Natural Deduction: Some recent Developments

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Structural proof analysis in natural deduction, in contrast to sequent calculus, soon leads to very complicated considerations. This difficulty is mainly due to the complex notion of normal derivation. In recent work, what are known as "general elimination rules" have been introduced. These rules permit a simple definition of normality and a straightforward proof of normalization. They also lead to a full understanding of the relation between natural deduction and sequent calculus. The improved control over the structure of normal derivations enables, for example, proofs of underivability in intuitionistic logic that so far have been possible only throught sequent calculus.

I Intuitionistic natural deduction

Gentzen's rules of natural deduction for intuitionistic logic have proved to be remarkably stable. There has been variation in the way the discharge of assumptions is handled. Then in 1984, Peter Schroeder-Heister changed the rule of conjunction elimination so that it admitted an arbitrary consequence similarly to the disjunction elimination rule. We shall do the same for the rest of the elimination rules, then prove normalization and show some applications of the system of "natural deduction with general elimination rules."

1. Introduction rules as determined by the BHK-conditions

As explained in Gentzen, the introduction rules formalize natural conditions on direct proofs of propositions of the different logical forms. These are often referred to as the BHK-conditions (for Brouwer, Heyting, and Kolmogorov). The rules are:

$$\frac{A \quad B}{A\&B}\,\&I \qquad \frac{A}{A \vee B}\,\vee I_1 \qquad \frac{B}{A \vee B}\,\vee I_2 \qquad \frac{\begin{matrix}\overset{1}{[A^m]}\\ \vdots \\ B\end{matrix}}{A \supset B}\,\supset\! I,1$$

In rule $\supset I$, $m \geq 0$ copies of formula A are discharged. We may leave undiscarged copies as well. The number next to the rule is a **discharge label** and those on top of formulas **assumption labels**.

The introduction rules for the quantifiers are

$$\frac{A(y/x)}{\forall x A}\,\forall I \qquad \frac{A(t/x)}{\exists x A}\,\exists I$$

Rule $\forall I$ has the standard variable restriction: y not free in any assumptions $A(y/x)$ depends on.

2. Inversion principles. Determination of elimination rules

Gentzen noticed that the elimination rules of natural deduction (E-rules) somehow repeat what was already contained in derivations with corresponding introduction rules (I-rules), and speculated that it should be possible to actually determine E-rules from I-rules. The idea is captured by the principle that "whatever follows from the direct conditions for introducing a formula, must follow from that formula." The principle determines the following **general elimination rules**, with a slight proviso on implication:

$$\frac{A\&B \quad \begin{matrix}\overset{1}{[A^m]},\overset{1}{[B^n]}\\ \vdots \\ C\end{matrix}}{C}\,\&E,1 \qquad \frac{A \vee B \quad \begin{matrix}\overset{1}{[A^m]}\\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overset{1}{[B^n]}\\ \vdots \\ C\end{matrix}}{C}\,\vee E,1 \qquad \frac{A \supset B \quad A \quad \begin{matrix}\overset{1}{[B^n]}\\ \vdots \\ C\end{matrix}}{C}\,\supset\! E,1$$

Here any numbers $m, n \geq 0$ of assumptions A and B can be discharged. If $m = 0$ or $n = 0$, there is a **vacuous** discharge, if $m > 1$ or $n > 1$, there is a **multiple** discharge. Otherwise a discharge is **simple**. Each instance of a rule must have a fresh discharge label.

The standard elimination rules of natural deduction follow by setting, in turn, $C \equiv A$ or $C \equiv B$ in $\&E$, and $C \equiv B$ in $\supset E$.

A direct proof of $A \supset B$ consists in a derivation of B from the assumption A. Thus, our inversion principle dictates that C follows from $A \supset B$ if C follows from the existence of such a derivation. First-order logic cannot express this, so rule $\supset E$ only shows how arbitrary consequences of B reduce to arbitrary consequences of A under the major premiss $A \supset B$. Schroeder-Heister, instead, used a higher-order rule, and so does type theory.

The propositional part of intuitionistic natural deduction is completed by adding an elimination rule for $\bot$ and by defining negation and equivalence:

$$\frac{\bot}{C}\,{\scriptstyle \bot E} \qquad \sim A \equiv A \supset \bot \qquad A \supset\subset B \equiv (A \supset B)\&(B \supset A).$$

The elimination rules for the quantifiers are

$$\frac{\forall x A \qquad \begin{matrix} \overset{1}{[A(t/x)^m]} \\ \vdots \\ C \end{matrix}}{C}\,{\scriptstyle \forall E,1} \qquad \frac{\exists x A \qquad \begin{matrix} \overset{1}{[A(y/x)^m]} \\ \vdots \\ C \end{matrix}}{C}\,{\scriptstyle \exists E,1}$$

The standard variable restriction holds for rule $\exists E$.

3. Discharge principle. Definition of derivations

"Compulsory discharge" dictates that one must discharge if one can. But look at

$$\cfrac{\cfrac{\overset{1}{[A]}}{B \supset A}\,{\scriptstyle \supset I}}{A \supset (B \supset A)}\,{\scriptstyle \supset I,1}$$

Assumption A is discharged at the second step, the first being a vacuous discharge. If it happened that B is identical to A, compulsory discharge would require a discharge of A at the first step, so something that looked like a syntactically correct derivation under the "compulsory" idea turned out not to be so. We adopt instead the following:

Discharge principle. *Each rule instance must have a fresh discharge label.*

We can now give a formal definition of the notion of a **derivation of formula A from the open assumptions Γ**. The open assumptions are counted with multiplicity, so they are multisets of formulas. The base case of a derivation is the derivation of a formula A from the open assumption A:

$$A$$

Now the rest is defined inductively according to the last rule applied, straightforward for rules that do not change the assumptions, and exemplified by $\&E$ for the rest:

Given derivations of $A\&B$ from Γ and C from A^m, B^n, Δ,

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\&B\end{matrix} \qquad \begin{matrix}[\overset{1}{A^m},\overset{1}{B^n}],\Delta\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle \&E,1}$$

is a derivation of C from Γ, Δ.

The full definition is given in *Structural Proof Theory, SPT* for short, pp. 167–170. We observe that the **composition of derivations** is justified by the definition: Given derivations of A from Γ and of C from A, Δ with no clash on labels, they can be put together into a derivation of C from Γ, Δ. The discharge of assumptions needs to be treated explicitly for composition to produce a correct derivation.

4. Normal derivations

Definition. *A derivation is* **normal** *if all major premisses of elimination rules are assumptions.*

In particular, the major premiss of rule $\bot E$ is an assumption in a normal derivation, analogously to the situation in sequent calculus in the derivations of which the corresponding rule is always a topmost rule.

5. Translation from sequent calculus to natural deduction

We define a translation from derivations in the sequent calculus **G0i** to derivations in natural deduction. The former is like Gentzen's original intuitionistic single-succedent calculus **LJ**, except that it has independent contexts in all two-premiss rules. Weakening and contraction remain explicit rules and only cut is eliminable.

Definition. *A formula is* **used** *in a sequent calculus derivation if it is active in a left rule.*

For the translation, it is essential to assume that the sequent calculus derivation has **no unused formulas principal in weakening or contraction**. The full translation can be found in *SPT*, ch. 8. Here are some example translations:

If the last rule is $L\&$, we have the translation

$$\dfrac{\begin{matrix}\vdots\\ A,B,\Gamma\to C\end{matrix}}{A\&B,\Gamma\to C}\,{\scriptstyle L\&} \quad\rightsquigarrow\quad \dfrac{A\&B \qquad \overset{1}{[A]},\overset{1}{[B]},\Gamma \overset{\vdots}{\to} C}{C}\,{\scriptstyle \&E,1}$$

Note how labels and brackets have been added in a hybrid expression that is translated next.

If the last rule is $L\supset$, we have the translation

$$\dfrac{\begin{matrix}\vdots & \quad & \vdots\\ \Gamma\to A & \quad & B,\Delta\to C\end{matrix}}{A\supset B,\Gamma,\Delta\to C}\,{\scriptstyle L\supset}\qquad\rightsquigarrow\qquad\dfrac{A\supset B\quad\begin{matrix}\vdots\\ \Gamma\to A\end{matrix}\quad\begin{matrix} & \vdots\\ \overset{1}{[B]},\Delta & \to C\end{matrix}}{C}\,{\scriptstyle \supset E,1}$$

The **principal formula** of the left rule, the one with the connective, becomes a **major premiss** of the corresponding E-rule. Note that the translation produces major premisses that are not derived but open assumptions, as required by our notion of normal derivations.

