} \notin dom(f_{k',k})$. Hence $k' = k$. So we have shown in all that $k' = k+1$ if and only if t_k is a leaf node in $T(a_k)$. In this case the extension to t_k comes from the next tree $T(a_{k+1})$.

Next we show that T is strongly extensional in the sense that it has no two bisimilar subtrees starting from the same node. So assume that there is a node $\vec{t} \in T$ and its immediate successors $\vec{t^1}$ and $\vec{t^2}$ such that $T\vec{t^1}$ and $T\vec{t^2}$ are bisimilar subtrees of T. Let B be the bisimulation. Let $\pi_n : T \to T(a_n)$, $n < \omega$ be the projection mapping, i.e.

$$\pi_n((t_i)_{i \geq k}) = \begin{cases} \text{undefined} & \text{if } n < k, \\ t_n & \text{otherwise.} \end{cases}$$

By the above we have that the first index of both $\vec{t^1}$ and $\vec{t^2}$ is the same, say k. We show first that for every $n \geq k$, and $t \in T(a_n)\pi_n(\vec{t^1})$ there is $\vec{t'} \in T\vec{t^1}$ such that $\pi_n(\vec{t'}) = t$ by induction on the nodes of the tree. So assume the claim holds for $\vec{t'}$ and let $t''_n \leftarrow \pi_n(\vec{t'})$ in $T(a_n)$. We build a sequence $\vec{t''}$ around t''_n such that $\vec{t''} \leftarrow \vec{t'}$, so $\vec{t''} \in T\vec{t^1}$. Let $m > n$. Because $f_{mn}(t'_m) = t'_n$ and $t''_n \leftarrow t'_n$, there is some $t''_m \in T(a_m)$ such that $f_{mn}(t''_m) = t''_n$ and $t''_m \leftarrow t'_m$. Let $m \leq n$. Assume we have found t''_m. If $t_m \notin dom(f_{m,m-1})$, then by the above, $m \leq k+1$. Otherwise let $t''_{m-1} = f_{m,m-1}(t''_m) \leftarrow t'_{m-1}$.

We claim that for every $n \geq k$, $T(a_n)\pi_n(\vec{t^1})$ and $T(a_n)\pi_n(\vec{t^2})$ are bisimilar subtrees of $T(a_n)$, hence $\pi_n(\vec{t^1}) = \pi_n(\vec{t^2})$, because they have a common immediate predecessor. Let

$$B_n = \{(\pi_n(\vec{h^1}), \pi_n(\vec{h^2})) \mid (\vec{h^1}, \vec{h^2}) \in B\}.$$

Assume $(\pi_n(\vec{h^1}), \pi_n(\vec{h^2})) \in B_n$ and $h \leftarrow \pi_n(\vec{h^1})$, so by the above, there is some $\vec{h'} \in T\vec{t^1}$ such that $\pi_n(\vec{h'}) = h$ and $\vec{h'} \leftarrow \vec{h^1}$. Since B is a bisimulation, there is $\vec{h''} \leftarrow \vec{h^2}$ such that $(\vec{h'}, \vec{h''}) \in B$. Hence $\pi_n(\vec{h''}) \leftarrow \pi_n(\vec{h^2})$ and $(\pi_n(\vec{h}'), \pi_n(\vec{h}'')) \in B_n$. So B_n is a bisimulation between $T(a_n)\pi_n(\vec{t^1})$ and $T(a_n)\pi_n(\vec{t^2})$.

Hence $\pi_n(\vec{t^1}) = \pi_n(\vec{t^2})$ for all $n \geq k$, and so $\vec{t^1} = \vec{t^2}$. From this it follows that if we take a to be the unique set pictured by T, then $T(a)$ and T are isomorphic. It is clear that a is an upper bound for all a_i, $i < \omega$, since $\pi_i : T \to T(a_i)$ is in fact an epimorphism. We call the above construction of a the *inverse limit* of $\langle a_i \mid i < \omega \rangle$, and denote it by $a = \text{inv lim}_{i<\omega} a_i$. Also we denote $T = \text{inv lim}_{i<\omega} T(a_i)$.

3.6 A Domain of non–wellfounded sets

Now let

$$\begin{aligned} C &= \{a \mid \mathrm{ht}(a) < \omega\}, \\ D &= \{\text{inv } \lim_{i<\omega} \vec{b} \mid \vec{b} \text{ is an increasing sequence in } C\}. \end{aligned}$$

We have shown that C is a conditional upper semi lattice. We show that D is isomorphic to the domain completion $\overline{C}$ of the class of compact elements C.

Lemma 55 $\overline{C} \cong D$.

Proof. Assume that $I \subseteq C$ is an ideal. If there is some $a \in I$ such that for all $a' \in I$, $a' \sqsubseteq a$, then I is the principal ideal generated by a. Assume that for all $a \in I$ there is $a' \in I$ such that $a \sqsubset a'$. So I is infinite and contains a strictly increasing sequence, $(a_i)_{i<\omega}$. Now for all $a \in I$ there is some n such that $a \sqsubseteq a_n$. Now if we assign inv $\lim_{i<\omega} a_i$ to I then we have the isomorphism from $\overline{C}$ to D. □

There is a canonical way to obtain a limiting sequence of sets, a_n, $n < \omega$, for any set a. But we need a lemma first.

