LASCAR TYPES AND LASCAR AUTOMORPHISMS IN ABSTRACT ELEMENTARY CLASSES

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Abstract. We study Lascar strong types and Galois types and especially their relation to notions of type which have finite character. We define a notion of a strong type with finite character, so called Lascar type. We show that this notion is stronger than Galois type over countable sets in simple and superstable finitary AECs. Furthermore we give an example where the Galois type itself does not have finite character in such a class.

1. Introduction

Saharon Shelah defined *abstract elementary classes* (AECs) in [21] as a platform to study model theory in a very general framework. In particular, this framework unified and generalized the study of infinitary languages previously done by Shelah and others. Since the appearance of [21], many important tools of model theory have been generalized to AECs, such as Ehrenfeucht-Mostowski-model techniques and studying categoricity using saturation with respect to *Galois types*. For more, see the books by John Baldwin [1] or Shelah [22].

One generalization of an important first order tool, development of a well-behaved independence calculus, has turned out to be one of the most difficult tasks. Actually, it is *not possible* to develop a notion of independence in the most general framework which has all the properties *forking* does in stable elementary classes. It is not necessarily possible even if the class is categorical in all uncountable cardinals and *homogeneous*; see an example in Hyttinen and Kesälä [9].

Buechler and Lessmann [6] introduced an independence calculus under the assumption of *simplicity* in a *homogeneous* framework. Hyttinen and Lessmann [13] gave another definition for simplicity, which is closer to the one used here, for an $\aleph_0$-stable *excellent* class of atomic models of a first order theory. There, as in this paper, the notion of independence has a built-in extension property. Hyttinen and Shelah [15] introduced this kind of a notion of independence first for a stable homogeneous framework. Each of these papers use *Lascar strong type* as a notion comparable to strong type in elementary classes. Lascar strong type was introduced by Daniel Lascar ([20],[19]) and it is equivalent to strong type in stable elementary classes.

The notion of a *finitary abstract elementary class* was defined in the second author's Ph.D. thesis [17]. The idea was to study which kind of properties are needed for an AEC to admit the construction of the model-theoretic tools needed for *geometric stability theory.* To start with, we wanted to assume that the AEC $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ has the amalgamation property (AP), the joint embedding property (JEP) and arbitrarily large models (ALM). The amalgamation property gives us Galois

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type over a model as a useful notion of type. Furthermore, with these assumptions we are able to work inside a *monster model*. This greatly simplifies the notation and makes also a notion of a type over a *set* unambiguous, since we can restrict to types realized in the monster model. Then we have to assume *simplicity* to be able to have an independence calculus over finite sets. Furthermore, since we wanted to study independence and especially dependencies of *finite* sets, we assumed also that $\mathrm{LS}(\mathbb{K}) = \aleph_0$ and a property called *finite character*: For any two models $N, M \in \mathbb{K}$ with $N \subseteq M$, we have that

$$N \preccurlyeq_{\mathbb{K}} M \text{ iff}$$

$$\text{for every finite sequence } \bar{a} \in N \text{ there is a } \mathbb{K}\text{-embedding } f : N \to M \text{ fixing } \bar{a}.$$

These assumptions enable us to build models out of chains of finite subsets of the monster model and give us control on types over *countable models*. More concretely, the authors show in [17] that if a finitary AEC is $\aleph_0$-stable, Galois types over countable models have *finite character:*

$$\mathrm{tp}^{\mathrm{g}}(\bar{a}/M) = \mathrm{tp}^{\mathrm{g}}(\bar{b}/M) \text{ if and only if } \mathrm{tp}^{\mathrm{g}}(\bar{a}/A) = \mathrm{tp}^{\mathrm{g}}(\bar{b}/A) \text{ for each finite } A \subset M.$$

This is almost the case also in the simple and superstable case studied in this paper, but we find that if we know all *Lascar strong types* over finite subsets of a countable set A, then the Galois type over A is determined, see Theorem 5.2. This result heavily uses the stability-theoretic machinery developed in [17], especially the fact that *Lascar types* are stationary. We define *Lascar type*, written Lt, to denote a weak version of Lascar strong type, written Lstp, namely

$$\mathrm{Lt}(\bar{a}/A) = \mathrm{Lt}(\bar{b}/A) \text{ if and only if } \mathrm{Lstp}(\bar{a}/A_0) = \mathrm{Lstp}(\bar{b}/A_0) \text{ for each finite } A_0 \subset A.$$

Our framework generalizes the excellent and homogeneous frameworks, but in contrast to general finitary AECs, in those frameworks Galois type over a model is always determined by a syntactic type and hence Galois type has finite character, not only over countable models but models of arbitrary size.

The motivation for this paper arose when we started to take advantage of the developed machinery and proved a result [11] of geometric stability theory about interpreting groups and fields on a geometry existing on the realizations of a regular type, generalizing Hrushovski [7]. The motivation is twofold: first we needed to improve the results in the thesis [17] and secondly we wanted to collect a 'toolbox' in one paper, where we would list the results in an easily accessible form and specify the assumptions used for any specific part of the theory. In [17] and the subsequent papers [8], [10] and [9], the assumptions differ, maybe a bit confusingly, but at least it is always assumed that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a finitary AEC. However, not even this is needed for the very basic properties of independence or the basic properties of Lascar types. We want to emphasize this since these results might be applicable also in some non-finitary frameworks, say classes definable in $L_{\omega_1\omega}(Q)$. Abstract elementary classes definable in $L_{\omega_1\omega}(Q)$ in general do not satisfy the finite character property, although classes definable in any fragment of $L_{\infty\omega}$ do, when $\preccurlyeq_{\mathbb{K}}$ is taken as elementary substructure with respect to that fragment.

The assumptions of AP, JEP and ALM however do not necessarily hold in a class definable with an arbitrary sentence of $L_{\infty\omega}$. These assumptions can be criticized since they do not follow from some natural assumption (for example categoricity) or form any known 'dividing lines' among abstract elementary classes. The possible *amalgamation spectra* for an AEC has been studied for example in Baldwin, Kolesnikov and Shelah [4]. We make the assumptions AP, JEP and ALM mainly for practical reasons. These assumptions hold for elementary classes, homogenous classes and excellent classes which are known to admit a plausible model theory.

Examples of such classes are listed for example in the introduction of Hyttinen, Lessmann and Shelah [14].

Simplicity is another strong assumption, which is not necessarily assumed in the study of homogeneous or excellent classes. However, in elementary classes it follows from stability, and also non-elementary examples of superstable but non-simple AECs are hard to construct, the authors only know the examples in Hyttinen and Lessmann [12] and Baldwin and Kolesnikov [3]. For *simple* finitary AECs, superstability follows from categoricity.

Model-theoretic examples studying the relations between some of these assumptions can be found in [9]. A model-theoretic example of a simple and superstable finitary AEC, which is not excellent or homogeneous, is found in the last section of this paper. Some finitary AECs arising from 'general mathematics' have been studied. Baldwin, Eklof and Trlifaj study abstract elementary classes induced by tilting and cotilting modules [2], [23]. These are finitary and although most of them appear to be simple, none of them are superstable. Another interesting example of an excellent class is given by covers of multiplicative groups of algebraically closed fields, see Bays and Zilber [5].