If a step of weakening is met we have, by the condition of no unused weakening formulas, some assumption label n and the translation

$$\dfrac{\begin{matrix}\vdots\\ \Gamma\to C\end{matrix}}{\overset{n}{[A]},\Gamma\to C}\,{\scriptstyle Wk}\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ \Gamma\to C\end{matrix}$$

If a step of contraction is met, the translation is

$$\dfrac{\begin{matrix}\vdots\\ A,A,\Gamma\to C\end{matrix}}{\overset{n}{[A]},\Gamma\to C}\,{\scriptstyle Ctr}\qquad\rightsquigarrow\qquad\begin{matrix}\vdots\\ \overset{n}{[A]},\overset{n}{[A]},\Gamma\to C\end{matrix}$$

In the end, initial sequents of the forms $A\to A$ or $\bot\to C$ are met, translated into an assumption A and an instance of $\bot E$, respectively. Formulas in the antecedents can have labels and brackets that are maintained, so, for example, $\overset{n}{[A]}\to A$ turns into $\overset{n}{[A]}$.

Observation. *The order of logical rules is maintained in the translation, with left rules giving E-rules and right rules I-rules. Cut-free derivations turn into normal derivations.*

6. Interpretation of weakening and contraction in natural deduction

Theorem. *Assume a derivation of $\Gamma\to C$ in* **G0i** *with no unused weakening or contraction formulas. Then:*

(i) *If A is principal in weakening, it is vacuously discharged in the translated derivation in natural deduction.*

(ii) *If A is principal in contraction, it is multiply discharged in the translated derivation in natural deduction.*

The result goes also in the other direction: vacuous discharges give weakenings, and multiple discharges contractions. The order of logical rules in normal and cut-free derivations, respectively, is the same. The latter has steps of weakening and contraction the position of which can vary. It is possible to change the calculus **G0i** a bit, into a "sequent calculus in natural deduction style," exemplified by the rule

$$\dfrac{A^m,B^n,\Gamma\to C}{A\&B,\Gamma\to C}\,{\scriptstyle L\&}$$

Any numbers $m, n \geq 0$ of active formulas are permitted, with number 0 marking implicit weakening and number > 1 contraction. Thus, this calculus has no structural rules (see *SPT*, ch. 5.2).

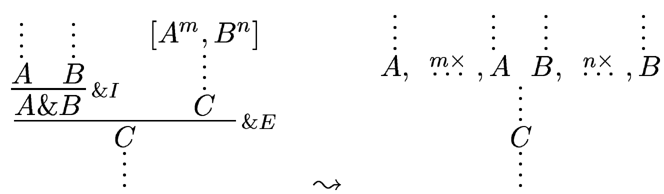
II NORMALIZATION

The standard normalization procedure of Gentzen and Prawitz consists of the removal of consecutive introduction-elimination pairs. Such pairs are known as "detours," or "Umwege" in Gentzen's terminology. Later Prawitz considered **permutation convertibilities**, instances of $\vee E$ or $\exists E$ that conclude a major premiss of an E-rule, and **simplification convertibilities**.

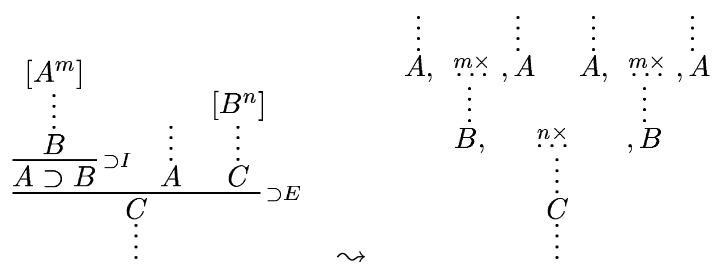
7. Normalization

Definition 7.1. *An E-rule with a major premiss derived by an I-rule is a* **detour convertibility**.

A detour convertibility on $A\&B$ and the result of the conversion are, with obvious labels left unwritten,



A detour convertibility on disjunction is quite similar. A detour convertibility on $A \supset B$ and the result of the conversion are



There is no I-rule for $\bot$ so no detour convertibility either.

Definition 7.2. *An E-rule with a major premiss derived by an E-rule is a* **permutation convertibility**.

The novelty of general elimination rules is that permutation conversions apply to all cases in which a major premiss of an E-rule has been derived. With six E-rules, this gives 36 convertibilities of which we show a couple:

A permutation convertibility on major premiss $C\&D$ derived by $\&E$ on $A\&B$ and its conversion are

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\&B\end{matrix}\quad \begin{matrix}[A^m,B^n]\\ \vdots\\ C\&D\end{matrix}}{C\&D}{\scriptstyle \&E}\qquad \begin{matrix}[C^k,D^l]\\ \vdots\\ E\end{matrix}}{\begin{matrix}E\\ \vdots\end{matrix}}{\scriptstyle \&E}\quad\rightsquigarrow\quad \dfrac{\begin{matrix}\vdots\\ A\&B\end{matrix}\qquad \dfrac{\begin{matrix}[A^m,B^n]\\ \vdots\\ C\&D\end{matrix}\qquad \begin{matrix}[C^k,D^l]\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle \&E}}{\begin{matrix}E\\ \vdots\end{matrix}}{\scriptstyle \&E}$$

A permutation convertibility on major premiss $C \supset D$ derived by $\vee E$ on $A \vee B$ obtains whenever a derivation has the part

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \begin{matrix}[A^m]\\ \vdots\\ C\supset D\end{matrix}\quad \begin{matrix}[B^n]\\ \vdots\\ C\supset D\end{matrix}}{C\supset D}{\scriptstyle \vee E}\qquad \begin{matrix}\vdots\\ C\end{matrix}\quad \begin{matrix}[D^l]\\ \vdots\\ E\end{matrix}}{\begin{matrix}E\\ \vdots\end{matrix}}{\scriptstyle \supset E}$$

After the permutation conversion the part is

$$\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \dfrac{\begin{matrix}[A^m]\\ \vdots\\ C\supset D\end{matrix}\quad \begin{matrix}\vdots\\ C\end{matrix}\quad \begin{matrix}[D^l]\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle \supset E}\quad \dfrac{\begin{matrix}[B^n]\\ \vdots\\ C\supset D\end{matrix}\quad \begin{matrix}\vdots\\ C\end{matrix}\quad \begin{matrix}[D^l]\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle \supset E}}{\begin{matrix}E\\ \vdots\end{matrix}}{\scriptstyle \vee E}$$

Finally, we have permutation convertibilities in which the conversion formula is $\bot$ derived by $\bot E$. Since $\bot E$ has only a major premiss, a permutation conversion just removes one of these instances:

$$\dfrac{\dfrac{\bot}{\bot}{\scriptstyle \bot E}}{C}{\scriptstyle \bot E}\qquad\rightsquigarrow\qquad \dfrac{\bot}{C}{\scriptstyle \bot E}$$

Definition 7.3. *A* **simplification convertibility** *in a derivation is an instance of an E-rule with no discharged assumptions, or an instance of $\vee E$ with no discharges of at least one disjunct.*

As with permutation conversions, also simplification conversions apply to all E-rules when general elimination rules are used. A simplification convertibility can prevent the normalization of a derivation, as is shown by the following:

$$\dfrac{\dfrac{\dfrac{\overset{1}{[A]}}{A\supset A}{\scriptstyle \supset I,1}\quad \dfrac{\overset{2}{[B]}}{B\supset B}{\scriptstyle \supset I,2}}{(A\supset A)\&(B\supset B)}{\scriptstyle \&I}\qquad \dfrac{\overset{3}{[C]}}{C\supset C}{\scriptstyle \supset I,3}}{C\supset C}{\scriptstyle \&E}$$

There is a detour convertibility but the pieces of derivation do not fit together in the right way to remove it. Instead, a simplification conversion will remove the detour convertibility:

$$\frac{\overset{3}{[C]}}{C \supset C}{\scriptstyle \supset I,3}$$

Threads in place of branches. Due to the form of the general E-rules we consider subformula structure along **threads** in a derivation (a term suggested to us by Dag Prawitz), instead of branches of the derivation tree as would be the case for the special elimination rules in the $\vee, \exists$-free fragment. These threads are constructed starting with the endformula of a derivation:

1. For I-rules with conclusion $A \circ B$, the threads are

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{A\&B} \qquad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A\&B} \qquad \frac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B} \qquad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B} \qquad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A \supset B}$$

2. For rules $\&E$, $\vee E$, and $\supset E$ with conclusion C, the thread continues up from the minor premiss C, with two threads produced for $\vee E$:

$$\frac{\begin{array}{c}\vdots\\ C\end{array}}{C}$$

3. If the last formula is an open assumption A or an assumption A discharged by $\supset I$, the thread ends with topformula A.

