Interpreting Groups and Fields in Simple, Finitary AECs☆,☆☆

Tapani Hyttinen[a], Meeri Kesälä[a,*]

[a] *Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 (Gustaf Hllstrmin katu 2b), FI-00014 University of Helsinki, Finland*

Abstract

We prove a version of Hrushovski's 1989 results on almost orthogonal regular types in the context of simple and superstable finitary abstract elementary classes: from a certain expression of 'non-orthogonality' we can conclude the existence of a group acting on the geometry obtained on the set of realizations of a regular Lascar strong type, and if we rule out the presence of a non-classical group we can classify the situation to be one of the classical cases of Hrushovski's.

We give two examples of classes of structures in this framework, which clearly demonstrate the phenomena described in the main theorem.

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1. Introduction

The motivation for *geometric stability theory* encompasses the idea that one can recover fundamental mathematical structures from assumptions of purely 'logical' nature. A predecessor for such an idea can be found in the construction for interpreting a field in a Pappian (or a skew-field in a Desarquesian) projective plane, where the axioms for such a plane only consider, 'lines', 'points' and an incidence relation between them. This paper further studies this phenomenon.

Most of the research of geometric stability theory has been done for *elementary classes*, i.e. classes of structures definable in first order logic. Recently there have been several attempts to generalize the required stability-theoretical machinery to some non-elementary frameworks. The benefit of this agenda is not only to widen the concept of a 'class of stuctures' beyond first-order definable classes but also to analyse further the logical tools and ideas.

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*Corresponding author

Email addresses: `tapani.hyttinen@helsinki.fi` (Tapani Hyttinen), `meeri.kesala@helsinki.fi` (Meeri Kesälä)

We work in the framework of *abstract elementary classes* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ (AECs), where $\mathbb{K}$ is a class of structures of a given, countable, similarity type and $\preccurlyeq_{\mathbb{K}}$ is a relation between these structures, where the class and the relation are defined axiomatically. The axioms capture the most important properties of the *elementary substructure relation* between structures definable with a complete theory in first order logic, but allow many more possibilities.

Extending the ideas by Zilber [18], Hrushovski [6] showed that we can interpret a group on the geometry induced on the set of realizations of a regular type when working in a monster model of a stable elementary class and assuming a certain configuration concerning *orthogonality* holds for the type. Furthermore, he studied further this group and found it must fall into three categories, one of which resembles the case of the projective plane.

In the paper [13] Hyttinen, Lessmann and Shelah partially generalize the theorem by Hrushovski to certain non-elementary frameworks; to *homogeneous model theory* and atomic ω-stable *excellent classes*. This result showed that it is possible to do geometric stability theory without *compactness*, a crucial property of elementary model theory not available in most non-elementary frameworks, and that the stability-theoretic machinery developed for these particular frameworks is adequate and exploitable. In this paper we want to investigate further this approach: how much of this work can be carried out without any trace of compactness (for example induced by homogeneity), but only with an appropriate independence calculus, which is available in *simple finitary AECs*? This is a natural question since we should be able to work with only *geometric* tools. Furthermore, we want to prove the theorem in the context of *regular* types, while Hyttinen, Lessmann, Shelah [13] only worked with *quasiminimal* types. Our result is thus analogous to the first order result due to Hrushovski. For more history of geometric stability theory and this particular problem, see the introduction of Hyttinen, Lessmann and Shelah [13].

Our main results is the following.

Theorem 1.1. *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple, superstable finitary AEC and let $\mathfrak{M}$ be the monster model for $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. Assume that A is a finite set, p is an unbounded regular Lascar strong type over A and $\mathbf{Q}$ is an A-invariant subset of $\mathfrak{M}$. Assume that there exists an integer $0 < n < \omega$ such that*

1. *For any independent sequence $(a_1, ..., a_n)$ of realizations of p and any finite subset C of $\mathbf{Q}$ we have*

$$\dim(a_1, ..., a_n/A) = \dim(a_1, ..., a_n/A \cup C).$$

2. *For some independent sequence $a_1, ..., a_{n+1}$ of realizations of p there is C a finite subset of $\mathbf{Q}$ such that $(a_1, ..., a_n)$ dominates $(a_1, ..., a_{n+1})$ over $A \cup C$.*

Then $\mathfrak{M}$ interprets a group G which acts on the geometry $\mathbf{P}/E$ induced on the set $\mathbf{P}$ of realizations of p. Furthermore, either $\mathfrak{M}$ interprets a nonclassical group or $n \in \{1, 2, 3\}$ and

- *If $n = 1$, then G is abelian and acts regularly on $\mathbf{P}/E$.*
- *If $n = 2$, the action of G on $\mathbf{P}/E$ is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebrically closed field K.*
- *If $n = 3$, the action of G on $\mathbf{P}/E$ is isomorphic to the action of $PGL_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

The main difference to Hrushovski's theorem is that we cannot rule out the possibility of a *nonclassical group* - a nonabelian group, which carries an ω-homogeneous geometry. The existence of such groups is an open question. The properties of these groups were studied in Hyttinen [8] and in [13].