Lemma 56 *Let T be a tree picture. There is an equivalence relation $\sim$ on T such that $T/\sim$ is isomorphic to $T(a)$, where a is the set pictured by T, and a surjective homomorphism $\eta : T \to T/\sim$ such that if $t \sim t'$, then $\eta(t) = \eta(t')$.*

Proof. Let $\sim$ be an equivalence relation on T defined as $t_1 \sim t_2$ iff there is $t \to t_1, t_2$ and the subtrees Tt_1 and Tt_2 are bisimilar. Let $\eta(t) = \{t' \in T \mid t \sim t'\}$. Define $\eta(t) \to \eta(t')$ if there is $t'' \sim t'$ such that $t \to t''$. If $t_1 \sim t_2$ and $\eta(t_1) \to \eta(t')$, then for some $t'' \sim t'$, $t_1 \to t''$. Because $t_1 \sim t_2$, there is $t_2' \sim t'' \sim t'$ and $t_2 \to t_2'$, so $\eta(t_2) \to \eta(t')$.

Let $T' = T/\sim$. Because T' is reduced by bisimulation, it is isomorphic to the canonical tree picture of the set a. We have also shown that $\eta : T \to T'$ is a surjective homomorphism. (It is also an epimorphism in the sense of definition 46.) □

Lemma 57 *Let a be a set. Then there is a sequence of sets $a_0 \sqsubseteq a_1 \sqsubseteq \cdots \sqsubseteq a$ such that* $\mathrm{approx}(a) = \{a_i \mid i < \omega\}$.

Proof. Let $T(a)$ be the canonical tree picture of a. Let $T_n = T(a) \restriction n$, and a_n be the set pictured by T_n. There is a partial isomorphism $h_n : T(a) \to T_n$, namely $id_{T(a)\restriction n}$.

Let $T_n' = T_n/\sim$ and let $g_n : T_n' \to T(a_n)$ be the isomorphism from the previous lemma. Let $f_n = g_n \circ \eta \circ h_n : T(a) \to T(a_n)$, where $\eta : T_n \to T_n'$ is the homomorphism from the proof of the previous lemma. We show that f_n is a partial epimorphism.

Assume $\mathrm{last}(f_n(t)) = \emptyset$ and there is $t' \in dom(f_n)$ such that $\ln(t') > \ln(t)$. Assume that $\mathrm{last}(t) \neq \emptyset$, i.e. there is some $t'' \leftarrow t$. Because $\ln(t'') \leq \ln(t') \leq n$, we have that $t'' \in dom(id_{T(a)\restriction n})$. But then $\eta(t'') \leftarrow \eta(t)$ in T'_n and $\mathrm{last}(f_n(t)) \neq \emptyset$. This proves the condition (v) of being a partial epimorphism. The other conditions are clear. □

Definition 58 *Let a be a set. The increasing sequence $a_1 \sqsubseteq a_2 \sqsubseteq \cdots$ of the proof of the previous lemma is called the canonical limiting sequence of the set a.*

Because D is a domain, we have that if $x \in D$, then $x = \bigsqcup \mathrm{approx}(x)$, i.e. x is the limit of its canonical limiting sequence.

3.7 Bisimulation in HF_1

Recall that bisimulation characterizes the identity for the non–wellfounded sets. Bisimulation can be also approximated by the length of how deep one sees. At limit stages it is required that two sets are equal in all the approximating bisimulations. In HF_1 it is enough to approximate only to length ω. As a corollary we can prove that $HF_1 \subseteq D$. This is made precise below.

Bisimulation can also be seen as a game played between two players, $\forall$ and $\exists$, and on two graphs or on two sets, a and b. The rules for this bisimulation game, $BG(a, b)$, are as follows. First the player $\forall$ chooses one of the sets a or b and an element x_1 of that chosen set. Then the player $\exists$ has to respond with an element, y_1 of the other set. Following that, $\forall$ chooses an element, x_2, from either x_1 or y_1 and $\exists$ responds with an element, y_2, from the other set. If $\forall$ moves an urelement, then $\exists$ has to respond with the same urelement from the other set. This way the game continues.

The player $\forall$ wins if $\exists$ is not able to respond with an element at some point of the game. Otherwise $\exists$ wins, i.e. the game continues arbitrarily long or $\forall$ is not able to move. A winning strategy for either of the players in the game $BG(a, b)$ is a function $\sigma : (TC(a) \cup TC(b))^{<\omega} \to TC(a) \cup TC(b)$ such that following that strategy the player wins the game. That is, given any legal sequence of moves, σ tells the next move in the game.

We may also restrict the length of the game $BG(a, b)$, i.e. the number of moves by $\forall$, obtaining games $BG_n(a, b)$. Similarly we can define the winning strategies σ_n by letting their domain be $(TC(a) \cup TC(b))^{\leq n}$. When $\exists$ wins $BG_n(a, b)$ we denote this also by $a \sim_n b$. It is shown in [2], cf. Theorem 12.6., that $\exists$ wins $BG(a, b)$ if and only if a and b are bisimilar.

Note that $\sim_n$ is an equivalence relation for every $n < \omega$. We begin with a characterization of the $\sqsubseteq$ ordering with the restricted length game G_n.

Lemma 59 *Let a and b be sets. Then $a \sqsubseteq b$ iff $a \sim_{\mathrm{ht}(a)} b$.*

Proof. Let $T(a)$ and $T(b)$ be the canonical tree pictures of a and b respectively. By corollary 50, $a \sqsubseteq b$ iff $T(a)$ and $T(b) \upharpoonright \mathrm{ht}(a)$ picture the same sets iff $a \sim_{\mathrm{ht}(a)} b$. $\Box$

Definition 60 *Let T be a tree and $t \in T$. We denote by $[t]^n$ the subtree Tt restricted to length n and reduced by bisimulation, i.e. $[t]^n = (Tt \upharpoonright n)/\sim$.*

We are going to prove that if $\exists$ wins the game $BG_n(a,b)$ for all $n < \omega$, where a is a hereditarily finite set, then $a = b$. For this we need a lemma providing us with a "uniform" set of winning strategies in the games $BG_n(a,b)$, $n < \omega$.