4. If the last formula is an assumption A or B discharged by $\&E$ or $\vee E$, the construction of the thread continues from the major premiss $A\&B$ or $A \vee B$:

$$\begin{array}{c}\vdots\\ \dfrac{A\&B}{A}\\ \vdots\\ \dfrac{C}{C}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{A\&B}{B}\\ \vdots\\ \dfrac{C}{C}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{A \vee B}{A}\\ \vdots\\ \dfrac{C}{C}\\ \vdots\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{A \vee B}{B}\\ \vdots\\ \dfrac{C}{C}\\ \vdots\end{array}$$

5. With $\bot E$, there is no minor premiss so the construction continues directly from the major premiss $\bot$:

$$\frac{\begin{array}{c}\vdots\\ \bot\end{array}}{C}$$

6. If the last formula is an assumption B discharged by $\supset E$, the construction continues with the major premiss $A \supset B$. A **new thread** begins with the minor premiss A as endformula:

$$\begin{array}{ccc} \vdots & & \\ \underline{A \supset B} & & \\ B & & \\ \vdots & & \\ \underline{C} & & \vdots \\ C & & A \\ \vdots & & \end{array}$$

Note that the construction of threads will not reach the parts of derivation that can be deleted in a simplification convertibility.

We can depict threads as follows, with a semicolon separating the ith major premiss of an E-rule A_{h_i} from its components C_{h_i} discharged by the elimination:

$$(A_1, \ldots, A_{h_1}; C_{h_1}, \ldots, A_{h_2}; C_{h_2}, \ldots, A_{h_i}; C_{h_i}, \ldots, A)$$

Height along threads: The **height** of a major premiss A_{h_i} in a thread is measured as follows. Let h_1 be the number of steps from the top formula to a first major premiss A_{h_1} and h_i the number of steps from the temporary assumption of the preceding major premiss $A_{h_{i-1}}$ to A_{h_i}. The height of A_{h_i} in the thread is the sum $h_1 + \ldots + h_i$.

From the construction of threads it is immediate that each formula in a derivation is in at least one thread. A thread is **normal** if it is a thread of a normal derivation. The height of each major premiss in normal threads is equal to zero. It is easily seen that the converse also holds. The formulas in a thread divide into an "E-part" of nested sequences of major premisses, each a subformula of the preceding formula, and an "I-part" in which formulas start building up in the other direction through introduction rules. Each formula in a normal thread is a subformula of the endformula or of an open assumption. (For a proof, not difficult, see *SPT*, p. 197).

Lemma 7.4. *A permutation conversion on major premiss A diminishes its height by one and leaves all other heights unaffected.*

Given a derivation, consider its conversion formulas in each thread, ordered by length into multisets.

Lemma 7.5. *Detour conversions on & and $\vee$ reduce the multiset ordering of conversion formulas in threads affected by the conversion.*

Note that permutation conversions do not create any new conversion formulas and therefore do not affect the multiset ordering. They can change a permutation convertibility into a detour convertibility. If this happens with implication, a new thread with the minor premiss as endformula is constructed.

The construction of threads is essential in lemma 7.5. It is seen from the detour conversion scheme for & that parts of the derivation get multiplied. These parts can contain conversion formulas of any length, so that the multiset of conversion formulas for the whole derivation is not necessarily reduced. For threads, instead, it is reduced.

For the case of detour conversion on implication, we cut the converted derivation into two parts:

1. The derivation of the minor premiss A, copied $m \times n$ times in the converted derivation.
2. The derivation to C from $m \times n$ copies of A, **changed into open assumptions**.

No conversion can create new major premisses of E-rules. Therefore only a bounded number of detour convertibilities on implication can be met, and the cutting into parts must terminate. Each part either is or becomes normal through conversions other than detour on implication. By the lemmas, this process stops independently of the order of conversions. We then have a number of normal derivations that can be composed into a derivation with the original conclusion. If new convertibilities are found, they are on components of the original detour convertibilities on impliction, thus, on strictly shorter formulas. Thus, the process of cutting, normalizing, and composing terminates and we have:

Theorem 7.6. *Natural deduction with general elimination rules is normalizing.*

Research problem. Show that the cutting into parts and normalization of the parts commutes.

8. Strong normalization

It is known that the commutation of the research problem holds, for Joachimski and Matthes (2003) proved directly strong normalization for natural deduction with general elimination rules, or the termination of conversions in any order whatsoever. Their proof uses a system of term assignment. The above proof of normalization is "almost strong," in that the only restriction on conversions concerns the cutting into parts of derivations at detour convertibilities on implication. It would be interesting to find a simple proof of commutation based on the conversion schemes and their combinatorial behaviour.

9. Applications of normalization

We give applications of normalization for natural deduction with general elimination rules to proofs of underivability, Harrop's theorem, and Mints' theorem on proper assumptions.

(a) Proofs of underivability. If A is a theorem in intuitionistic logic, the last rule in a normal derivation must be an I-rule. The reason is that an E-rule would leave its major premiss as an open assumption. We can show underivability of the standard classical formulas, double negation, excluded third, Dummett law $(A \supset B) \vee (B \supset A)$, Peirce's law $((A \supset B) \supset A) \supset A$, and so on, by showing the underivability of these when A and B are atomic formulas P and Q.

(b) Harrop's theorem. Harrop formulas are defined as follows: Atomic formulas and $\perp$ are Harrop, and if A and B are Harrop, also $A\&B$ is Harrop. If B is Harrop, also $A \supset B$ is

Harrop. The idea is that there are no cases (disjunctions) among Harrop formulas, nor any cases "hidden" inside implications, such as in $A \supset B \vee C$.

Theorem. *If Γ consists of Harrop formulas and $A \vee B$ is derivable from Γ, then A or B is derivable from Γ.*

Proof: The proof is by induction on the height (max. number of consecutive steps of inference) of a normal derivation. $A \vee B$ cannot be an assumption for then $\Gamma \equiv A \vee B$. The last rule can be $\vee I$ or an E-rule. With $\vee I$, leave out the last step. With $\&E$ and $\supset E$ and major premisses $C\&D$ and $C \supset D$, we have

$$\dfrac{C\&D \qquad \begin{matrix}[\overset{1}{C^m}, \overset{1}{D^n}], \Gamma \\ \vdots \\ A \vee B\end{matrix}}{A \vee B}\,{\scriptstyle \&E,1} \qquad\qquad \dfrac{C \supset D \quad C \quad \begin{matrix}[\overset{1}{D^n}], \Gamma \\ \vdots \\ A \vee B\end{matrix}}{A \vee B}\,{\scriptstyle \supset E,1}$$

The minor premiss $A \vee B$ is derivable from C^m, D^n, Γ and D^n, Γ, respectively. The major premiss $C\&D$ is an assmption and therefore a Harrop formula. Then also C and D are Harrop formulas and by the inductive hypothesis, A or B is derivable from C^m, D^n, Γ, and $\&E$ gives the conclusion. With major premiss $C \supset D$, D is a Harrop formula and by the inductive hypothesis, A or B is derivable from D^n, Γ. If the last rule is $\bot E$, change the conclusion $A \vee B$ into one of A or B. The last rule cannot be $\vee E$, for a major premiss $C \vee D$ is not a Harrop formula and therefore not in Γ. QED.

(c) Mints' theorem. Formula A is a **proper assumption** if it is underivable. Mints' theorem states that principal formulas in left rules of sequent calculus derivations can be restricted to proper assumptions. We show the corresponding result for natural deduction:

Theorem. *If C is derivable from Γ, it has a derivation in which all major premisses of E-rules are proper assumptions.*

Proof: Consider a derivable assumption A. The last rule in its (normal) derivation is an I-rule, so a substitution in a normal derivation of C from Γ creates a detour convertibility. Conversions do not produce new major premisses of E-rules. Detour conversions produce shorter conversion formulas. Therefore the substitution of derivable major premisses with their derivations terminates. QED.