The paper is constructed as follows. Sections 2 and 3 define finitary AECs and Lascar types, and recall some of their properties. In section 4 we introduce our definition for a notion of independence, simplicity and the stability assumptions. We first list the properties of the notion of independence in a simple, weakly stable abstract elementary class with amalgamation, joint embedding and arbitrarily large models, see Proposition 4.3. The properties are usual such as extension, symmetry, stationarity for certain types etc, but we mostly have to restrict to types over finite sets. We give another list assuming that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is one of the following:

- a simple superstable finitary AEC or
- a class as in Proposition 4.3 but assuming also that the notion of independence has *local character*.

Then we get all the usual properties over arbitrary sets, where stationarity is for Lascar types over sets, see Theorem 4.8 and Remark 4.9. Section 5 studies the simple and superstable finitary framework. We show that *Lascar type* determines Galois type over countable sets but also give an example of a class where weak type does not determine Galois type even over a countable model in $\mathbb{K}$.

Some of the results improve those given in [17], and the proofs are genuinely new. Those proofs which are strongly similar to those in [17] are left out but clear references are given. Since the first part of the thesis, (and the consequent papers [8] and [9]) deal with the $\aleph_0$-stable case, most of the results refer to the paper [10] which is in the superstable context.

2. Finitary AECs

A finitary abstract elementary class was introduced in Hyttinen, Kesälä [8], but there the definition was slightly less general than in the consequent papers Hyttinen, Kesälä [10] and [9]. A finitary AEC is an abstract elementary class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ with a countable Löwenheim-Skolem number, amalgamation, joint embedding, arbitrarily large models and *finite character:*[1] For any two models $N, M \in \mathbb{K}$ with $N \subseteq M$, we have that

$$N \preccurlyeq_{\mathbb{K}} M \text{ iff}$$

for every finite sequence $\bar{a} \in N$ there is a $\mathbb{K}$-embedding $f : N \to M$ fixing $\bar{a}$.

[1]This formulation of finite character is due to Kueker [18].

In this paper we always assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an abstract elementary class of structures in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\mathbb{K})$ and that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ satisfies the properties amalgamation, joint embedding and arbitrarily large models. The finite character property and $\mathrm{LS}(\mathbb{K}) = \aleph_0$ are not needed until section 4.2. We mention specifically where they are used.

We work inside $\mathfrak{M}$, which is the the κ-universal and κ-model homogeneous monster model of the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. We say that a subset $A \subset \mathfrak{M}$ is *bounded*, if $|A| < \kappa$. We assume that κ is sufficiently large.

We use the notion of Galois type over (bounded) subsets of the monster model: For tuples $\bar{a}, \bar{b} \in \mathfrak{M}$ and $A \subset \mathfrak{M}$

$$\mathrm{tp}^{\mathrm{g}}(\bar{a}/A) = \mathrm{tp}^{\mathrm{g}}(\bar{b}/A)$$

if there is $f \in \mathrm{Aut}(\mathfrak{M}/A)$ mapping $\bar{a}$ to $\bar{b}$, where $\mathrm{Aut}(\mathfrak{M}/A)$ denotes the automorphisms of $\mathfrak{M}$ fixing A pointwise. Furthermore, we use a weaker notion of a *weak type* :

$$\mathrm{tp}^{\mathrm{w}}(\bar{a}/A) = \mathrm{tp}^{\mathrm{w}}(\bar{b}/A)$$

if for each finite subset $A_0 \subset A$ we have that $\mathrm{tp}^{\mathrm{g}}(\bar{a}/A_0) = \mathrm{tp}^{\mathrm{g}}(\bar{b}/A_0)$.

Clearly these two notions agree over finite subsets of the monster model. They agree over all bounded subsets, if $\mathfrak{M}$ is *homogeneous.* Furthermore, they agree over models in $\mathbb{K}$, if the class is $\aleph_0$-stable and $\aleph_0$-tame (see [8]), hence especially if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an *excellent* class of atomic models of a first order theory. Weak type often corresponds to some notion of a syntactic type, see Kueker [18]. We note that when we talk about types over *sets*, it is important that we restrict to types of elements in a given monster model $\mathscr{M}$. There might be for example some syntactic notion of type over a set A which agrees with, say, weak type on elements of $\mathfrak{M}$ but not all types can be simultaneously realized in one model of $\mathbb{K}$, see example 18.9 of Baldwin [1]. However, if we restrict to types over the empty set or over *models* $\mathscr{A} \preccurlyeq_{\mathbb{K}} \mathfrak{M}$, all such types are realized in $\mathscr{M}$ if and only they are realized in some $\preccurlyeq_{\mathbb{K}}$-extension of $\mathscr{A}$[2].

For any bounded ordinal α, we say that a sequence $(\bar{a}_i)_{i<\alpha}$, of tuples is *strongly A-indiscernible* in $\mathscr{M}$, if for any other bounded ordinal $\beta \geq \alpha$ we can extend the sequence to $(\bar{a}_i)_{i<\beta}$ such that for any partial order-preserving $f : \beta \to \beta$ we can find $F \in \mathrm{Aut}(\mathscr{M}/A)$ mapping $\bar{a}_i$ to $\bar{a}_{f(i)}$ for each $i \in \mathrm{dom}(f)$.

The following Lemma is important, since most of the proofs needed for this paper work with the interplay between the concept of boundedness and on the other hand indiscernible sequences, which represent 'unboundedness'. This replaces the role of compactness and the interplay between finite and infinite in elementary classes. A similar technique for non-elementary classes was used already by Keisler [16].

The proof of this lemma is based on Shelah's Representation Theorem and Ehrenfeucht-Mostowski-model techniques that are available in Abstract Elementary Classes. The proof is skipped and also all those proofs where it is used. However, we want to state it here since it is the major reason for why the concepts of Lascar type and Lascar splitting work in this framework. Especially, finite character is not needed, but we only need to assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is an AEC with amalgamation, joint embedding and arbitrary large models. For the details in the case $\mathrm{LS}(\mathbb{K}) = \aleph_0$, see Hyttinen and Kesälä [10] or [9]. To see how the same is done for larger Löwenheim-Skolem numbers, see for example Baldwin [1].

[2]It is possible to define Galois type not referring to automorphisms of $\mathscr{M}$ but referring to $\preccurlyeq_{\mathbb{K}}$-embeddings between elements of $\mathbb{K}$, see for example Baldwin [1].

Lemma 2.1. *(Shelah) For every bounded cardinal κ there exists a cardinal $\mathrm{H}(\kappa)$ such that the following holds. Whenever A is a set of size κ and $(\bar{a}_i)_{i<\mathrm{H}(\kappa)} \subset \mathscr{M}$ are distinct tuples, there exists a strongly A-indiscernible sequence $(\bar{b}_i)_{i<\omega}$ in $\mathscr{M}$ such that for each $n<\omega$ there are $i_0 < ... < i_n < \mathrm{H}(\kappa)$ such that*

$$\mathrm{tp^g}(\bar{b}_0, ..., \bar{b}_n/A) = \mathrm{tp^g}(\bar{a}_{i_0}, ..., \bar{a}_{i_n}/A).$$

Furthermore, if I is any (bounded) linear ordering, there exists a $(\bar{a}_i)_{i\in I}$ in $\mathfrak{M}$ such that for any $n<\omega$ and $j_0 < ... < j_n \in I$ there are $i_0 < ... < i_n < \mathrm{H}(\kappa)$ such that

$$\mathrm{tp^g}(\bar{b}_0, ..., \bar{b}_n/A) = \mathrm{tp^g}(\bar{a}_{i_0}, ..., \bar{a}_{i_n}/A).$$

We know that when $\mathrm{LS}(\mathbb{K}) = \lambda$, then $\mathrm{H}(\aleph_0) = \beth_{(2^\lambda)^+}$, which is often referred to as the Hanf number of abstract elementary classes with $\mathrm{LS}(\mathbb{K}) = \lambda$. We will always assume that the concept of boundedness related to the monster model is closed under the operation $\mathrm{H}(\cdot)$, that is, when a set A is bounded in $\mathfrak{M}$, also the cardinal $\mathrm{H}(|A|)$ is bounded in $\mathfrak{M}$.