Lemma 61 *Let T and T' be trees such that $T \sim_n T'$ for all $n < \omega$. Then there are following kind of winning strategies σ_n, $n < \omega$ for the player $\exists$ in the games $BG_n(T,T')$, $n < \omega$.*

Assume $t \in T \cup T'$ and S is a maximal set of immediate successors of t such that $t' \sim_n t''$ for all $t', t'' \in S$ and $n < \omega$. For all $i, j < \omega$, if $\vec{s} \in dom(\sigma_i)$, $\vec{s'} \in dom(\sigma_j)$, $\sigma_i(\vec{s}) \in S$, and $\sigma_j(\vec{s'}) \in S$, then $\sigma_i(\vec{s}) = \sigma_j(\vec{s'})$.

Proof. Let $n < \omega$. Because $T \sim_n T'$, let σ'_n be a winning strategy for $\exists$ in $BG_n(T,T')$. We construct the winning strategy σ_n as follows. Let us well–order the nodes of T and T' and let t and S be as above. Now if $\sigma'_n(\vec{s}) \in S$, let $\sigma_n(\vec{s})$ be the least $t' \in S$. Otherwise let $\sigma_n(\vec{s}) = \sigma'_n(\vec{s})$. This is a winning strategy, since $t \sim_n t'$ for all $t, t' \in S$. $\Box$

Recall that $HF_1 = \{x \mid \forall y \in TC(x) \cup \{x\}(|y| < \omega)\}$ is the class of all hereditarily finite sets.

Lemma 62 *Assume $a \in HF_1$, b is a set and for all $n < \omega$, $\exists$ wins the game $BG_n(a,b)$, then $\exists$ wins the game $BG(a,b)$.*

Proof. Let σ_n, $n < \omega$, be winning strategies for $\exists$ in $BG_n(a,b)$ that also satisfy the conditions of the previous lemma. We define inductively on the sequence of moves the winning strategy σ for $\exists$ in $BG(a,b)$. Let $\vec{s}$ be a sequence of moves of length n. Assume first that $\mathrm{last}(\vec{s}) \in T(b)$, then there is an infinite set $X(\vec{s}) \subseteq X(\vec{s} \upharpoonright n-1) \subseteq \omega$ such that $\sigma_p(\vec{s}) = \sigma_{p'}(\vec{s})$ for all $p, p' \in X(\vec{s})$, because there is only a finite number of possible moves for $\exists$ in any node of $T(a)$. Then we define $\sigma(\vec{s}) = \sigma_p(\vec{s})$ for some $p \in X(\vec{s})$.

Assume then that $t = \mathrm{last}(\vec{s}) \in T(a)$. Let $t' \in T(b)$ be the node from which $\exists$ has to choose the corresponding node for t. If t' has finitely many successors, then we can do as above. Assume t' has infinitely many successors. There are two cases.

1°: The number of successors of the root node in the tree $[t']^n$ increases as n increases. Denote by r the predecessor of t. So $\exists$ wins $BG_n(T(a)r, T(b)t')$ for all $n < \omega$. But at some $n < \omega$, the root of $[t']^n$ has more successors than the root of $[r]^n$ since $T(a)$ is hereditarily finite. Hence $[r]^n \not\cong [t']^n$ and as these trees are reduced by bisimulation, $[r]^n \not\sim [t']^n$, i.e. $r \not\sim_n t'$. Thus $\exists$ does not win $BG_n(T(a)r, T(b)t')$, a contradiction.

2°: There is $k < \omega$ such that the root of $[t']^n$ has at most k immediate successors for all $n < \omega$. Let S_n be the set of $\sim_n$ equivalence classes of the immediate successors of the root of $[t']^n$. By the fact that if for some $n < \omega$ and $t'_1, t'_2 \leftarrow t'$, $t'_1 \not\sim_n t'_2$ implies $t'_1 \not\sim_m t'_2$ for any $m \geq n$, we have that S_n becomes finer as n increases. Since the root of $[t']^n$ has at most k immediate successors for all $n < \omega$, there is some $n < \omega$, such that $S_n = S_m$ for all $m \geq n$. Let us denote $S' = S_m$ for any such $m \geq n$ and let $S = \bigcup S' \cap \bigcup_{i<\omega} ran(\sigma_i)$. The previous lemma guarantees that $S = \{t'_1, \ldots, t'_k\}$ is a finite set of representatives of the equivalence classes in S'.

Thus we can find an infinite set $X(\vec{s}) \subseteq X(\vec{s} \restriction (n-1))$ such that $\sigma_p(\vec{s}) = \sigma_q(\vec{s})$ for all $p, q \in X(\vec{s})$. So we can define $\sigma(\vec{s}) = \sigma_p(\vec{s})$ for some $p \in X(\vec{s})$. $\Box$

Note that from the previous lemma it follows that then also $b \in HF_1$ and $a = b$.

Lemma 63 *Assume a is a set. If b is an upper bound for* approx(a)*, then $\exists$ wins $BG_n(a, b)$ for all $n < \omega$.*

Proof. Let $T(a)$ be the canonical tree picture for a, and a_i, $i < \omega$ be its canonical limiting sequence. Because $a_n \sqsubseteq b$ for all $n < \omega$, by corollary 50 (ii), it follows that $T(a_n)$ and $T(b) \restriction n$ picture the same set, i.e. they are bisimilar. So $\exists$ wins $BG_n(a, b)$ for all $n < \omega$. $\Box$

So we have now achieved that all the hereditarily finite sets can be approximated in the ordering $\sqsubseteq$.