In Mints' original proof of 1993, a derivable assumption (a formula in the antecedent part of a sequent) is removed by a cut, say, the cut

$$\dfrac{\to A \quad A, \Gamma \to C}{\Gamma \to C}\,{\scriptstyle Cut}$$

It is not obvious that the repetition of cuts on derivable assumptions and their elimination terminates. Our proof, in comparison, is almost trivial, because of the knowledge that the substitution of a derivable assumption produces a detour convertibility. This would correspond to a cut in which both cut formulas are principal in the derivation of the premisses of the cut.

There is an essential difference between Harrop's and Mints' theorems, not visible because we have not spent time on treating quantifier rules. Namely, Harrop's theorem gives an effective proof transformation, but Mints' theorem does not. The reason is that it is not decidable if a formula is a proper assumption.

10. Translations from natural deduction to sequent calculus

The translation we defined in paragraph 5 is easily defined also in the direction from normal natural deduction derivations to cut-free sequent calculus derivations. Thus, an **isomorphic** translation is established. Translations in this direction have been defined already by Gentzen, and by Prawitz in his book *Natural Deduction* of 1965. Gentzen's translation produces cuts whenever the standard $\&E$ and $\supset E$ rules are translated. The "reason" for these cuts is that the special elimination rules produce derivations that are not normal in our sense, hence, not isomorphic to cut-free derivations. So there must be these cuts. With Prawitz' translation, the sequent calculus derivation is cut free, even if the natural deduction derivation was not normal in our sense. It therefore follows that there must be a cut-elimination procedure hidden in Prawitz' translation. We show these phenomena through some examples:

Gentzen's translation: In Gentzen's translation, each rule is translated in two stages. In the first stage, the open assumptions of a formula C in a derivation are collected into a multiset (sequence in Gentzen) Γ and the expression $\Gamma \to C$ replaces the line on which C occurred. This suffices for I-rules. For E-rules, a second stage inserts a cut. Here are two examples:

Stage 1.

$$\begin{array}{c}\Gamma\\ \vdots\\ \dfrac{A\&B}{A}\,{\scriptstyle \&E}\end{array} \quad\rightsquigarrow\quad \dfrac{\Gamma \to A\&B}{\Gamma \to A}$$

$$\dfrac{\begin{array}{cc}\Gamma & \Delta\\ \vdots & \vdots\\ A \supset B & A\end{array}}{B}\,{\scriptstyle \supset E} \quad\rightsquigarrow\quad \dfrac{\Gamma \to A \supset B \quad \Delta \to A}{\Gamma, \Delta \to B}$$

Stage 2.

$$\dfrac{\Gamma \to A\&B \quad \dfrac{A \to A}{A\&B \to A}\,{\scriptstyle L\&}}{\Gamma \to A}\,{\scriptstyle Cut}$$

$$\dfrac{\Gamma \to A \supset B \quad \dfrac{\Delta \to A \quad B \to B}{A \supset B, \Delta \to B}\,{\scriptstyle L\supset}}{\Gamma, \Delta \to B}\,{\scriptstyle Cut}$$

The next example shows a combination of two eliminations with Gentzen's $\&E$-rule:

Stage 1.

$$\dfrac{\dfrac{(A\&B)\&C}{A\&B}\,{\scriptstyle \&E}}{A}\,{\scriptstyle \&E} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{(A\&B)\&C \to (A\&B)\&C}{(A\&B)\&C \to A\&B}}{(A\&B)\&C \to A}$$

Stage 2.

$$\dfrac{\dfrac{(A\&B)\&C \to (A\&B)\&C \qquad \dfrac{A\&B \to A\&B}{(A\&B)\&C \to A\&B}\,{\scriptstyle L\&}}{(A\&B)\&C \to A\&B}\,{\scriptstyle Cut} \qquad \dfrac{A \to A}{A\&B \to A}\,{\scriptstyle L\&}}{(A\&B)\&C \to A}\,{\scriptstyle Cut}$$

Next eliminate the trivial upper cut with an initial sequent to get

$$\dfrac{\dfrac{A\&B \to A\&B}{(A\&B)\&C \to A\&B}\,{\scriptstyle L\&} \qquad \dfrac{A \to A}{A\&B \to A}\,{\scriptstyle L\&}}{(A\&B)\&C \to A}\,{\scriptstyle Cut}$$

Neither premiss of the remaining cut is an initial sequent. The overall conclusion is that Gentzen's translation produces an essential cut whenever the derivation is not normal in our sense, that is, whenever a major premiss of an E-rule has been derived.

Prawitz' translation:

The translation is defined inductively by the height of a derivation. Assumptions A turn into initial sequents $A \to A$. We show by a few examples how the logical rules are translated.

Example 1. The last rule is $\supset I$:

$$\dfrac{\begin{matrix}[A], \Gamma \\ \vdots \\ B\end{matrix}}{A \supset B}\,{\scriptstyle \supset I}$$

Translation of the premiss

$$\begin{matrix}A, \Gamma \\ \vdots \\ B\end{matrix}$$

is by assumption

$$\begin{matrix}\vdots \\ A, \Gamma \to B\end{matrix}$$

and an application of rule $R\supset$ gives

$$\dfrac{\begin{matrix}\vdots \\ A, \Gamma \to B\end{matrix}}{\Gamma \to A \supset B}\,{\scriptstyle R\supset}$$

Example 2. The last rule is the standard $\supset\! E$ rule with the major premiss an assumption:

$$\cfrac{A \supset B \quad \begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{\begin{array}{c}B\\ \vdots\\ C\end{array}}\;{\scriptstyle \supset E}$$

By assumption, a derivation of $\Gamma \to A$ is at hand. Leaving out step $\supset\! E$ we get a shorter derivation of C from B taken as an assumption and formulas Γ' from Γ, not necessarily all. Then also a derivation of $B, \Gamma' \to C$ is at hand. Weakening is an admissible rule in Prawitz' sequent calculus, so also a derivation of $B, \Gamma \to C$ is obtained. Now rule $L\!\supset$ derives $A \supset B, \Gamma \to C$, modulo possible duplications of formulas in Γ. (Prawitz treats contexts as sets so duplication will not be visible.)

Example 3. Last step is $\&E$ with major premiss derived:

$$\cfrac{\cfrac{(A\&B)\&C}{A\&B}\;{\scriptstyle \&E}}{A}\;{\scriptstyle \&E}$$

The translation is in two parts:

$$\begin{array}{c}\cfrac{(A\&B)\&C}{A\&B}\;{\scriptstyle \&E}\\ \vdots\end{array} \quad\rightsquigarrow\quad \cfrac{\begin{array}{c}\vdots\\ A\&B \to A\end{array}}{(A\&B)\&C \to A}\;{\scriptstyle L\&} \qquad\qquad \cfrac{\begin{array}{c}\vdots\\ A\&B\end{array}}{A}\;{\scriptstyle \&E} \quad\rightsquigarrow\quad \begin{array}{c}\cfrac{A \to A}{A\&B \to A}\;{\scriptstyle L\&}\\ \vdots\end{array}$$

Next the parts are put together to produce

$$\cfrac{\cfrac{A \to A}{A\&B \to A}\;{\scriptstyle L\&}}{(A\&B)\&C \to A}\;{\scriptstyle L\&}$$

Our isomorphic translation produces (as shown in the next paragraph), after elimination of a trivial cut,

$$\cfrac{\cfrac{A\&B \to A\&B}{(A\&B)\&C \to A\&B}\;{\scriptstyle L\&} \qquad \cfrac{A \to A}{A\&B \to A}\;{\scriptstyle L\&}}{(A\&B)\&C \to A}\;{\scriptstyle Cut}$$

It is not surprising that a cut is produced at the formula $A\&B$. That is where the original derivation in natural deduction was cut into parts in Prawitz' translation. We get the result of the latter translation from isomorphic translation by one step of cut elimination.