3. Lascar types

We use the following definition for Lascar strong type.

Definition 3.1 (Lascar strong type)**.** *We say that $\bar{a}$ and $\bar{b}$ have the same* Lascar strong type *over A, written*

$$\mathrm{Lstp}(\bar{a}/A) = \mathrm{Lstp}(\bar{b}/A),$$

if $\ell(\bar{a}) = \ell(\bar{b})$ and $\mathbf{E}(\bar{a}, \bar{b})$ holds for any A-invariant equivalence relation $\mathbf{E}$ of $\ell(\bar{a})$-tuples with a bounded number of classes.

Since *weak type* was a notion derived from Galois type adding finite character, we first called the finite character version of Lascar strong type with the name *weak Lascar strong type*, written $\mathrm{Lstp^w}$. Since this name is quite awkward, we will rename the notion as just Lascar type. The name should still indicate that the notion of type is weaker than Lascar strong type. Also an automorphism of $\mathfrak{M}$ preserving Lascar types is called a *Lascar-automorphism* oppose to the notion of strong automorphism $f \in \mathrm{Saut}(\mathfrak{M})$ preserving Lascar strong types. We denote the Lascar type of an element a over a set A as $\mathrm{Lt}(a/A)$ and give the following definitions.

Definition 3.2 (Lascar type)**.** *For finite tuples $\bar{a}, \bar{b}$ and A a subset of $\mathfrak{M}$, we write*

$$\mathrm{Lt}(\bar{a}/A) = \mathrm{Lt}(\bar{b}/A)$$

if $\mathrm{Lstp}(\bar{a}/A_0) = \mathrm{Lstp}(\bar{b}/A_0)$ for each finite $A_0 \subset A$.

We write $f \in \mathrm{Laut}(\mathfrak{M}/A)$ if f is an automorphism of $\mathfrak{M}$ and for each finite tuple $\bar{a}$ in the monster model,

$$\mathrm{Lt}(\bar{a}/A) = \mathrm{Lt}(f(\bar{a})/A).$$

Remark 3.3. *If A is a finite set, $\mathrm{Lt}(\bar{a}/A)$ equals $\mathrm{Lstp}(\bar{a}/A)$ and $\mathrm{Laut}(\mathfrak{M}/A)$ equals $\mathrm{Saut}(\mathfrak{M}/A)$.*

The following facts are shown in Hyttinen and Kesälä [10]. They follow from the definition of Lascar strong type and Lemma 2.1.

Remark 3.4. *There is only a bounded number of Lascar strong types over a bounded set. Let $\mathrm{S_{Lstp}}(A)$ denote the set of Lascar strong types over A. For finite sets, the following equation holds:*

$$\sup\{|\,\mathrm{S_{Lstp}}(A)| : A \text{ finite}\} < \beth_{(2^{\mathrm{LS}(\mathbb{K})})^+}.$$

Lemma 3.5. *Let $\bar{a}$, $\bar{b}$ be finite tuples and A be bounded. The following are equivalent*

(1) $\mathrm{Lstp}(\bar{a}/A) = \mathrm{Lstp}(\bar{b}/A)$
(2) *The exists $0 \le n < \omega$, strongly A-indiscernible sequences J_i, for $0 \le i \le n$ and finite tuples $\bar{a}_i$, for $0 \le i \le n+1$ such that $\bar{a} = \bar{a}_0$, $\bar{b} = \bar{a}_{n+1}$ and $\bar{a}_n, \bar{a}_{n+1} \in J_n$.*
(3) *There exists a strong automorphism $f \in \mathrm{Saut}(\mathfrak{M}/A)$ with $f(\bar{a}) = \bar{b}$.*

4. Independence

We say that a weak type $\mathrm{tp}^{\mathrm{w}}(\bar{a}/B)$ *Lascar splits* over a subset $A \subseteq B$ if there is a strongly A-indiscernible sequence $(\bar{b}_i)_{i<\omega}$ such that $\bar{b}_0, \bar{b}_1 \in B$ and $\mathrm{tp}^{\mathrm{w}}(\bar{b}_0/A \cup \bar{a}) \neq \mathrm{tp}^{\mathrm{w}}(\bar{b}_1/A \cup \bar{a})$. Then we define a notion of independece based on Lascar-splitting and with a built-in extension property as follows: We say that $\bar{a}$ is independent of C over B, written

$$\bar{a} \downarrow_B C$$

if there is a finite subset $A \subseteq B$ such that for each $D \supseteq C$ there is $\bar{b}$ realizing $\mathrm{tp}^{\mathrm{w}}(\bar{a}/B \cup C)$ such that $\mathrm{tp}^{\mathrm{w}}(\bar{b}/D)$ does not Lascar-split over A.

4.1. **Simplicity and weak stability.** We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is *weakly stable* in a cardinal λ, if whenever $|A| \le \lambda$ and $(\bar{a}_i)_{i<\lambda^+}$ are finite tuples, there are $i < j < \lambda^+$ such that $\mathrm{tp}^{\mathrm{w}}(\bar{a}_i/A) = \mathrm{tp}^{\mathrm{w}}(\bar{a}_j/A)$. We note that here it is possible to talk about types over sets since $(\bar{a}_i)_{i<\lambda^+}$ are chosen to exists in the monster model. If we want a definition independent of the monster model, we should talk about weak types over models in $\mathbb{K}$ realized in some $\preccurlyeq_{\mathbb{K}}$-extension of the particular model.

Definition 4.1 (Weak stability). *We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is* weakly stable *if there is a cardinal λ such that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is weakly stable in λ.*

Definition 4.2 (Simplicity). *We say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is* simple, *if $\bar{a} \downarrow_A A$ for each tuple $\bar{a}$ and finite set A.*

The following Theorem is proved in section 2.2 of Hyttinen and Kesälä [10]. Although [10] assumes that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a finitary AEC, the proofs do not use finite character of $\preccurlyeq_{\mathbb{K}}$ or the countable Löwenheim-Skolem number. Invariance, Monotonicity and Restricted Local Character follow from the definition. Finite Reflexivity is Lemma 2.9 and Finite extension is Corollary 2.12 of [10], where the proofs only use the definitions. Symmetry is proved using Simplicity, Weak stability and Lemma 2.1 and then the further properties are proved using the previous ones and the definitions. However, not all of the proofs are straightforward, but we first prove some Lemmas, a finite version of the Pairs Lemma for example, and get the properties in several stages.