Corollary 64 *For any hereditarily finite set a, $a = \bigsqcup$*approx(a)*. Thus $HF_1 \subseteq D$.*

Proof. Assume $a \in HF_1$ is a set. Let $T(a)$ be the canonical tree picture for a, and a_i, $i < \omega$ be its canonical limiting sequence. Let $T = \text{inv}\lim_{i<\omega} T(a_i)$. Because T is an upper bound for all $T(a_i)$, $\exists$ wins the game $BG(T(a), T)$, by lemma 62. So $T(a)$ and T are bisimilar and hence picture the same sets. Similarly for any other upper bound b for a, we have that $b = a$. Hence $a = \bigsqcup \text{approx}(a)$. $\Box$

The next example shows that there are sets that cannot be approximated in the ordering $\sqsubseteq$.

Example 65 *There is a set a such that $a \neq \text{inv}\lim_{i<\omega} a_i$, where a_i, $i < \omega$ is the canonical limiting sequence of a.*

Proof. We define sets x_i, $i < \omega$ as follows: Let $x_0 = \emptyset$, and $x_{i+1} = \{x_i\}$. Let $a = \{x_i \mid i < \omega\}$ and let a_i, $i < \omega$ be its canonical limiting sequence. So $a_i = \{x_n \mid n \leq i\}$. Let us consider $T = \text{inv}\lim_{i<\omega} T(a_i)$ and $T(a)$. Let

b be the unique set pictured by T. We show that b is non–wellfounded. Let $f_n : T(a) \to T(a_n)$, for $n < \omega$, be the epimorphisms.

Let t be the root of $T(a)$. Let $t_{x_i} \leftarrow t$ be the unique node in $T(a)$ such that $d(t_{x_i}) = x_i$, where d is the unique decoration of T. Then let $t^j_{x_i}$, where $j \le i$, be the unique node such that $t^j_{x_i} \leftarrow^* t_{x_i}$ and $\ln(t^j_{x_i}) = j+1$. The node can be chosen uniquely, since the successors of every t_{x_i} are linearly ordered.

Let $\vec{t_i} = \langle f_n(t^i_{x_n}) \mid n \ge i \rangle$. Now $f_n(t^i_{x_n}) \to f_n(t^{i+1}_{x_n})$ for all $n \ge i+1$, so $\vec{t_i} \in T$ and $\vec{t_i} \to \vec{t}_{i+1}$, for all $i < \omega$. Let c_i be the set assigned to $\vec{t_i}$ by the decoration of T. Then $c_{i+1} \in c_i$ for all $i < \omega$, and so c_i, $i < \omega$ is a non–wellfounded sequence in b. Because a is wellfounded, $a \ne b$. So approx(a) has two upper bounds, namely a and b, which are incomparable. □

The previous example shows also that $\exists$ having a winning strategy in $BG_n(a,b)$ for all $n < \omega$ does not imply that $\exists$ has a winning strategy in $BG(a,b)$. Next we consider when a set a is the same as $\bar{a}$.

3.8 A characterization of the sets in D

The domain D consists of those sets that can be approximated in the ordering $\sqsubseteq$. We show that there is another condition characterizing this.

Let T be a tree and $t \in T$. The notation $t \to T'$, where T' is a tree means that for some $t' \leftarrow t$, $Tt' \sim T'$, i.e. t has an immediate successor such that the tree beginning from that successor is bisimilar to T'.

Definition 66 *Let a be a set. We say that a is* inv lim*–closed if there is no $t \in T(a)$ and its immediate successors t_i, $i < \omega$ such that $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$ and $t \not\to \text{inv}\lim_{i<\omega}[t_i]^i$.*

Definition 67 *Let a be a set and a_i, $i < \omega$ its canonical limiting sequence. Define $\bar{a}$ to be the unique set pictured by the tree* $\text{inv}\lim_{i<\omega} T(a_i)$*, or equivalently, $\bar{a} = \text{inv}\lim_{i<\omega} a_i$.*

Theorem 68 *Let a be a set, a_i, $i < \omega$ its canonical limiting sequence, $f_i : T(a) \to T(a_i)$ be the partial epimorphisms, and let $T = \text{inv}\lim_{i<\omega} T(a_i)$. The following are equivalent:*

(i) the function $f : T(a) \to T$, $f(t) = (f_i(t))_{i \ge \ln(t)}$, is an epimorphism,

(ii) $a = \bar{a}$,

(iii) a is inv lim*–closed.*

Proof. Note that T is a picture of $\bar{a}$. We may assume that $\text{ht}(a) = \omega$ since otherwise the claim is clear.

(i) $\to$ (ii): Because $\mathrm{ht}(T(a)) = \mathrm{ht}(T)$, and there is a partial epimorphism between them, we have by corollary 50, that $a = \bar{a}$.

(ii) $\to$ (iii): Assume $a = \bar{a}$. Then there is an isomorphism $g : T \to T(a)$, because $T = \text{inv}\lim_{i<\omega} T(a_i)$ and $T(a)$ picture the same set a and T is already reduced by bisimulation in the sense of lemma 56. Assume $t \in T(a)$, $t_i \leftarrow t$, and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$ for $i < \omega$. Let $T' = \text{inv}\lim_{i<\omega}[t_i]^i$ and let $f'_{i+1,i} : [t_{i+1}]^{i+1} \to [t_i]^i$, $i < \omega$, be the epimorphisms. We show that $t \to T'$. Let $k = \ln(t) + 1$.

First we show that there is a natural way to embed the subtree $[t_i]^i$ into $T(a_{k+i})$. The idea is to find the corresponding node for t_i in $T(a_{k+i})$ and then map $[t_i]^i$ surjectively onto that node's successors. Let us call this kind of embedding a canonical embedding.