Example 4. Step to be translated is $\supset\! E$ with major premiss derived:

$$\cfrac{\cfrac{A \supset (B \supset C) \quad A}{B \supset C}\;{\scriptstyle \supset E} \qquad B}{C}\;{\scriptstyle \supset E}$$

Prawitz' translation is again produced in two stages. The result is

$$\frac{A \to A \qquad \dfrac{B \to B \quad A \to A}{B \supset C, B \to C}\,{\scriptstyle L\supset}}{A \supset (B \supset C), A, B \to C}\,{\scriptstyle L\supset}$$

Isomorphic translation:

$$\frac{\dfrac{A \to A \quad B \supset C \to B \supset C}{A \supset (B \supset C), A \to B \supset C}\,{\scriptstyle L\supset} \qquad \dfrac{B \to B \quad C \to C}{B \supset C, B \to C}\,{\scriptstyle L\supset}}{A \supset (B \supset C), A, B \to C}\,{\scriptstyle Cut}$$

One step of cut elimination gives Prawitz' translation.

Derivations are cut into two parts whenever the translation reaches a major premiss of an elimination rule that is not an assumption. The parts are translated separately and then put together. This process corresponds to a step of cut elimination.

11. Non-normal derivations and derivations with cuts

This is a vast topic. Convertibilities in natural deduction translate into cuts in which the cut formula is principal in the right premiss (permutation convertibility) or in both premisses (detour convertibility). This is quite natural, for what would you do with an assumption A if it had not been analyzed into its components by an E-rule?

We give a couple of examples of the translation of non-normal derivations. They show how major premisses of E-rules always become cut formulas.

1. Non-normality with a detour convertibility:

$$\frac{\dfrac{\begin{matrix}\Gamma & \Delta \\ \vdots & \vdots \\ A & B\end{matrix}}{A\&B}\,{\scriptstyle \&I} \qquad \begin{matrix}[A],[B],\Theta \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \&E}$$

This is translated into

$$\frac{\dfrac{\begin{matrix}\Gamma & \Delta \\ \vdots & \vdots \\ A & B\end{matrix}}{\Gamma, \Delta \to A\&B}\,{\scriptstyle R\&} \qquad \dfrac{\begin{matrix}A, B, \Theta \\ \vdots \\ C\end{matrix}}{A\&B, \Theta \to C}\,{\scriptstyle L\&}}{\Gamma, \Delta, \Theta \to C}\,{\scriptstyle Cut}$$

The translation now continues from the premisses of $R\&$ and $L\&$.

2. Non-normality with a permutation convertibility:

$$\frac{\dfrac{A\&B \qquad \begin{matrix}[A],[B],\Gamma \\ \vdots \\ C\&D\end{matrix}}{C\&D}\,{\scriptstyle \&E} \qquad \begin{matrix}[C],[D],\Delta \\ \vdots \\ E\end{matrix}}{E}\,{\scriptstyle \&E}$$

The translation is

$$\dfrac{\dfrac{\begin{array}{c}A,B,\Gamma\\ \vdots\\ C\&D\end{array}}{A\&B,\Gamma\to C\&D}{\scriptstyle L\&} \qquad \dfrac{\begin{array}{c}C,D,\Delta\\ \vdots\\ E\end{array}}{C\&D,\Delta\to E}{\scriptstyle L\&}}{A\&B,\Gamma,\Delta\to E}{\scriptstyle Cut}$$

It happens that the cut formula is always principal in the right premiss of cut. Thus, a non-normality becomes a left rule and a cut in sequent calculus. The existence of an independent rule of cut in place of non-normality makes cut elimination more complicated than normalization, and weaker in properties (failure of what would correspond to strong normalization).

For more, see *SPT*, ch. 8.

III Classical natural deduction

We give first a system of natural deduction for the full language of classical propositional logic, then prove normalization and the subformula property. Attempts at finding a corresponding system for classical predicate logic have not resulted in a natural finitary system of rules, contrary to the situation in sequent calculus.

12. Natural deduction for classical propositional logic

A *rule of excluded middle* for atomic formulas $P, Q, R, \dots$ is added to the system of intuitionistic natural deduction for propositional logic with general elimination rules:

$$\dfrac{\begin{array}{c}\overset{1}{[P^m]}\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\overset{1}{[\sim P^n]}\\ \vdots\\ C\end{array}}{C}{\scriptstyle EM_0,1}$$

In the first subderivation, $m \geq 0$ copies of the formula P and in the second, $n \geq 0$ copies of the formula $\sim P$, are discharged.

The rule of indirect proof, used by Prawitz in 1965, is derivable with EM_0: Assume there is a derivation of $\perp$ from $\sim P$. We then have the derivation

$$\dfrac{\overset{1}{[P]} \qquad \dfrac{\begin{array}{c}\overset{1}{[\sim P]}\\ \vdots\\ \perp\end{array}}{P}{\scriptstyle \perp E}}{P}{\scriptstyle EM_0,1}$$

Contrary to the rule of indirect proof, the premiss $\sim P$ is not discharged after $\perp$ has been derived, but one step later. The rule of indirect proof does not convert to the components if it is applied to a disjunctive formula. For this reason, Prawitz had to consider the $\vee$-free fragment. Rule *EM* for arbitrary propositional formulas A instead,

$$\dfrac{\begin{matrix}\overset{1}{[A^m]}\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\overset{1}{[\sim A^n]}\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle EM,1}$$

is admissible:

Theorem. *Rule EM is admissible for arbitrary propositional formulas.*

Proof: We show that application of rule *EM* to a formula D converts to applications of rule EM_0 to the atoms of D. Consider the case in which indirect proof is insufficient, that of a disjunction $A \vee B$: We assume given the two derivations

$$\begin{matrix}A \vee B\\ \vdots \\ C\end{matrix} \qquad\qquad \begin{matrix}\sim(A \vee B)\\ \vdots \\ C\end{matrix}$$

We can assume that $A \vee B$ and its negation are simply discharged in rule *EM*:

$$\dfrac{\begin{matrix}\overset{1}{[A \vee B]}\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\overset{1}{[\sim(A \vee B)]}\\ \vdots\\ C\end{matrix}}{C}\,{\scriptstyle EM,1}$$

This derivation is converted into a derivation with *EM* applied to A and B:

$$\dfrac{\begin{matrix}\dfrac{\overset{4}{[B]}}{A \vee B}\,{\scriptstyle \vee I}\\ \vdots \\ C\end{matrix} \qquad \dfrac{\begin{matrix}\dfrac{\overset{3}{[A]}}{A \vee B}\,{\scriptstyle \vee I}\\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\dfrac{\dfrac{\overset{2}{[A \vee B]} \qquad \dfrac{\overset{3}{[\sim A]} \quad \overset{1}{[A]}}{\bot}\,{\scriptstyle \supset E} \qquad \dfrac{\overset{4}{[\sim B]} \quad \overset{1}{[B]}}{\bot}\,{\scriptstyle \supset E}}{\bot}\,{\scriptstyle \vee E,1}}{\sim(A \vee B)}\,{\scriptstyle \supset I,2}\\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle EM,3}}{C}\,{\scriptstyle EM,4}$$

The other cases of conversions are similar to those for indirect proof. In the end, atoms of formula D or $\bot$ are reached. The false formula $\bot$ is not an atom, but applications of rule *EM* can be eliminated in this case: If a derivation of formula C from the assumption $\sim\bot$ is given, the assumption is derivable by $\bot E$ and $\supset I$. QED.

13. Normal derivations and the subformula property

Definition. *A derivation in intuitionistic natural deduction* $+EM_0$ *is* **normal** *if no instance of* EM_0 *is followed by a logical rule and all subderivations up to instances of* EM_0 *are normal intuitionistic derivations.*

Lemma. *Rule EM_0 commutes down with the logical rules, modulo possible multiplications of open assumptions.*

Proof: A routine verification. QED.

Thus, if formula C is classically derivable, the corresponding normal natural deduction derivation has intuitionistic subderivations followed by instances of EM_0. In the system of Prawitz, the rule of indirect proof for atoms is applied after the eliminative part and before the introductory part of a normal derivation.

Theorem. Subformula property.

(i) *In a normal derivation of C from open assumptions Γ in intuitionistic natural deduction $+EM_0$, instances of EM_0 are on atoms of Γ, C.*

(ii) *In a derivation of C from open assumptions Γ in intuitionistic natural deduction $+EM_0$, instances of EM_0 can be restricted to atoms of C.*

Proof: We first note that applications of EM_0 to atoms not in Γ, C can be permuted so that they come right after the intuitionistic subderivation. For (i), assume there is an instance of EM_0 on an atom not in Γ, C. Let the first of these be on an atom P:

$$\dfrac{\begin{matrix}\overset{1}{[P]},\Gamma' \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[\sim \overset{1}{P}],\Gamma'' \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle EM,1}$$

Both subderivations are intuitionistic and by assumption normal. How can P be active in the first derivation? It cannot be a premiss in an introduction rule, nor is it a major premiss of an elimination rule, so it must be the minor premiss of $\supset E$. Then the major premiss is of the form $P \supset A$ and this is either an open assumption or a subformula of a major premiss that is an open assumption or a discharged major premiss or a subformula of a discharged major premiss. In the latter cases the major premiss is a subformula of C.