Proposition 4.3. *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is simple and weakly stable in some infinite cardinal. Then the relation $\downarrow$ has the following properties, where $\bar{a}$ and $\bar{b}$ are finite tuples and A, B, C, D (bounded) subsets of the monster model.*

(1) ***Invariance:*** *If $A \downarrow_C B$ and f is an automorphism of the monster model, then $f(A) \downarrow_{f(C)} f(B)$.*
(2) ***Monotonicity:*** *If $A \downarrow_B D$ and $B \subseteq C \subseteq D$ then $A \downarrow_C D$ and $A \downarrow_B C$.*
(3) ***Restricted Local Character:*** *Assume that $\bar{a} \downarrow_A C$. Then there is finite $A' \subseteq A$ such that $\bar{a} \downarrow_{A'} C$.*
(4) ***Finite Reflexivity:*** *Assume that A is finite. The type $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A)$ is unbounded if and only if $\bar{a} \not\downarrow_A \bar{a}$.*

(5) ***Finite Extension:*** *Assume that* A *and* B *are finite and that* $\bar a \downarrow_A B$. *Then for any* D *there is* $\bar b$ *realizing* $\mathrm{Lt}(\bar a/A \cup B)$ *such that* $\bar b \downarrow_A D$.
(6) ***Simplicity:*** *Assume that* A *is finite. Then* $\bar a \downarrow_A A$.
(7) ***Finite Symmetry:*** *Let* A *be finite. Then* $\bar a \downarrow_A \bar b$ *if and only if* $\bar b \downarrow_A \bar a$.
(8) ***Restricted Finite Character:*** *Let* A *be finite. Then* $\bar a \downarrow_A B$ *if and only if* $\bar a \downarrow_A \bar b$ *for all finite tuples* $\bar b \in B$.
(9) ***Transitivity:*** *Assume that* $B \subseteq C \subseteq D$. *Then* $A \downarrow_B D$ *if and only if* $A \downarrow_B C$ *and* $A \downarrow_C D$.
(10) ***Stationarity of Finite Lascar Types:*** *Let* A *be finite. Assume that* $\mathrm{Lt}(\bar a/A) = \mathrm{Lt}(\bar b/A)$, $\bar a \downarrow_A B$ *and* $\bar b \downarrow_A B$. *Then* $\mathrm{Lt}(\bar a/B) = \mathrm{Lt}(\bar b/B)$.

4.2. **Superstability.** In this section we assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a finitary AEC, i.e. also $\mathrm{LS}(\mathbb{K}) = \aleph_0$ and $\preccurlyeq_{\mathbb{K}}$ has the finite character property. The following definition of superstability is used in Hyttinen and Kesälä [10] and clearly it makes no sense if $\mathrm{LS}(\mathbb{K})$ is uncountable.

Definition 4.4 (Superstability)**.** *We say that the class* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is* superstable *if it is weakly stable in at least one cardinal and the following holds.*

Let $(A_n)_{n<\omega}$ *be an increasing sequence of finite sets such that* $\bigcup_{n<\omega} A_n$ *is a model, and let* $\bar a$ *be a tuple. Then there is* $n < \omega$ *such that* $\bar a \downarrow_{A_n} A_{n+1}$.

In Hyttinen and Kesälä [10] the authors find some equivalent definitions of superstability, show that this definition implies that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is weakly stable starting from a cardinal and show that superstability is implied by $\aleph_0$-stability in simple finitary AECs. Those proofs use also the so called Tarski Vaught property, but the use of that can be replaced by the results of this paper.

We want to prove that superstability implies local character for $\downarrow$. For that, we prove a stronger version of superstability in Proposition 4.6, which is proved using the Tarski-Vaught property as Proposition 3.11 of [10]. Before that we recall a lemma (Lemma 3.4 of [8]) which uses the finite character of $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$.

Lemma 4.5. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a finitary AEC. Let* $(A_n : n < \omega)$ *be an increasing sequence of sets such that* $\bigcup_{n<\omega} A_n$ *is a model in* $\mathbb{K}$. *Let* $(\bar b_n)_{n<\omega}$ *be a sequence of finite tuples of the same length, such that*

$$\mathrm{tp}^{\mathrm{g}}(\bar b_m/A_n) = \mathrm{tp}^{\mathrm{g}}(\bar b_n/A_n), \quad \text{for each } n < m < \omega.$$

Then there exists a tuple $\bar a$ *such that*

$$\mathrm{tp}^{\mathrm{g}}(\bar a/A_n) = \mathrm{tp}^{\mathrm{g}}(\bar b_n/A_n), \quad \text{for each } n < \omega.$$

Proposition 4.6. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, superstable, finitary AEC. Let* A_i, $i < \omega$ *be an increasing sequence of finite sets and let* $\bar a_i$, $i < \omega$ *be finite tuples such that* $\mathrm{Lstp}(\bar a_{i+1}/A_i) = \mathrm{Lstp}(\bar a_i/A_i)$ *for each* $i < \omega$. *Then there exists* $n < \omega$ *such that* $\bar a_{n+1} \downarrow_{A_n} A_{n+1}$.

Proof. We assume towards a contradiction, that A_i and $\bar a_i$ are as in the proposition, but $\bar a_{i+1} \not\downarrow_{A_i} A_{i+1}$ for each $i < \omega$.

We write $A = \bigcup_{j<\omega} A_j$.

We fix a bijection $\pi : \omega \to \omega \times \omega$ and denote by π_0 and π_1 the projections such that $\pi \circ \pi_i = \mathrm{Id}$. furthermore, we choose such bijection that $\pi(0) = (0, 0)$, the maps π_0 and π_1 are increasing and $\pi_1(i) \leq i$ for all $i < \omega$.

Then we will define countable models $\mathscr{A}_i$, with enumerations $\{a^i_j : j < \omega\}$, elements b_i, $i < \omega$ and automorphisms $f_i \in \mathrm{Aut}(\mathfrak{M}/A_i)$, $i < \omega$, such that

(1) $A_n \cup f_n(\mathscr{A}_n) \subset \mathscr{A}_{n+1}$

(2) $b_0,...,b_n \downarrow_{A_n} A\cup\bigcup_{i<\omega}\bar{a}_i$,
(3) $f_n\circ...\circ f_{\pi_1(n)}(a^{\pi_1(n)}_{\pi_0(n)})=b_n$ and
(4) f_n fixes A_n and the elements $b_i=f_i\circ...\circ f_{\pi_1(i)}(a^{\pi_1(i)}_{\pi_0(i)})$ for $i<n$.

For $n=0$, choose $\mathscr{A}_0$ a countable model containing A_0 with an enumeration $\{a^0_i : i<\omega\}$, and b_0 by simplicity and extension, realizing $\mathrm{tp}^{\mathrm{w}}(a^0_0/A_0)$ with $b_0\downarrow_{A_0} A\cup\bigcup_{i<\omega}\bar{a}_i$. Then let $f_0\in\mathrm{Aut}(\mathfrak{M}/A_0)$ map a^0_0 to b_0.

Assume we have defined $\mathscr{A}_i$ and b_i for $i\le n$. Let $\mathscr{A}_{n+1}$ be a countable model containing A_{n+1} and $f_n(\mathscr{A}_n)$ and choose an enumeration of $\mathscr{A}_{n+1}$ as above.