We build the embedding $h_i : [t_i]^i \to T(a_{k+i})$ inductively as follows: Let $\eta : (Tt_i \restriction i) \to [t_i]^i$ be the $\sim$–epimorphism from lemma 56. Let $h_i(t_i) = f_{k+i}(t_i) \in T(a_{k+i})$. Assume $h_i(t)$ is defined and $t \to t'$. Let $t'' \in (Tt_i \restriction i)$ be such that $\eta(t'') = t'$. Define $h_i(t') = f_{k+i}(t'')$. If there is another $s \in (Tt_i \restriction i)$ such that $\eta(s) = t'$, then $s \sim_{i-\ln(s)} t''$ and hence also $f_{k+i}(s) = f_{k+i}(t'')$, so h_i is well–defined. If $h_i(t) = h_i(t')$ then $f_{k+i}(s) = f_{k+i}(s')$ for the corresponding s and s'. But then also $t = \eta(s) = \eta(s') = t'$ and we have that h_i is an injection. We show that it is also a homomorphism. Assume $t \to t'$, and let s and s' be such that $\eta(s) = t$ and $\eta(s') = t'$. Then $\eta(s) \to \eta(s')$, and since η is a homomorphism, $s \to s'$. But then $h_i(t) = f_{k+i}(s) \to f_{k+i}(s') = h_i(t')$. Similarly to the other direction.

We have that $f'_{i+1,i}(t_{i+1}) = t_i$ because t_{i+1} is the root of $[t_{i+1}]^{i+1}$ and t_i is the root of $[t_i]^i$ and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$. We also have that $f_{k+i+1,k+i}(f_{k+i+1}(t_{i+1})) = f_{k+i}(t_{i+1})$, because $f_{k+i+1,k+i} \circ f_{k+i+1} : T(a) \to T(a_{k+i})$ is an epimorphism and hence equal to f_{k+i} since epimorphisms are unique. Furthermore $f_{k+i}(t_{i+1}) = f_{k+i}(t_i)$, because t_{i+1} and t_i have a common immediate predecessor and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$ means that $t_i \sim_i t_{i+1}$. So $f_{k+i+1,k+i}(h_{i+1}(t_{i+1})) = f_{k+i+1,k+i}(f_{k+i+1}(t_{i+1})) = f_{k+i}(t_{i+1}) = f_{k+i}(t_i) = h_i(t_i) = h_i(f'_{i+1,i}(t_{i+1}))$. And so on for all other $t \in [t_{i+1}]^{i+1}$. So the following diagram commutes

$$
\begin{array}{ccc}
T(a_{k+i}) & \xleftarrow{f_{k+i+1,k+i}} & T(a_{k+i+1}) \\
\uparrow{\scriptstyle h_i} & & \uparrow{\scriptstyle h_{i+1}} \\
[t_i]^i & \xleftarrow[f'_{i+1,i}]{} & [t_{i+1}]^{i+1}
\end{array}
$$

To show that $t \to T'$, we are going to construct an embedding $g' : T' \to T$ such that $g'(T')$ becomes a subtree of $g^{-1}(t)$, and furthermore g' is a surjection onto the successors of $g'(r)$ where r is the root of T'. So let $\vec{s} = (s_i)_{i \geq k'} \in T'$, where k' is the length of $\vec{s}$ in T'. We have that for all $i \geq k'$, $f'_{i+1,i}(s_{i+1}) = s_i$. Because the above diagram commutes, we have that $f_{k+i+1,k+i}(h_{i+1}(s_{i+1})) = h_i(f'_{i+1,i}(s_{i+1})) = h_i(s_i)$ for all $i \geq k'$. Furthermore, $h_{k'}(s_{k'}) \notin dom(f_{k'+k,k'+k-1})$ since otherwise $s_{k'} \in dom(f'_{k',k'-1})$. So let $g'(\vec{s}) = (h_i(s_i))_{i \geq k'} \in T$ but then $g(g'(\vec{s})) \in T(a)$.

Hence we have shown that $g(g'(T')) \subseteq T(a)$. Also, $f_i(t) \to f_i(t_i)$ for $i < \omega$, and so $t \to T'$.

(iii) $\to$ (i): Assume a is inv lim–closed. We show that $f : T(a) \to T$, $f(t) = (f_i(t))_{i \geq \ln(t)}$ is an epimorphism by showing that it is surjective. So let $\vec{t} \in T$. We can write $\vec{t} = (f_i(t_i))_{i \geq k}$, where $t_i \in T(a)$, and $f_{i+1,i}(f_{i+1}(t_{i+1})) = f_i(t_i)$ for all $i \geq k$. We show that there is some $s \in T(a)$ such that $f_i(s) = f_i(t_i)$ for all $i \geq k$.

We may assume that there is an infinite number of different t_i's, otherwise the claim follows immediately. There is some $t' \in T(a)$ and an infinite number of its immediate successors t'_i, $i \geq k$, such that for every $i \geq k$, $t'_i \to^* t_i$. For some $i, i{+}1$, it may happen that $t'_i = t'_{i+1}$, but in that case also $f_{i+1,i}(f_{i+1}(t'_{i+1})) = f_i(t'_i)$, because $f_{i+1,i}$ as a homomorphism preserves predecessors. We can find an infinite number of different t'_i's because there are only finitely many levels above t_i's.