For (ii), consider a derivation with a first atom P active in EM_0 but not in C. The subderivation of C from P, Γ' is transformed into a derivation of $\sim P$ from $\sim C, \Gamma'$ which is then substituted for the assumption of $\sim P$ in the second subderivation, followed by an application of EM to C:

$$\dfrac{\overset{2}{[C]} \qquad \begin{matrix} \dfrac{\dfrac{[\sim \overset{2}{C}] \quad \begin{matrix}\overset{1}{[P]},\Gamma' \\ \vdots \\ C\end{matrix}}{\bot}\,{\scriptstyle \supset E}}{\sim P}\,{\scriptstyle \supset I,1} \qquad \Gamma'' \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle EM,2}$$

By the lemma of the preceding paragraph, the application of EM to C converts to atoms of C. The proof transformation is repeated for the remaining atoms that are not atoms in C. QED.

Note that (ii) does not require normality.

14. Interpretation of classical propositional logic

Each instance of the rule of excluded middle can be presented as an instance of $\vee E$ in which the law of excluded middle is assumed for the atom in question:

$$\dfrac{P\vee\sim P \quad \begin{matrix}\overset{1}{[P]}\\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overset{1}{[\sim P]}\\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \vee E,1}$$

Given a derivation of the formula C in classical propositional logic, we do the above transformation for each instance of EM_0. Then we have a derivation of C from the original assumptions plus those instances of excluded middle on which the rule was applied. Collecting together these instances $P_1\vee\sim P_1,\dots,P_m\vee\sim P_m$, we have the result that

$$(P_1\vee\sim P_1)\&\dots\&(P_m\vee\sim P_m)\supset C$$

is derivable in intuitionistic natural deduction. This gives us an interpretation:

Classical propositional logic is the special case of intuitionistic propositional logic in which the atomic formulas are assumed to be decidable.

15. Infinitary natural deduction

The conditions for direct derivability of formulas by introduction rules bear a remarkable similarity to the definition of truth in model-theoretic semantics. Thus, it has been suggested that one gets the proof-theoretic reading by simply changing "truth" for "proof" in the truth conditions for formulas. However, the similarity breaks down in the case of the universal quantifier. The truth condition for $\forall xA$ is simply that $A(a/x)$ be true for each element a of the domain of discourse. The provability condition, instead, requires a proof of $A(y/x)$ for an arbitrary y, which means that there is a *uniform* proof for each of the instances $A(a/x)$. In the classical semantics, nothing of the kind is required. The solution we propose to the discrepancy is to bring the syntax closer to the classical semantics, through an infinitary introduction rule for the universal quantifier. We assume a well-defined formal syntax with a denumerable infinity of expressions for individual constants $a_1, a_2, \dots$ that make up the domain of quantification $\mathcal{D}$. The rule of universal introduction is

$$\dfrac{A(a_1/x) \quad A(a_2/x) \quad \dots}{\forall xA}\,{\scriptstyle \forall I_\omega}$$

The introduction rule for the existential quantifier is

$$\dfrac{A(a_i/x)}{\exists xA}\,{\scriptstyle \exists I}$$

The standard universal introduction rule is admissible: Assume that there is a derivation of $A(y/x)$ for y arbitrary. A substitution of y in this derivation by a_i, for $i = 1, 2, \ldots$, gives derivations of $A(a_i/x)$ for $i = 1, 2, \ldots$, and rule $\forall I_\omega$ concludes $\forall x A$.

It is possible to formulate the calculus so that free variables are not used.

We now determine the general elimination rules for the above quantifiers by the inversion principle. A strict obedience to the inversion principle would give the following universal elimination rule:

$$\dfrac{\forall xA \qquad \begin{matrix} \overset{1}{[A(a_1/x)^{m_1}]}, \overset{1}{[A(a_2/x)^{m_2}]}, \ldots \\ \vdots \\ C \end{matrix}}{C}\ \forall E_\omega,1$$

We allow instead only one instance of A and obtain a finitary rule:

$$\dfrac{\forall xA \qquad \begin{matrix} \overset{1}{[A(a_i/x)^{m}]} \\ \vdots \\ C \end{matrix}}{C}\ \forall E,1$$

For $\exists$, an infinitary elimination rule is determined:

$$\dfrac{\exists xA \qquad \begin{matrix} \overset{1}{[A(a_1/x)^{m_1}]} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \overset{1}{[A(a_2/x)^{m_2}]} \\ \vdots \\ C \end{matrix} \qquad \ldots}{C}\ \exists E_\omega,1$$

The rule has instances for any $m_1 \geq 0, m_2 \geq 0, \ldots$. As for universal introduction, the standard existential elimination rule is admissible.

With the above rules, a perfect duality of universality and existence is achieved.

We shall designate the standard system of intuitionistic natural deduction by *ND* and the one with the above quantifier rules by ND_ω. There are no free variables in the quantifier rules of ND_ω, thus, no variable restrictions to care about, either.

The mirror image duality of universal and existential rules of ND_ω is displayed better if the rules are written as sequent calculus rules:

$$\dfrac{\Gamma \to A(a_i/x)}{\Gamma \to \exists x A}\ R\exists \qquad \dfrac{A(a_i/x), \Gamma \to C}{\forall x A, \Gamma \to C}\ L\forall$$

$$\dfrac{\Gamma_1 \to A(a_1/x) \quad \Gamma_2 \to A(a_2/x) \quad \ldots}{\Gamma_1, \Gamma_2, \ldots \to \forall x A}\ R_\omega\forall \qquad \dfrac{A(a_1/x), \Gamma_1 \to C \quad A(a_2/x), \Gamma_2 \to C \quad \ldots}{\exists x A, \Gamma_1, \Gamma_2, \ldots \to C}\ L_\omega\exists$$

The infinitary universal elimination rule becomes

$$\dfrac{A(a_1/x), A(a_2/x), \ldots, \Gamma \to C}{\forall x A, \Gamma \to C}\ L_\omega\forall$$

Theorem 1. *A closed formula is derivable in ND if and only if it is derivable in ND_ω.*

Proof: The intuitionistic rule of universal introduction was shown admissible under rule $\forall I_\omega$, and an analogous argument shows standard existence elimination admissible. In the other direction, consider a first instance of rule $\forall I_\omega$ in a derivation with the conclusion $\forall x A$. Each of its premisses $A(a_i/x)$ is derivable by the rules of *ND*. For each valuation v, then, $v(A(a_i/x)) = 1$ so by definition $v(\forall x A) = 1$ and by the completeness of *ND*, $\forall x A$ is derivable in *ND*. If the first rule is $\exists E_\omega$, the intuitionistic subderivations are transformed into

$$\frac{\begin{matrix}\overset{1}{[A(a_i/x)^{m_i}]}\\ \vdots\\ C\end{matrix}}{A(a_i/x)\supset C}\,{\supset}I,1$$

followed by an application of rule $\forall I_\omega$ to conclude $\forall x(A \supset C)$. By above, this is derivable in *ND* so C is derivable from $\exists x A$ in *ND*. QED.

Theorem 2. *Derivations in ND_ω are normalizing.*

Proof: The new cases of detour convertibility are on the quantifiers. We have the conversions

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ A(a_1/x)\end{matrix}\quad\begin{matrix}\vdots\\ A(a_2/x)\end{matrix}\quad\dots\quad\begin{matrix}\vdots\\ A(a_i/x)\dots\end{matrix}}{\forall x A}\,\forall I_\omega\qquad\begin{matrix}\overset{1}{[A(a_i/x)]}\\ \vdots\\ C\end{matrix}}{C}\,\forall E,1\quad\rightsquigarrow\quad\begin{matrix}\vdots\\ A(a_i/x)\\ \vdots\\ C\end{matrix}$$

$$\cfrac{\cfrac{\begin{matrix}\vdots\\ A(a_i/x)\end{matrix}}{\exists x A}\,\exists I\qquad\begin{matrix}\overset{1}{[A(a_1/x)]}\\ \vdots\\ C\end{matrix}\quad\begin{matrix}\overset{1}{[A(a_2/x)]}\\ \vdots\\ C\end{matrix}\quad\dots\quad\begin{matrix}\overset{1}{[A(a_i/x)]}\dots\\ \vdots\\ C\end{matrix}}{C}\,\forall E_\omega,1\quad\rightsquigarrow\quad\begin{matrix}\vdots\\ A(a_i/x)\\ \vdots\\ C\end{matrix}$$

These conversions work in the same way as the standard detour conversions on $\forall$ and $\exists$. The same holds for permutation convertibilities: If a major premiss of an E-rule has been derived by $\exists E$, with the conclusion C, the elimination is permuted up in the auxiliary derivations of C from the assumption $A(a_1/x)$, from the assumption $A(a_2/x), \dots$. QED.