Now $f_n\circ...\circ f_{\pi_1(n+1)}(a^{\pi_1(n+1)}_{\pi_0(n+1)})$ is an element in $f_n\circ...\circ f_{\pi_1(n+1)}(\mathscr{A}_{\pi_1(n+1)})$, where we extend this impression to denote $a^{n+1}_{\pi_0(n+1)}$ and $\mathscr{A}_{n+1}$ if $\pi_1(n+1)=n+1$. Then by simplicity,

$$f_n\circ...\circ f_{\pi_1(n+1)}(a^{\pi_1(n+1)}_{\pi_0(n+1)})\downarrow_{(A_{n+1}\cup b_0,...,b_n)} A_{n+1}\cup b_0,...,b_n,$$

and hence by extension, there is b_{n+1} realizing $\mathrm{tp}^{\mathrm{w}}(f_n\circ...\circ f_{\pi_1(n+1)}(a^{\pi_1(n+1)}_{\pi_0(n+1)})/A_{n+1}\cup b_0,...,b_n)$ such that

$$b_{n+1}\downarrow_{(A_{n+1}\cup b_0,...,b_n)} A\cup\bigcup_{i<\omega}\bar{a}_i.$$

Then item 2 follows by symmetry and transitivity. Furthermore, let f_{n+1} be an automorphism mapping $f_n\circ...\circ f_{\pi_1(n+1)}(a^{\pi_1(n+1)}_{\pi_0(n+1)})$ to b_{n+1} and fixing $b_0,...,b_n\cup A_{n+1}$. Now we are done with the construction.

We claim that the set $\mathscr{B}=\bigcup_{i<\omega}b_i$ is a model. For each $n<\omega$ the set

$$B_n=\bigcup_{\pi_1(i)=n}b_i=\bigcup_{\pi_1(i)=n}f_i\circ f_{i-1}\circ...\circ f_{n+1}\circ f_n(a^n_{\pi_0(i)})$$

is a model, since by the finite character property of $\mathbb{K}$, the increasing partial maps

$$f_i\circ f_{i-1}\circ...\circ f_{n+1}\circ f_n\restriction(a^n_0,...,a^n_{\pi_0(i)}),\ \pi_1(i)=n,$$

extend to an automorphism mapping $\mathscr{A}_n$ to B_n. Furthermore, since $f_n(\mathscr{A}_n)\subseteq\mathscr{A}_{n+1}$, we have that each $f_n(a^n_i)$ is a^{n+1}_j for some j and hence $B_n\subseteq B_{n+1}$ for each $n<\omega$. Then $\mathscr{B}=\bigcup_{i<\omega}B_i$ is a model as a union of an increasing chain of models.

Secondly we claim that $\mathscr{B}$ contains A. It is enough to show that it contains each A_i. For a given $n<\omega$, A_n is included in $\mathscr{A}_n$ as some $a^n_{i_0},...,a^n_{i_k}$. But for all $m\ge n$, f_m fixes A_n pointwise, and hence $b_m=a^n_{i_p}$ for m such that $\pi(m)=(i_p,n)$.

Then we define $B_n=A_n\cup b_0,...,b_n$ for each $n<\omega$. Now since $\mathrm{Lstp}(\bar{a}_{n+1}/A_n)=\mathrm{Lstp}(\bar{a}_n/A_n)$ and $\bar{a}_n,\bar{a}_{n+1}\downarrow_{A_n}B_n$ by 2, monotonicity and finite symmetry, stationarity of finite Lascar types implies that $\mathrm{Lstp}(\bar{a}_{n+1}/B_n)=\mathrm{Lstp}(\bar{a}_n/B_n)$ for each $n<\omega$. Since $\bigcup_{n<\omega}B_n=\mathscr{B}$ is a model, by Lemma 4.5 there exists $\bar{a}$ realizing each $\mathrm{Lstp}(\bar{a}_n/B_n)$.

Finally we show that $\bar{a}\not\downarrow_{B_n}B_{n+1}$ for each $n<\omega$. This will contradict superstability, and then we are done with the proof.

Let $n<\omega$. By the construction and monotonicity,

$$B_n\downarrow_{A_n}\bar{a}_{n+1}.$$

However, since $\bar{a}_{n+1}\not\downarrow_{A_n}A_{n+1}$, we get that $\bar{a}_{n+1}\not\downarrow_{A_n}B_{n+1}$ by monotonicity. Since $\bar{a}_{n+1}\downarrow_{A_n}B_n$ by symmetry, transitivity implies that $\bar{a}_{n+1}\not\downarrow_{B_n\cup A_n}B_{n+1}$. But now since $A_n\subseteq B_n\subseteq B_{n+1}$ and $\bar{a}\models\mathrm{Lstp}(\bar{a}_{n+1}/B_{n+1})$, it follows that $\bar{a}\not\downarrow_{B_n}B_{n+1}$. This proves the claim and hence we are done with the proof. □

Now *local character* of $\downarrow$ is a straightforward corollary using finite character of $\downarrow$.

Corollary 4.7 (Local character)**.** *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, superstable, finitary AEC. Let* A *be any set and* $\bar{a}$ *a finite tuple. There is a finite subset* $D \subset A$ *such that* $\bar{a} \downarrow_D A$*.*

Proof. Assume that $\bar{a}$ and A are a counterexample. Then using finite character of $\downarrow$ we can define an increasing sequence of finite subsets A_i such that $A_0 = \emptyset$ and $\bar{a} \not\downarrow_{A_i} A_{i+1}$ for each $i < \omega$. This contradicts the previous proposition. □

Then as in [10], we get the full list of properties of independence. This theorem improves Theorem 3.13 of [10], since we can omit the assumption 'Tarski-Vaught property'. The proof is identical, using the restricted properties and Local character. We list also a stronger version of superstability here to note that under the other assumptions (especially weak stability), this stronger version is equivalent with Definition 4.4.

Theorem 4.8. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is simple and superstable finitary AEC. Let* A, B, C *and* D *be bounded subsets of the Monster model. Then the relation* $\downarrow$ *has the following properties.*

(1) ***Invariance:*** *If* $A \downarrow_C B$ *and* f *is an automorphism of the monster model, then* $f(A) \downarrow_{f(C)} f(B)$*.*
(2) ***Monotonicity:*** *If* $A \downarrow_B D$ *and* $B \subset C \subseteq D$ *then* $A \downarrow_C D$ *and* $A \downarrow_B C$*.*
(3) ***Transitivity:*** *Let* $B \subseteq C \subseteq D$*. If* $A \downarrow_B C$ *and* $A \downarrow_C D$*, then* $A \downarrow_B D$*.*
(4) ***Symmetry:*** $A \downarrow_C B$ *if and only if* $B \downarrow_C A$*.*
(5) ***Extension:*** *For any* $\bar{a}$ *and* $C \subseteq B$ *there is* $\bar{b}$ *such that* $\mathrm{Lt}(\bar{b}/C) = \mathrm{Lt}(\bar{a}/C)$ *and* $\bar{b} \downarrow_C B$*.*
(6) ***Finite character:*** $A \downarrow_C B$ *if and only if* $\bar{a} \downarrow_C \bar{b}$ *for every finite* $\bar{a} \in A$ *and* $\bar{b} \in B$*.*
(7) ***Local character:*** *For any finite* $\bar{a}$ *and any* B *there exists a finite* $E \subseteq B$ *such that* $\bar{a} \downarrow_E B$*.*
(8) ***Reflexivity:*** *The weak type* $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A)$ *is not bounded if and only if* $\bar{a} \not\downarrow_A \bar{a}$*.*
(9) ***Stationarity:*** *If* $\mathrm{Lt}(\bar{a}/C) = \mathrm{Lt}(\bar{b}/C)$*,* $\bar{a} \downarrow_C B$ *and* $\bar{b} \downarrow_C B$*, then* $\mathrm{Lt}(\bar{a}/B) = \mathrm{Lt}(\bar{b}/B)$*.*
(10) ***Superstability:*** *For any increasing sequence of finite sets* A_i*,* $i < \omega$*, and any finite sequence* $\bar{a}$*, there is* $n < \omega$ *with* $\bar{a} \downarrow_{A_n} A_{n+1}$*.*

The proof of Local character is the first place where we use the finite character property of $\preccurlyeq_{\mathbb{K}}$ and it is not needed elsewhere in the proof of Theorem 4.8. Hence if we would strengthen the assumption 'simplicity' or 'superstability' to imply local character for arbitrary sets (not only for finite sets and models), we would get the analogue of Theorem 4.8 without using the property finite character of $\preccurlyeq_{\mathbb{K}}$. However, we use finite character of $\preccurlyeq_{\mathbb{K}}$ again to prove Theorem 5.2. We formulate the following remark.