So for all $i \geq k$, $f_{i+1,i}(f_{i+1}(t'_{i+1})) = f_i(t'_i)$, hence $[t'_i]^{l+i} \sqsubseteq [t'_{i+1}]^{l+i+1}$, where $l = \text{ht}(t_i) - \text{ht}(t'_i)$ for some (any) $i \geq k$. Because a is inv lim–closed, we have that $t' \to T'$ where $T' = \text{inv lim}_{i \geq k}[t'_i]^{l+i}$. Then $[t'_i]^{l+i} \sqsubseteq T'$ for all $i \geq k$. Let $h_i : [t'_i]^{l+i} \to T(a_{k'+i})$ be the canonical embedding, where $k' = \ln(t_i)$ for any $i \geq k$. Because for every $i \geq k$, $t'_i \to^* t_i$, there are $s_i \in [t'_i]^{l+i}$, $i \geq k$, such that $h_i(s_i) = f_{k'+i}(t_i)$. Let $f'_{i+1,i} : [t'_{i+1}]^{l+i+1} \to [t'_i]^{l+i}$, $i < \omega$ be the epimorphisms. As above, we have that the following diagram commutes:

$$\begin{array}{ccc} T(a_{k'+i}) & \xleftarrow{f_{k'+i+1,k'+i}} & T(a_{k'+i+1}) \\ {\scriptstyle h_i}\uparrow & & \uparrow{\scriptstyle h_{i+1}} \\ [t'_i]^{l+i} & \xleftarrow[f'_{i+1,i}]{} & [t'_{i+1}]^{l+i+1} \end{array}$$

Hence $h_i(f'_{i+1,i}(s_{i+1})) = f_{k'+i+1,k'+i}(h_{i+1}(s_{i+1})) = f_{k'+i+1,k'+i}(f_{k'+i+1}(t_{i+1})) = f_i(t_i) = h_i(s_i)$, and so $f'_{i+1,i}(s_{i+1}) = s_i$, because h_i is injective. Hence $s = (s_i)_{i \geq k} \in T'$, and $f'_i(s) = s_i$ for all $i \geq k$, where $f'_i : T' \to [t'_i]^{i+l}$, $i \geq k$ are the epimorphisms. There is a canonical embedding $h : T' \to T(a)$ such that the following diagram commutes:

$$\begin{array}{ccc} T' & \xrightarrow{h} & T(a) \\ {\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle f_{k'+i}} \\ [t'_i]^{i+l} & \xrightarrow[h_i]{} & T(a_{k'+i}) \end{array}$$

Hence we have that $f_{k'+i}(h(s)) = h_i(f'_i(s)) = h_i(s_i) = f_{k'+i}(t_i)$, for all $i \geq k$. So $f(h(s)) = (f_i(t_i))_{i \geq k}$, and we have shown that f is a surjection. It is straightforward to see that it is an epimorphism also. $\square$

We next show that in the case of pure sets, corollary 64 is the best we can have. A set is called pure if its transitive closure contains no urelements.

Lemma 69 *Assume that a is a pure, well-founded, and infinite set. Then $a \notin D$.*

Proof. Let a_i, $i < \omega$ be the canonical limiting sequence of a, and let $f_{i+1,i} : T(a_{i+1}) \to T(a_i)$ be the epimorphisms. Also let $f_i : T(a) \to T(a_i)$ be the epimorphisms witnessing $a_i \sqsubseteq a$, for $i < \omega$. We are going to show that a is not inv lim–closed.

For every $i < \omega$, a_i is a finite set since there are only a finite number of pure sets of height $i-1$. Let t be the root of $T(a)$. We are going to find $t_i \leftarrow t$, $i < \omega$ such that $\mathrm{ht}[t_i]^i = i$ and $[t_i]^i \sqsubseteq [t_{i+1}]^{i+1}$, i.e. $f_{i+1,i}(f_{i+1}(t_{i+1})) = f_i(t_i)$. We show the claim by induction on $i < \omega$, but we require also that for every $i < \omega$, the set

$$A_i = \{t' \in T(a) \mid f_{i+1,i}(f_{i+1}(t')) = f_i(t_i)\}$$

is infinite.

Let $t_0 \leftarrow t$ be arbitrary. The choice can be arbitrary since $[t_0]^0$ pictures the empty set. Assume that t_i which satisfies the above conditions has been found. Let A_i be the infinite set guaranteed by the induction condition. The set $B = \{f_{i+1}(t') \in T(a_{i+1}) \mid t' \in A\}$ is on the other hand finite, since a_{i+1} is finite. But the set $B' = \{t'' \in T(a) \mid f_{i+2,i+1}(f_{t+2}(t'')) \in B\}$ is infinite since a was infinite. Hence for some $t_{i+1} \leftarrow t$, there is an infinite number of $t'' \in B'$ such that $f_{i+2,i+1}(f_{i+2}(t'')) = f_{i+1}(t_{i+1})$. Gather those into a set A_{i+1}. From this it also follows that $\mathrm{ht}([t_{i+1}]^{i+1}) = i+1$. So this proves the induction step.

Now we have the strictly increasing infinite sequence t_i, $i < \omega$. We show that $T(a)$ is not inv lim–closed. Assume towards a contradiction that $T(a)$ is inv lim–closed. We show that then a is non–wellfounded.

We build a non–wellfounded sequence s_i, $i < \omega$ of nodes in $T(a)$. Let r be the root of $T(a)$. We have that $(t_i)_{i \geq 0} \in T'$. By theorem 68, there is some $t' \leftarrow r$ such that $f_i(t') = f_i(t_i)$ for all $i \geq 0$. Let $s_0 = t'$. Now if s_0 has finitely many immediate successors $\{t'_1, \dots, t'_l\}$, then for some $1 \leq l' \leq l$ there are infinitely many t'' such that $f_i(t'') = f_i(t'_{l'})$. Then let $s_1 = t'_{l'}$.