Another difference is how the group interpreted in the monster model reflects to smaller models of the class. In elementary classes, the structures of the class are elementary substructures of the monster model and hence their structure is very similar. In abstract elementary classes, the relation of the structures and the monster is given as an abstract relation $\preccurlyeq_{\mathbb{K}}$, which we might not know so much about. Hence the interpreted objects are given in every member of the class $\mathbb{K}$, but their properties might change. In Example 7.6 of this paper, the monster model interprets an algebraically closed field, but the field interpreted in some $\preccurlyeq_{\mathbb{K}}$-substructure can be any field of given characteristic.

We now explain further the chosen framework. Abstract elementary classes are a standard framework for extending first order model theory. As explained, we want to work with regular types and we want to reduce the homogeneity assumptions in [13]. The price we pay are assumptions of *simplicity, finite character* and *domination* for finite sequences. We chose the framework of simple, finitary AECs for the following reasons: In these classes, the only homogeneity assumption is the amalgamation property. This is much less than, for example, in excellent classes, since we cannot conclude tameness. However, we assume *finite character* for the elementary substructure relation. This connects the Galois types of finite sequences of a model to the elementary submodel relation and gives many tools to manage types over finite and countable sets. This assumption holds for classes definable in $L_{\infty,\omega}$, with $\preccurlyeq_{\mathbb{K}}$ given in the corresponding fragment of $L_{\infty,\omega}$, and there is a fundamental connection between this property and definability, see Kueker [14].

Another assumption is simplicity. In non-elementary classes, it is not guaranteed that even categoricity in all uncountable cardinals would imply the existence of a well-behaved notion of independence, see Hyttinen and Kesälä [9]. Since geometric stability theory studies the applications of an independence calculus, it seems reasonable to assume simplicity to guarantee that we have such a calculus. Notice also that if a type p over A is regular, then this guarantees that the type is also simple, i.e. for any set $B \supseteq A$ and a realizing p, the type of a over B is free from the empty set. Examples of stable but not simple classes are few, see Hyttinen and Lessmann [12] and Baldwin and Kolesnikov [3]. In elementary classes simplicity follows from stability. The properties of types and the independence calculus needed are listed in section 2 and the rest of the paper relies only on these properties, not the details of the definition of the framework.

Our third assumption in need of further explaining is item 2 of the main Theorem. Instead of just assuming that we find a finite subset C of $\mathbf{Q}$ such that $\dim(a_1,...,a_{n+1}/C) = n$, we assume that C witnesses the n-subsequence $a_1,...,a_n$ *dominating* the sequence $a_1,...,a_{n+1}$. This stronger assumption is needed in order to analyse the structure of the interpreted group G, and we don't know how to prove the theorem without the assumption. To be precise, domination is needed to show that if $g \in \mathbf{G}$ is *generic* over X, then it is 'free' of X, see the proof of Proposition 5.11. When p is quasiminimal, we get domination from just $\dim(a_1,...,a_{n+1}/C) = n$, hence this stronger form is not needed in [13]. In the first order theorem by Hrushovski [6], the sets $\mathbf{P}$ and $\mathbf{Q}$ are modified using tools available in $\mathfrak{M}^{eq}$ to get this and more. We have not yet developed the machinery of $\mathfrak{M}^{eq}$ for simple finitary AECs, although that could very likely be done. But now, as also in [13], we work in the original context.

The main examples of finitary AECs are excellent and homogeneous classes, which both origin form a 'model-theoretic' background, but we hope this framework turns out fruitful to study also classes arising 'outside' model-theory. Some examples of such finitary AECs are covers of multiplicative groups of algebraically closed fields, see Bays and Zilber [4], and classes induced by tilting and cotilting modules, see Baldwin, Eklof and Trlifaj [2], [17], where the latter are not homogeneous or excellent. In the last section of this paper we give two examples of classes in the framework of the main theorem. However, these classes are 'model-theoretic' and illustrate the phenomena of the main theorem, not so much 'practical applications' of the theorem. However, Example 7.6 is the class of Pappian projective planes of given characteristic, and hence binds the theorem to the history of geometric stability theory.