Remark 4.9. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is an AEC with amalgamation, joint embedding and arbitrarily large models. Furthermore assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is weakly stable and that for every tuple* $\bar{a}$ *and* every *set* A*, there is finite* $E \subset A$ *such that* $\bar{a} \downarrow_E A$*. Then the notion* $\downarrow$ *satisfies all the properties listed in Theorem 4.8.*

But we note that Local character is a strong property, for example it is *not* implied by categoricity alone. However it does follow from categoricity in any uncountable cardinal if we assume also simplicity, $\mathrm{LS}(\mathbb{K}) = \aleph_0$ and finite character of $\preccurlyeq_{\mathbb{K}}$. Actually then weak $\aleph_0$-stability implies superstability, see Hyttinen and Kesälä [10].

5. Lascar types and Galois types

In this section $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple, superstable, finitary AEC. We show that equality of Lascar types over a countable set implies the equality of Galois types. The proof uses the independence calculus provided by Theorem 4.8, but it also uses heavily the assumptions that $LS(\mathbb{K}) = \aleph_0$ and that $\preccurlyeq_{\mathbb{K}}$ has finite character. This theorem improves Theorem 3.19 of Hyttinen and Kesälä [10], since we do not need the Tarski-Vaught property. At first we prove a lemma.

Lemma 5.1. *Let A be any set and $\bar{a}, \bar{b}, \bar{c}$ finite tuples such that $\mathrm{Lt}(\bar{a}/A) = \mathrm{Lt}(\bar{b}/A)$, $\bar{a}\bar{c} \downarrow_{A_0} A$ and $\bar{b} \downarrow_{A_0} A$ for some finite set $A_0 \subseteq A$. Then there exists $\bar{d}$ such that $\bar{b}\bar{d} \models \mathrm{Lt}(\bar{a}, \bar{c}/A)$ and $\bar{b}, \bar{d} \downarrow_{A_0} A$.*

Proof. By extension, there are $\bar{a}', \bar{c}'$ realizing $\mathrm{Lt}(\bar{a}, \bar{c}/A_0)$ such that

$$\bar{a}', \bar{c}' \downarrow_{A_0} \bar{a}, \bar{c}, \bar{b}, A.$$

By restricted symmetry and monotonicity, $\bar{b} \downarrow_{A_0 \cup A} \bar{a}', \bar{c}'$ and then by transitivity, $\bar{b} \downarrow_{A_0} \bar{a}', \bar{c}', A$.

By simplicity, $\bar{c} \downarrow_{A_0 \cup \bar{a}', \bar{c}', \bar{b}} A_0 \cup \bar{a}', \bar{c}', \bar{b}$. Again by extension, there is $\bar{d}$ realizing $\mathrm{Lt}(\bar{c}/A_0 \cup \bar{a}', \bar{c}', \bar{b})$ such that $\bar{d} \downarrow_{A_0 \cup \bar{a}', \bar{c}', \bar{b}} A$. Then by invariance, monotonicity and symmetry $\bar{d} \downarrow_{A_0 \cup \bar{b}} \bar{a}', \bar{c}'$ and furthermore by transitivity, $\bar{d} \downarrow_{A_0 \cup \bar{b}} \bar{a}', \bar{c}', A$. By restricted symmetry and transitivity,

$$\bar{d}, \bar{b} \downarrow_{A_0} \bar{a}', \bar{c}', A.$$

Since $\bar{b}, \bar{d} \models \mathrm{Lt}(\bar{a}', \bar{c}'/A_0) = \mathrm{Lt}(\bar{a}, \bar{c}/A_0)$, we get that $\bar{b}, \bar{d} \models \mathrm{Lt}(\bar{a}, \bar{c}/A)$ by stationarity. □

Theorem 5.2. *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple, superstable, finitary AEC. Let A be a countable set and let $\bar{a}$, $\bar{b}$ be finite tuples such that $\mathrm{Lt}(\bar{a}/A) = \mathrm{Lt}(\bar{b}/A)$. Then there is $f \in \mathrm{Aut}(\mathfrak{M}/A)$ such that $f(\bar{a}) = \bar{b}$.*

Furthermore, if p_i, $i < \omega$, are countably many Lascar types over subsets $D_i \subseteq A$, we can choose f such that $f(p_i) = p_i$ for all $i < \omega$.

Proof. We prove the second claim. We construct A_j $j < \omega$, an increasing sequence of finite subsets of A and for each $j < \omega$ we choose finite tuples $\bar{a}_j, \bar{b}_j$ and two increasing sequences of countable models $\mathscr{B}_j, \mathscr{C}_j$ containing A as follows:

1. $\bar{a} = \bar{a}_0$, $\bar{b} = \bar{b}_0$ and $\mathscr{B}_0 = \mathscr{C}_0$ is some countable model containing A.
2. $\bigcup_{j<\omega} \bar{a}_j = \bigcup_{j<\omega} \mathscr{B}_j$, $\bigcup_{j<\omega} \bar{b}_j = \bigcup_{j<\omega} \mathscr{C}_j$ and $A = \bigcup_{j<\omega} A_j$
3. For each $j < \omega$, $\bar{a}_0, ..., \bar{a}_j \downarrow_{A_j} A$ and $\bar{b}_0, ..., \bar{b}_j \downarrow_{A_j} A$.
4. The types $p_i, i < \omega$ are realized in each $\mathscr{B}_j$ for $j > 0$.
5. For each finite $A_j \subset A$ and $j < \omega$ there is a Lascar automorphism $f \in \mathrm{Laut}(\mathfrak{M}/A_j)$ such that $f(\bar{a}_0, ..., \bar{a}_j) = \bar{b}_0, ..., \bar{b}_j$.

First we show how this construction implies the claim. By 5 we have Lascar automorphisms $f_j \in \mathrm{Laut}(\mathfrak{M}/A_j)$ mapping $\bar{a}_0, ..., \bar{a}_j$ to $\bar{b}_0, ..., \bar{b}_j$ for each $j < \omega$. The union

$$f_j \restriction (A_j \cup \bar{a}_0, ..., \bar{a}_j) \to A_j \cup \bar{b}_0, ..., \bar{b}_j$$

defines and isomorphism between $\bigcup_{j<\omega} \mathscr{B}_j$ and $\bigcup_{j<\omega} \mathscr{C}_j$ fixing A pointwise and mapping $\bar{a}$ to $\bar{b}$. Furthermore, this map is a $\preccurlyeq_{\mathbb{K}}$-embedding by finite character of $\mathbb{K}$ and hence it extends to an automorphism $f \in \mathrm{Aut}(\mathfrak{M}/A)$.