On the other hand, if s_0 has infinitely many immediate successors, then we can do as above, i.e. find infinitely many immediate successors t'_i, $i < \omega$, of s_0 that form an increasing sequence. Because $T(a)$ is assumed to be inv lim–closed, there is some $t' \leftarrow s_0$ such that $f_i(t'_i) = f_i(t')$ for all $i \geq \ln(s_0)$. Then let $s_1 = t'$. This way we can continue infinitely long finding a sequence $r \to s_0 \to s_1 \to \cdots$. This shows that a is non–wellfounded, which is a contradiction. □

Lemma 70 *Let a be a set.*

(i) If $a \subseteq D$, then $a \subseteq \bar{a}$,

(ii) $\bar{\bar{a}} = \bar{a}$,

(iii) $a \in D$ iff $a = \bar{a}$.

Proof. (i) Let r be the root of $T(a)$ and let $\vec{r}$ be the root of $T(\bar{a})$. Let $x \in a$ and let $t \leftarrow r$ be the unique node such that $T(a)t$ pictures x. Now for every $t' \in T(a)t$, let $\vec{t'} = (f_i(t'))_{i \geq k}$. Then $\vec{t'} \in T(\bar{a})$. We also have that $T(a)t$ and $T(\bar{a})\vec{t}$ are isomorphic, since $a \subseteq D$, and $\vec{t} \leftarrow \vec{r}$. From this it follows that $x \in \bar{a}$.

(ii) Let a_i, $i < \omega$ be the approximation sequence of $\bar{a}$. But then a_i, $i < \omega$ is also the approximation sequence for a. Hence both $\bar{\bar{a}}$ and $\bar{a}$ are decorations of the same tree inv $\lim_{i<\omega} T(a_i)$. □

(iii) If $a \in D$, then a is of the form inv $\lim_{i<\omega} \vec{b}$, where $\vec{b}$ is an increasing sequence of wellfounded sets. But we have that $\vec{b}$ is cofinal in $\vec{a}$, where $\vec{a}$ is the canonical limiting sequence of a. Hence $a = \bar{a}$.

If $a = \bar{a} =$ inv $\lim \vec{a}$, then a is a limit of wellfounded sets. □

3.9 The axioms of *ZFA* in D

Although D does not satisfy some important axioms of *ZFA*, it satisfies some of them. Since D has some resemblance to *HF*, which satisfies *ZFC*− infinity, it is somewhat interesting to study this question.

When considering the axioms of *ZFA*, we are going to use the bisimulation games as well as the definition of inv lim–closedness.

To prove that D is extensional, we show that D is transitive. So assume $x \in D$, and $y \in x$. So x is inv lim–closed. If y was not inv lim–closed, then x would not be either, since $T(y)$ is a subtree of $T(x)$. Thus the axiom of strong extensionality holds in D. The axiom of urelements, $\forall x \forall y (\mathcal{U}(x) \rightarrow y \notin x)$ holds in D.

Let us consider pairing. Assume $x, y \in D$. We immediately see that $\{x, y\}$ is inv lim–closed, since there are no new strictly increasing infinite sequences in $\{x, y\}$ which were not already either in x or in y. Hence $\{x, y\}$ is inv lim–closed, so $\{x, y\} \in D$. The axiom of choice also holds in D.

We show that the union axiom fails in D. For $i < \omega$, let x_i be as in the example 65 and let u_i be an urelement such that if $i \neq j$, then $u_i \neq u_j$. Let $y_i = \{u_i, x_0, \ldots, x_i\}$ and let $a = \{y_i \mid i < \omega\}$. We have that $a \in D$ since the sets y_i, $i < \omega$ do not form an increasing sequence. On the other hand $\bigcup a \notin D$, since $x_i \in \bigcup a$, and $x_i \sqsubseteq x_{i+1}$ for every $i < \omega$, but inv $\lim_{i<\omega} x_i \notin \bigcup a$. Hence $\bigcup a$ is not inv lim–closed and therefore cannot belong to D.

Next we show the infinity axiom. Let us consider the unique set x such that $x = \omega \cup \{x\}$. We are going to show that $x = \bar{\omega}$ from which it follows that $\bar{x} = \bar{\bar{\omega}} = \bar{\omega} = x$, and thus $x \in D$. Moreover, x is an inductive set and thus the infinity axiom will hold.

For every $n \in \omega$, $\bar{n} = n$, since $\mathrm{ht}(n) < \omega$. We describe the winning strategy for $\exists$ in $BG(T(\bar{\omega}), T(x))$. For $y \in x$, let t_y be the node $\langle xy \rangle$ in $T(x)$. Let a_i, $i < \omega$ be the canonical limiting sequence for ω, and let $f_i : T(\omega) \rightarrow T(a_i)$ be the epimorphisms. We view inv $\lim_{i<\omega} T(a_i)$ and $T(\bar{\omega})$ as the same trees.

Let r be the root of $T(x)$. First if $\forall$ chooses some $t_n \leftarrow r$, then let $\exists$ respond with $\vec{t} = (f_i(t_n))_{i\geq 0} \in \text{inv lim}_{i<\omega} T(a_i)$. $\exists$ wins in this case since $T(n)$ and $T(\bar{\omega})\vec{t}$ are pictures of the set n. If $\forall$ chooses $t_x \leftarrow r$, then let $\exists$ choose $(f_i(t_i))_{i\geq 0} \in T(\bar{\omega})$. After this $\forall$ and $\exists$ are in the same position as in the beginning.