16. Natural deduction for classical predicate logic

We assume that all quantified formulas are in prenex normal form. The rule of excluded middle of propositional logic is generalized into infinitary rules for each quantifier prefix class Π_n, Σ_n. The simplest cases are *universal* and *existential* excluded middle. For a lighter notation, we assume that each discharged formula is simply discharged:

$$\cfrac{\begin{matrix}\overset{1}{[P(a_1), P(a_2), \dots]}\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}\overset{1}{[\sim P(a_i)]}\\ \vdots\\ C\end{matrix}}{C}\,\forall EM_0,1\qquad\qquad\cfrac{\begin{matrix}\overset{1}{[P(a_i)]}\\ \vdots\\ C\end{matrix}\qquad\begin{matrix}\overset{1}{[\sim P(a_1), \sim P(a_2), \dots]}\\ \vdots\\ C\end{matrix}}{C}\,\exists EM_0,1$$

In the case that the $P(a_i)$ are the same constant proposition P, the schemes reduce to the propositional rule. A few examples will show the naturalness of these rules of inference. A typical classical mode of inference is to consider the two cases given by the formula $\forall xA \vee \exists x \sim A$. We have with a rule of universal excluded middle the derivation:

$$\dfrac{\dfrac{\dfrac{[A(a_1/x), \overset{1}{A}(a_2/x), \dots]}{\forall xA}\,\forall I_\omega}{\forall xA \vee \exists x \sim A}\,\vee I \qquad \dfrac{\dfrac{[\sim \overset{1}{A}(a_i/x)]}{\exists x \sim A}\,\exists I}{\forall xA \vee \exists x \sim A}\,\vee I}{\forall xA \vee \exists x \sim A}\,\forall EM{,}1$$

The dual formula $\exists xA \vee \forall x \sim A$ is derived analogously by rule $\exists EM$. A further example shows a classical derivation of the existential formula $\exists x(A(x) \supset B)$, in which B is assumed to be a constant proposition. For better readability, we leave out the substitution notation and write arbitrary formulas as if they were atomic formulas:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall xA(x) \supset B \qquad \dfrac{[A(a_1), \overset{2}{A}(a_2), \dots]}{\forall xA(x)}\,\forall I_\omega}{B}\,\supset E}{A(a_i) \supset B}\,\supset I}{\exists x(A(x) \supset B)}\,\exists I \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\sim \overset{2}{A}(a_j)] \quad [\overset{1}{A}(a_j)]}{\bot}\,\supset E}{B}\,\bot E}{A(a_j) \supset B}\,\supset I{,}1}{\exists x(A(x) \supset B)}\,\exists I}{\exists x(A(x) \supset B)}\,\forall EM{,}2$$

The other classical prenex formulas are derived similarly. The generalizations of rules of excluded middle to an arbitrary number of universal or existential quantifiers, that is, the prenex classes Π_1, Σ_1, is straightforward: For a formula $\forall x_1 \dots \forall x_n P(x_1, \dots, x_n)$, the corresponding rule of excluded middle requires a derivation of C from all instances of P, and a derivation of C from at least one counterinstance to P. Dually, the rule for $\exists x_1 \dots \exists x_n P(x_1, \dots, x_n)$ requires a derivation of C from at least one instance of P, and a derivation of C from the negations of each instance of P.

The next quantifier classes are Π_2 and Σ_2. The rule for Π_2 is:

$$\dfrac{\begin{array}{c}[P(a_1, a_{i_1}), \overset{1}{P}(a_2, a_{i_2}), \dots]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[\sim P(a_j, a_1), \overset{1}{\sim} P(a_j, a_2), \dots]\\ \vdots \\ C\end{array}}{C}\,\forall\exists EM_0{,}1$$

The rule for Σ_2 is:

$$\dfrac{\begin{array}{c}[P(a_i, a_1), \overset{1}{P}(a_i, a_2), \dots]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[\sim P(a_1, a_j), \overset{1}{\sim} P(a_2, a_j), \dots]\\ \vdots \\ C\end{array}}{C}\,\exists\forall EM_0{,}1$$

The general case of rules of excluded middle is seen from the next cases, the prenex classes Π_3 and Σ_3. To simplify notation, we write i for a_i. Rule $\forall\exists\forall EM_0$ requires derivations of C from

$$\begin{array}{l} P(1,i_1,1), P(1,i_1,2), \dots \\ P(2,i_2,1), P(2,i_2,2), \dots \\ \vdots \end{array}$$

and from

$$\sim P(j,1,k), \sim P(j,2,k), \dots$$

for some j,k. Rule $\exists\forall\exists EM_0$ requires the duals of these derivations. It is now straightforward to write rules of excluded middle for arbitrary prenex classes Π_n, Σ_n. The corresponding rules are designated by $\Pi_n EM_0$ and $\Sigma_n EM_0$, and the infinite collctions by ΠEM_0 and ΣEM_0.

Theorem 1. *Rule EM is admissible for arbitrary formulas.*

Proof: The propositional part reduces as in paragraph 12. For quantificational formulas, we consider the case of $\forall x\exists y A(x,y)$ which shows how the general case is handled. The derivation

$$\dfrac{\begin{matrix}\overset{1}{[\forall x\exists y A(x,y)]} \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\overset{1}{[\sim\forall x\exists y A(x,y)]} \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle EM,1}$$

converts, with discharges indicated only by labels to save horizontal space, into the derivation

$$\dfrac{\begin{matrix}\dfrac{\dfrac{\overset{4}{A(1,i_1)}}{\exists y A(1,y)}{\scriptstyle \exists I} \quad \dfrac{\overset{4}{A(2,i_2)}}{\exists y A(2,y)}{\scriptstyle \exists I} \quad \dots}{\forall x\exists y A(x,y)}{\scriptstyle \forall I_\omega} \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\dfrac{\dfrac{\overset{3}{\forall x\exists y A(x,y)} \quad \dfrac{\overset{2}{\exists y A(j,y)} \quad \dfrac{\overset{4}{\sim A(j,1)} \quad \overset{1}{A(j,1)}}{\bot}{\scriptstyle \supset E} \quad \dfrac{\overset{4}{\sim A(j,2)} \quad \overset{1}{A(j,2)}}{\bot}{\scriptstyle \supset E} \quad \dots}{\bot}{\scriptstyle \exists E_\omega,1}}{\bot}{\scriptstyle \forall E,2}}{\sim\forall x\exists y A(x,y)}{\scriptstyle \supset I,3} \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \forall\exists EM,4}$$

The right branch shows typical nested major premisses of E-rules. The case $\exists x\forall y A(x,y)$ is dual to this one. It is now clear how the general case reduces to rules $\Pi_n EM$ and $\Sigma_n EM$ on propositional formulas. We still have to show that these convert further to atoms. We consider the case of the above conversion on $\forall\exists EM$. There are three cases according to the form of the propositional part:

1. $A\&B$: The relevant part of the derivation, with renamed labels, is

$$\dfrac{\overset{1}{[A(1,i_1)\&B(1,i_1)]} \quad \begin{matrix}\overset{1}{[A(2,i_2)\&B(2,i_2)]} \dots \\ \vdots \\ C\end{matrix} \quad \overset{1}{[\sim(A(j,1)\&B(j,1))]} \quad \begin{matrix}\overset{1}{[\sim(A(j,2)\&B(j,2))]} \dots \\ \vdots \\ C\end{matrix}}{C}\,{\scriptstyle \forall\exists EM,1}$$