We need to show that is preserves the types p_i. For this, let $\bar{d}$ in $\mathfrak{M}$ realize p_i for some $i < \omega$, where p_i is a type over a subset $D_i \subset A$. By local character, there is $A' \subset D_i$ such that $\bar{d}, f(\bar{d}) \downarrow_{A'} D_i$. By 4 the type p_i is realized in the union of the models $\mathscr{B}_j$. Let $n < \omega$ be an index such that $A' \subset A_n$ and $\bar{e} \subset \bar{a}_0, ..., \bar{a}_n$ realizes p_i. But now $\mathrm{Lt}(\bar{d}/A_n) = \mathrm{Lt}(\bar{e}/A_n)$ implies that $\mathrm{Lt}(f(\bar{d})/A') = \mathrm{Lt}(f(\bar{e})/A')$, and since f extends the Lascar automorphism f_n on $\bar{e} \cup A'$,

$$\mathrm{Lt}(f(\bar{d})/A') = \mathrm{Lt}(f_n(\bar{e})/A') = \mathrm{Lt}(\bar{e}/A') = \mathrm{Lt}(\bar{d}/A').$$

Then by stationarity we get that $\mathrm{Lt}(f(\bar{d})/D_i) = \mathrm{Lt}(\bar{d}/D_i) = p_i$, and hence we have shown that f preserves p_i.

Then we show how to construct the required sets. We choose an enumeration a'_j with $\{a'_j : j < \omega\} = A$ and will require that $a'_j \in A_j$ for each $j < \omega$. Furthermore, we will construct enumerations b^i_j, c^i_j for $j > 0$ such that $\{b^i_j : i < \omega\} = \mathscr{B}_j \setminus \mathscr{B}_{j-i}$ and $\{c^i_j : i < \omega\} = \mathscr{C}_j \setminus \mathscr{C}_{j-i}$ if these sets are nonempty.

By local character, we can choose A_0 containing a'_0 such that 1 and 3 hold. Now assume we have constructed the required sets and enumerations for $j < 2n$.

Let $\mathscr{B}_{2n+1} = \mathscr{B}_{2n}$. Let $\bar{a}_{2n+1}$ be a finite tuple containing b^i_j for $i, j \leq 2n+1$. By local character, there is a finite $A_{2n+1} \subset A$ containing a'_{2n+1} and A_{2n} such that

$$\bar{a}_0, ..., \bar{a}_{2n+1} \downarrow_{A_{2n+1}} A \text{ and } \bar{b}_0, ..., \bar{b}_{2n} \downarrow_{A_{2n+1}} A.$$

Then by Lemma 5.1, there is $\bar{b}_{2n+1}$ such that $\bar{b}_0, ..., \bar{b}_{2n}, \bar{b}_{2n+1}$ realizes

$$\mathrm{Lt}(\bar{a}_0, ..., \bar{a}_{2n}, \bar{a}_{2n+1}/A)$$

and 3 holds. Then we choose $\mathscr{C}_{2n+1}$ be some countable model containing $\mathscr{C}_{2n} \cup \bar{b}_0, ..., \bar{b}_{2n+1}$ and choose some enumeration of $\mathscr{C}_{2n+1} \setminus \mathscr{C}_{2n}$ as $\{c^i_{2n+1} : i < \omega\}$.

For $2n+1$ we do vice versa, let $\mathscr{C}_{2n+2} = \mathscr{C}_{2n+1}$ and let $\bar{b}_{2n+2}$ contain c^i_j for $i, j \leq 2n+2$. Then A_{2n+2} and $\bar{a}_{2n+2}$ are given by local character and Lemma 5.1. Finally we let $\mathscr{B}_{2n+2}$ be some countable model realizing each p_i and containing $\mathscr{B}_{2n+1} \cup \bar{a}_0, ..., \bar{a}_{2n+1}$ and choose the enumeration of $\mathscr{B}_{2n+2} \setminus \mathscr{B}_{2n+1}$. We are done with the construction. □

The following corollary applies if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is '$\aleph_0$-Lascar-stable'. However, we do not know if $\aleph_0$-stability with respect to weak types or Galois types implies $\aleph_0$-stability with respect to Lascar types in general. It does follow from weak $\aleph_0$-stability if $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is in addition categorical in some uncountable cardinal, *tame* or has *the extension property for splitting*, see Hyttinen and Kesälä [9].

Corollary 5.3. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, superstable, finitary AEC. Let* A *be a countable set and assume there are only countably many Lascar types over* A*. Let* $\bar{a}$*,* $\bar{b}$ *be finite tuples such that* $\mathrm{Lt}(\bar{a}/A) = \mathrm{Lt}(\bar{b}/A)$*. Then there is* $f \in \mathrm{Laut}(\mathfrak{M}/A)$ *such that* $f(\bar{a}) = \bar{b}$*.*

With only the results in this paper we cannot replace the Tarski-Vaught property in the results of section 4 of Hyttinen and Kesälä [10], since the property is there used to construct a-primary models. We can construct sets which are a-saturated and constructible with respect to the given notion of a-isolation, but without the Tarski-Vaught property we don't know if those sets are actually $\preccurlyeq_{\mathbb{K}}$-elementary submodels of the monster model. Especially we still need the property to show the 'a-categoricity transfer theorem' Theorem 4.15. However, we could replace the Tarski-Vaught property with an assumption that all a-saturated subsets of the monster are actually $\preccurlyeq_{\mathbb{K}}$-elementary submodels of the monster. We state this as a remark.

Remark 5.4. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, superstable finitary AEC. Assume that any a-saturated subset of* $\mathfrak{M}$ *is an* $\preccurlyeq_{\mathbb{K}}$*-elementary substructure of* $\mathfrak{M}$. *Then all results of Hyttinen and Kesälä* [10] *are true in* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *even if* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *does not have the Tarski-Vaught property.*

5.1. **Example.** We present an example of a simple, superstable, finitary AEC where weak type over a countable model is different of Galois type over a countable model. This shows that Theorem 5.2, is in a sense 'best possible', since we cannot determine Galois type from weak type alone, not even over countable models. We use ideas from Baldwin and Kolesnikov [3], where there are examples of classes where weak type does not determine Galois type, especially over bigger models even when the class is $\aleph_0$-stable. However, those examples are not simple.

Example 5.5. *Let* G *be* $\oplus_{i<\omega}(\mathbb{Z}/2\mathbb{Z})$, *i.e. the group of finite support functions from* ω *to* $\{0,1\}$ *with addition. We choose the vocabulary* L *to consist of unary predicate symbols* S, T *and* P_n *for* $n < \omega$, *unary function symbols* f_a *for each* $a \in G$ *and binary function symbols symbols* R_n, $n < \omega$. *We define the function to be partial, but they can also be defined total functions by requiring that they are identities or projections elsewhere.*

We describe the following properties of an L*-structure* M.