Assume $\forall$ chooses the first move $\vec{t}$ from $T(\bar{\omega})$. There are two cases: First if there is some $j < \omega$ and $n < \omega$ such that $f_i(t_n)$ appears in the sequence $\vec{t}$ for all $i \geq j$, then let $\exists$ choose $t_n \leftarrow r$. In this case $\vec{t} = (f_i(t_n))_{i<\omega}$ and hence $\exists$ wins since $T(\bar{\omega})\vec{t}$ and $T(x)t_n$ both picture the set n. Second, if there are no such j and n, then $\vec{t} = (f_n(t_n))_{n\geq 0}$. This is so because $f_{n+1,n}(f_{n+1}(t_{n+1})) = f_n(t_n)$ and $f_{n+1}(t_{n+1}) \in dom(f_{n+1,n})$ for all $n < \omega$, where $f_{n+1,n} : T(a_{n+1}) \to T(a_n)$ are the epimorphisms. In this case, let $\exists$ choose $t_x \leftarrow r$. Again after this move $\forall$ and $\exists$ are in a similar position as in the beginning. Hence $\exists$ wins the game $BG(T(\bar{\omega}), T(x))$. The set x is inductive, i.e. if $y \in x$, then $y \cup \{y\} \in x$. Since $x \in D$, D satisfies the axiom of infinity. Note that from the above, it also follows that $\omega \notin D$, since $\omega \neq x = \bar{\omega}$.

The separation axiom fails in D, because we can define the natural numbers from $\bar{\omega}$ by $\omega = \{n \in \bar{\omega} \mid n \neq \bar{\omega}\}$, and $\omega \notin D$ as we saw above. Next we show the collection axiom. Assume $x \in D$ and for every $y \in x$, there is $z \in D$ such that $\phi(x, y, z)$. Let $a = \{z \in D \mid \exists y \in x\phi(x, y, z)\}$. So $a \subseteq D$. But then $a \subseteq \bar{a} \in D$, by lemma 70 (i).

Let us consider the power set axiom. We need to show that $y = \mathcal{P}(x) \cap D \in D$ for every $x \in D$. We show that $y = \bar{y}$ from which the claim follows. For that, we are going to show that y is inv lim–closed. Assume $y_i \in y$, and $[y_i]^i$ is an increasing sequence of sets. Let $z_i \in y_i$, $i < \omega$ be an increasing sequence. But because $z_i \in x$, then $\text{inv lim}_{i<\omega}[z_i]^i \in x$, since x is inv lim–closed. Hence $\text{inv lim}_{i<\omega}[y_i]^i \subseteq x$ and y is inv lim–closed.

Considering AFA, we can reformulate it to deal only with trees, such that the tree and the inverse limit of its approximations are bisimilar. Restricted to that class of graphs, $AFA_{\text{inv lim}}$ holds in D.

So we have that $D \models ZFC^{-2} + SEA + AFA_{\text{inv lim}} -$ Separation $-$ Union.

3.10 Comparison to Boffa's work

Next we discuss briefly the earlier construction of Boffa [3] of the non–wellfounded sets as limits of their wellfounded approximations. The goal in [3] is not to show that this construction produces a domain structure and so the ordering $\sqsubseteq$ is not explicitly defined. Also the urelements were not assumed.

Definition 71 *Let i be a natural number.*

(i) $HF[i] = \{x \mid \text{ht}(x) < i\}$,

(ii) $HF = \bigcup_{i<\omega} HF[i]$,

(iii) $x[i]$, *the* i*th approximation of* x, *is the set which decorates the tree obtained by restricting the canonical tree picture of* x *to height* i.

We have that $x[0] = \emptyset$, $x[i+1] = \{y[i] \mid y \in x\}$. So the canonical limiting sequence of a set x is the same as $\langle x[i] \mid i < \omega \rangle$. Now we obtain a sequence of finite sets

$$HF[1] \stackrel{f_0}{\leftarrow} H[2] \stackrel{f_1}{\leftarrow} \cdots$$

where f_i is the function such that $f_i(x) = x[i]$. The inverse limit $\widehat{HF}$ consists of the limits $\langle x[i] \mid i < \omega \rangle$ and the $\in$–relation is defined as before.

Proposition 72 $V_{afa}[\emptyset] \cap D = \widehat{HF}$.

Proof. It is immediate that $\widehat{HF} \subseteq V_{afa}[\emptyset] \cap D$. Let $x \in V_{afa}[\emptyset] \cap D$. Since $TC(x)$ does not contain urelements, every $x[i]$ is finite, and hence in $HF[i]$. Thus $x = \text{inv} \lim_{i<\omega} x[i] \in \widehat{HF}$. □

Boffa mentions, that topologically viewed $\widehat{HF}$ is a compact and totally disconnected space. HF is open and dense in $\widehat{HF}$. All finitely branching graphs have decorations in $\widehat{HF}$. When there are no urelements, the construction above coincides with that of Boffa. Recall that $V_{afa}[A]$, where $A \subseteq \mathcal{U}$, is the class of all sets whose transitive closure may contain only the urelements listed in A.

Note that now $\sqsubseteq$ can be defined as $x \sqsubseteq y$ iff $y[\text{ht}(x)] = x$. When there are no urelements, the Domain D as a topological space actually looks very much like the Cantor space 2^ω. As it is known, $|D| = 2^{\aleph_0}$. We can readily define an ultra metric on D. Let $x, y \in D$. Then let $d(x,y) = 0$, if $x = y$, and $d(x,y) = 2^{-n}$, if $x \neq y$, where n is the least number such that $x[n] \neq y[n]$.

It is easy to see that d is an ultra metric on D.

3.11 Open problems

A problem left open in this study is to generalize the ordering $\sqsubseteq$ to the class of all non–wellfounded sets so that the result is a domain. One possibility is to try to take longer approximation sequences $(a_\alpha)_{\alpha<\gamma}$ for some ordinal $\gamma \geq \omega$.

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