We write the conversion in two parts, with the last step of $\forall\exists EM$ omitted. In this step, the open assumptions from both subderivations shown below are discharged. Brackets around the discharged A- and $\sim A$-formulas and symbols for rules that are obvious are left out from the first subderivation:

$$\dfrac{\begin{matrix} \dfrac{\overset{3}{A(1,i_1)} \quad B(1,i_1)}{A(1,i_1)\&B(1,i_1)} \quad \dfrac{\overset{3}{A(2,i_2)} \quad B(2,i_2)}{A(2,i_2)\&B(2,i_2)} \quad \ldots \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \dfrac{\dfrac{\overset{2}{A(j,1)\&B(j,1)} \quad \dfrac{\overset{3}{\sim A(j,1)} \quad \overset{1}{A(j,1)}}{\bot}{\scriptstyle \&E,1}}{\bot}}{\sim(A(j,1)\&B(j,1))}{\scriptstyle \supset I,2} \quad \dfrac{\dfrac{\overset{2}{A(j,2)\&B(j,2)} \quad \dfrac{\overset{3}{\sim A(j,2)} \quad \overset{1}{A(j,2)}}{\bot}{\scriptstyle \&E,1}}{\bot}}{\sim(A(j,2)\&B(j,2))}{\scriptstyle \supset I,2} \quad \ldots \\ \vdots \\ C \end{matrix}}{C}{\scriptstyle \forall\exists EM,3}$$

The right branch of the derivation is

$$\begin{matrix} \dfrac{\dfrac{\overset{5}{[A(k,1)\&B(k,1)]} \quad \dfrac{\sim B(k,1) \quad \overset{4}{[B(k,1)]}}{\bot}{\scriptstyle \supset E}}{\bot}{\scriptstyle \&E,4}}{\sim(A(k,1)\&B(k,1))}{\scriptstyle \supset I,5} \quad \dfrac{\dfrac{\overset{5}{[A(k,2)\&B(k,2)]} \quad \dfrac{\sim B(k,2) \quad \overset{4}{[B(k,2)]}}{\bot}{\scriptstyle \supset E}}{\bot}{\scriptstyle \&E,4}}{\sim(A(k,2)\&B(k,2))}{\scriptstyle \supset I,5} \quad \ldots \\ \vdots \\ C \end{matrix}$$

A last step of $\forall\exists EM$ ischarges the B- and $\sim B$-formulas. 2. $A \lor B$: The conversion is similar to case 1.

3. $A \supset B$: For typographical reasons, we show the conversion when rule $\forall EM$ has been used. Rule $\forall\exists EM$ works in just the same way. Discharge brackets and some rule symbols are left out:

$$\dfrac{\begin{matrix} \dfrac{\dfrac{\dfrac{\dfrac{\overset{6}{\sim A(1)} \quad \overset{4}{A(1)}}{\bot}}{B(1)}{\scriptstyle \bot E}}{A(1)\supset B(1)}{\scriptstyle 4} \quad \dfrac{\dfrac{\dfrac{\overset{6}{\sim A(2)} \quad \overset{5}{A(2)}}{\bot}}{B(2)}{\scriptstyle \bot E}}{A(2)\supset B(2)}{\scriptstyle 5} \ldots}{\forall x(A\supset B)}{\scriptstyle \forall I_\omega} \\ \vdots \\ C \end{matrix} \qquad \dfrac{\begin{matrix} \dfrac{\dfrac{\overset{3}{B(1)}}{A(1)\supset B(1)} \quad \dfrac{\overset{3}{B(2)}}{A(2)\supset B(2)} \ldots}{\forall x(A\supset B)}{\scriptstyle \forall I_\omega} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \dfrac{\dfrac{\overset{2}{\forall x(A\supset B)} \quad \dfrac{\overset{3}{\sim B(i)} \quad \dfrac{\overset{1}{A(i)\supset B(i)} \quad \overset{6}{A(i)}}{B(i)}}{\bot}}{\bot}{\scriptstyle \forall E,1}}{\sim\forall x(A\supset B)}{\scriptstyle \supset I,2} \\ \vdots \\ C \end{matrix}}{C}{\scriptstyle \forall EM,3}}{C}{\scriptstyle \exists EM,6}$$

QED.

Corollary 2. *The calculus* $ND_\omega+\Pi EM_0+\Sigma EM_0$ *is complete for classical predicate logic.*

Proof: By the theorem, rule *EM* is admissible for arbitrary formulas A. The law of excluded middle for A follows:

$$\dfrac{\dfrac{\overset{1}{[A]}}{A\lor\sim A}{\scriptstyle \lor I} \quad \dfrac{\overset{1}{[\sim A]}}{A\lor\sim A}{\scriptstyle \lor I}}{A\lor\sim A}{\scriptstyle EM,1}$$

QED.

The main task of constructing a classical calculus of natural deduction for the full language of predicate logic is now finished. Contrary to the propositional case, there is no complete separation of intuitionistic and classical inferences:

Theorem 3. *Rules* ΠEM_0 *and* ΣEM_0 *permute down with the logical rules except when an infinity of premisses in* $\forall I_\omega$ *or* $\exists E_\omega$ *have been derived by rule EM.*

Proof: Consider rule $\&I$ with the premisses A and B derived by $\forall EM$. The derivation is

$$\dfrac{\dfrac{\begin{matrix}[P(1),\overset{1}{P}(2),\ldots] \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[\sim\overset{1}{P}(i)] \\ \vdots \\ A\end{matrix}}{A}\,{\scriptstyle \forall EM,1} \qquad \dfrac{\begin{matrix}[Q(1),\overset{2}{Q}(2),\ldots] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[\sim\overset{2}{Q}(j)] \\ \vdots \\ B\end{matrix}}{B}\,{\scriptstyle \forall EM,2}}{A\&B}\,{\scriptstyle \&I}$$

We shall write the infinite assumptions as $\mathbf{P} \equiv P(1), P(2), \ldots$ and $\mathbf{Q} \equiv Q(1), Q(2), \ldots$. Rule $\forall EM$ permutes down:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[\overset{1}{\mathbf{P}}] \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[\overset{3}{\mathbf{Q}}] \\ \vdots \\ B\end{matrix}}{A\&B}\,{\scriptstyle \&I} \qquad \dfrac{\begin{matrix}[\sim\overset{1}{P}(i)] \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[\overset{3}{\mathbf{Q}}] \\ \vdots \\ B\end{matrix}}{A\&B}\,{\scriptstyle \&I}}{A\&B}\,{\scriptstyle \forall EM,1} \qquad \dfrac{\dfrac{\begin{matrix}[\overset{2}{\mathbf{P}}] \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[\sim\overset{3}{Q}(j)] \\ \vdots \\ B\end{matrix}}{A\&B}\,{\scriptstyle \&I} \qquad \dfrac{\begin{matrix}[\sim\overset{2}{P}(i)] \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[\sim\overset{3}{Q}(j)] \\ \vdots \\ B\end{matrix}}{A\&B}\,{\scriptstyle \&I}}{A\&B}\,{\scriptstyle \forall EM,2}}{A\&B}\,{\scriptstyle \forall EM,3}$$

Permutations for all the other finitary rules is easy. The generalization to arbitrary rules $\Pi_n EM_0$ and $\Sigma_n EM_0$ is straightforward for these rules. With rules $\forall I_\omega$ and $\exists I_\omega$, if an infinity of premisses has been derived by rules *EM*, the permutations make the depth of derivation grow indefinitely. QED.

References

Gentzen, G. (1934-35) Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift*, vol. 39, pp. 176-210 and 405-431.

Joachimski, F. and R. Matthes (2003) Short proofs of normalization for the simply typed λ-calculus, permutative conversions and Gödel's T, *Archive for Mathematical Logic*, vol. 42, pp. 59–87.

Mints, G. (1993) A normal form for logical derivations implying one for arithmetic derivations, *Annals of Pure and Applied Logic*, vol. 62, pp. 65–79.

Negri, S. and J. von Plato (2001) *Structural Proof Theory*, Cambridge University Press.

von Plato, J. (2001) Natural deduction with general elimination rules, *Archive for Mathematical Logic*, vol. 40 (2001), pp. 541–567.

von Plato, J. (2003) Translations from natural deduction to sequent calculus, *Mathematical Logic Quarterly*, vol. 49, pp. 435-443.

Prawitz, D. (1965) *Natural Deduction: A Proof-Theoretical Study.* Almqvist & Wicksell, Stockholm.

Schroeder-Heister, P. (1984) A natural extension of natural deduction, *The Journal of Symbolic Logic*, vol. 49, pp. 1284–1300.