(1) *Each unary predicate is disjoint and each* P_n^M *consists of exactly two elements.*
(2) *If* T^M *is nonempty, then* S^M *is nonempty.*
(3) *If* S^M *is nonempty, the functions* $f_a^M : S^M \to S^M$ *determine a regular action of* G *on* S^M.
(4) *The functions* $R_n^M : T^M \times S^M \to P_n^M$ *are such that*
   (a) *For all* $x \in T^M$, $y \in S^M$ *and* $a \in G$,
$$R_n^M(x,y) = R_n^M(x, f_a^M(y)) \text{ if and only if } a(n) = 0.$$
   (b) *For all* $x \in T^M$ *and* $y \in S^M$ *and* $X \subseteq \omega$ *there is at most one* $z \in T^M$ *such that*
$$\{n \in \omega : R_n^M(z,y) = R_n^M(x,y)\} = X.$$

Also elements outside the predicates are allowed. We define $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *to be the class of structures* M *satisfying the properties above and where* $\preccurlyeq_{\mathbb{K}}$ *is the substructure relation.*

We note that the previous properties can be written as axioms in $L_{\omega_1\omega}$. We show that this is an abstract elementary class by describing the models of $\mathbb{K}$ up to isomorphism. First let us consider the following structure $\mathfrak{M}$:

- $P_n^{\mathfrak{M}} = \{n\} \times \{0,1\}$,
- $S^{\mathfrak{M}} = \{0\} \times \{\mu : \omega \to \{0,1\} : \ \mu \text{ has finite support}\}$,
- $T^{\mathfrak{M}} = \{1\} \times \{\eta : \omega \to \{0,1\}\}$,
- for each $a \in G$, $f_a^{\mathfrak{M}}(\mu)(i) = \mu(i) + a(i)$ and
- for each $n < \omega$, $R_n^{\mathfrak{M}}(\mu, \eta) = (n, \mu(n) + \eta(n))$.
- Furthermore, there are κ many elements outside the predicates, where κ is some cardinal $\geq 2^{\aleph_0}$.

When referring to $T^{\mathfrak{M}}$ and $S^{\mathfrak{M}}$, we leave out the first coordinate, since it is in the definition only to make the two sets disjoint.

We claim that this structure is in the class $\mathbb{K}$ and furthermore that each structure of size $\leq \kappa$ in $\mathbb{K}$ is isomorphic to a substructure of $\mathfrak{M}$.

Clearly the structure $\mathfrak{M}$ satisfies the given properties. We let M be a structure in $\mathbb{K}$ of size $\leq \kappa$ and claim that there is an isomorphism f from M to a substructure of $\mathfrak{M}$. If T^M and S^M are empty, this is clearly the case. We fix a bijection between the unions

$$\bigcup_{n<\omega} P^M \text{ and } \bigcup_{n<\omega} (\{n\} \times \{0,1\})$$

and may assume that it is the identity. If S^M is nonempty and $\mu_0 \in S^M$, the functions f_a generate an isomorphism between S^M and $S^{\mathfrak{M}}$ mapping μ_0 to $\widehat{0}$, the function identically zero. Then we map each η in T^M to a function $\eta' \in T^{\mathfrak{M}}$ such that

$$R_n^M(\eta, \mu_0) = R_n^{\mathfrak{M}}(\eta', \widehat{0}) \text{ for each } n < \omega.$$

By 4(b), f is injective on T^M. Furthermore, the elements in M outside the predicates are mapped injectively to some subset of elements of $\mathfrak{M}$ outside the predicates. Clearly the result function is an embedding.

Now we can describe the models M in $\mathbb{K}$ up to isomorphism. The smallest model consists of only the predicates P_n^M, $n < \omega$. Then we can have that T^M is empty, but S^M is isomorphic to $S^{\mathfrak{M}}$. If T^M is nonempty, it is isomorphic to some substructure of $T^{\mathfrak{M}}$. In addition, there can be none or arbitrarily many elements outside the predicates. Furthermore, the model $\mathfrak{M}$ is a monster model for the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$.

Clearly the class has joint embedding, arbitrarily large models, $LS(\mathbb{K}) = \aleph_0$ and $\preccurlyeq_{\mathbb{K}}$ has finite character. Amalgamation is immediate by the proof that every member of $\mathbb{K}$ embedds to $\mathfrak{M}$.

We show that the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is simple. Clearly the weak type in $S^w_{\mathfrak{M}}(\emptyset)$ of an element in any of the predicates is bounded. Hence

$$\mathrm{bcl}(\emptyset) = \{\bar{a} \in \mathfrak{M} : \mathrm{tp}^{\mathrm{w}}(\bar{a}/\emptyset) \text{ is bounded}\}$$

consists of all the elements in the predicates. By the definition of independence, always $\bar{a} \downarrow_{\emptyset} \mathrm{bcl}(\emptyset)$. Thus we can describe the independence relation as follows:

$$\bar{a} \downarrow_A B \text{ if and only if } \bar{a} \cap (B \setminus (A \cup \mathrm{bcl}(\emptyset))) = \emptyset.$$

Hence the class is simple.

For the same reason there are no infinite sequences $(A_n)_{n<\omega}$ such that $\bar{a} \not\downarrow_{A_n} A_{n+1}$ would hold for each $n < \omega$. Furthermore, the class is clearly weakly stable in cardinals $\lambda \geq 2^{\aleph_0}$. Hence the class is superstable. We can also use Theorem 3.38 of Hyttinen and Kesälä [10]: if the class is a-categorical in big enough cardinals it is superstable. There a-categoricity in κ means that any two a-saturated models, i.e. models realizing all Lascar strong types over finite subsets, of size κ are isomorphic. Clearly if $\kappa > 2^{\aleph_0}$, an a-saturated model in $\mathbb{K}$ must contain the 'full' $T^{\mathfrak{M}}$ and κ many elements outside the predicates, hence the class is a-categorical in each $\kappa > 2^{\aleph_0}$.

Then we study the Lascar strong types of elements in the predicate $T^{\mathfrak{M}}$. We choose a model N to be the union of the predicates $P_n^{\mathfrak{M}}$. We notice that if η_2 realizes the strong type $\mathrm{Lstp}(\eta_1/\emptyset)$, we must have that $\eta_2 = \eta_1$ and hence also $\mathrm{tp}^{\mathrm{g}}(\eta_2/N) = \mathrm{tp}^{\mathrm{g}}(\eta_1/N)$.

However, the weak type $\mathrm{tp}^{\mathrm{w}}(\eta_1/N)$ does not determine $\mathrm{tp}^{\mathrm{g}}(\eta_1/N)$. If η_2 differs from η_1 in infinitely many $i < \omega$, it is not possible to find an automorphism $f \in \mathrm{Aut}(\mathfrak{M}/N)$ mapping η_2 to η_1. This is since when μ_0 denotes the identically zero function in $S^{\mathfrak{M}}$, we cannot find $\mu \in S^{\mathfrak{M}}$ such that $R_n(\eta_1, \mu)$ would get the same values than $R_n(\eta_2, \mu_0)$ for every $n < \omega$. However, for every finite subset of N we can find such an automorphism: Let A be contained in $P_0^{\mathfrak{M}}, ..., P_p^{\mathfrak{M}}$. There is $a \in G$ such that

$$R_n^{\mathfrak{M}}(\eta_1, \mu_0) = R_n^{\mathfrak{M}}(\eta_2, f_a^{\mathfrak{M}}(\mu_0)) \text{ for each } n \leq p.$$

Now the conditions

- $\eta_1 \mapsto \eta_2$,
- $\mu_0 \mapsto f_a^{\mathfrak{M}}(\mu_0)$ and
- $R_n^{\mathfrak{M}}(\eta_1, \mu_0) \mapsto R_n^{\mathfrak{M}}(\eta_2, f_a^{\mathfrak{M}}(\mu_0))$ for each $n < \omega$

determine an automorphism of $\mathfrak{M}$ fixing $P_0^{\mathfrak{M}}, ..., P_p^{\mathfrak{M}}$. Hence $\mathrm{tp}^{\mathrm{w}}(\eta_2/N) = \mathrm{tp}^{\mathrm{w}}(\eta_1/N)$ does not imply that $\mathrm{tp}^{\mathrm{g}}(\eta_2/N) = \mathrm{tp}^{\mathrm{g}}(\eta_1/N)$.

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