2. The framework

Finitary abstract elementary classes, specializations of Shelah's Abstract Elementary Classes [16], were introduced in Hyttinen, Kesälä [7], but there the definition was slightly less general than in the consequent papers Hyttinen, Kesälä [10], [9] and [11]. A finitary AEC is an abstract elementary class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ with a countable Löwenheim-Skolem number, amalgamation, joint embedding, arbitrarily large models and *finite character:*[1] For any two models $N, M \in \mathbb{K}$ with $N \subseteq M$, we have that

$$N \preccurlyeq_{\mathbb{K}} M \text{ iff}$$

$$\text{for every finite sequence } \bar{a} \in N \text{ there is a } \mathbb{K}\text{-embedding } f : N \to M \text{ fixing } \bar{a}.$$

We work inside $\mathfrak{M}$, which is the the κ-universal and κ-model homogeneous monster model of the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. We say that a subset $A \subset \mathfrak{M}$ is *bounded*, if $|A| < \kappa$. We assume that κ is sufficiently large.

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[1] This formulation of finite character is due to Kueker [14].

We can define a notion of a weak type tp^{w} and Lascar splitting and then deduce a notion of independence $\downarrow$ with built- in extension as follows:

$$A \downarrow_B C$$

if for every finite sequence $\bar{a} \in A$ there is a finite set $E \subseteq B$ such that for every extension $D \supseteq B \cup C$ there is $\bar{a}'$ realizing the weak type $\mathrm{tp}^{\mathrm{w}}(\bar{a}/B \cup C)$ such that $\mathrm{tp}^{\mathrm{w}}(\bar{a}'/D)$ does not Lascar- split over E. Then we say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is *simple*, if for every sequence and every *finite* set A,

$$\bar{a} \downarrow_A A.$$

In this paper work in the context of simple, finitary AECs $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ which are superstable in the following sense

Assumption 2.1 (Superstability). *The class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is weakly stable in some cardinal and there is no finite tuple $\bar{a}$ and an increasing sequence of finite sets A_i, $i < \omega$ such that*

- $\bar{a} \not\downarrow_{A_i} A_{i+1}$ *for each $i < \omega$ and*
- $\bigcup_{i<\omega} A_i$ *is a model.*

This notion of superstability is implied by $\aleph_0$-stability with respect to weak types (See Corollary 3.28 of [10]) and therefore also from categoricity in any uncountable cardinal. It also follows from a weaker form of categoricity, so called *a-categoricity* in a suitable cardinal, see [10]. Both implications use simplicity.

2.1. Lascar types and independence

We recall the notion of a *Lascar strong type* and *Lascar type*. Two finite tuples $\bar{a}$ and $\bar{b}$ have the same Lascar strong type over a bounded set C, written $\mathrm{Lstp}(\bar{a}/C) = \mathrm{Lstp}(\bar{b}/C)$ if $E(\bar{a}, \bar{b})$ holds for any C-invariant equivalence relation E with a bounded number of classes. An automorphism which preserves all Lascar strong types over A is called a *strong* automorphism. The group of these automorphisms is denoted by $\mathrm{Saut}(\mathfrak{M}/A)$, it is a normal subgroup of $\mathrm{Aut}(\mathfrak{M}/A)$ and we can show that $\mathrm{Lstp}(\bar{a}/A) = \mathrm{Lstp}(\bar{b}/A)$ if and only if there is $f \in \mathrm{Saut}(\mathfrak{M}/A)$ mapping $\bar{a}$ to $\bar{b}$.

Two tuples $\bar{a}$ and $\bar{b}$ have the same Lascar type over C, written $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C)$ if they have the same Lascar strong type over every *finite* subset C_0 of C, or equivalently, we have strong automorphisms $f \in \mathrm{Saut}(\mathfrak{M}/C_0)$ mapping $\bar{a}$ to $\bar{b}$ for any finite subset $C_0 \subseteq C$. Clearly if C is finite, $\mathrm{Lstp}(\bar{a}/C)$ equals $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C)$. For details about Lascar types in finitary AECs, see Hyttinen and Kesälä [11].

The following theorem is proved in [11]. We list also a stronger form of superstability, which will be used in the paper, although it is a straightforward application of the properties local character and finite character. A similar list of properties is stated in [10], but with an additional assumption called the 'Tarski-Vaught property'. In [11] the authors notice that this assumption is not needed.

Theorem 2.2. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is simple and superstable. Let* A, B, C *and* D *be bounded subsets of the monster model. Then the relation* $\downarrow$ *has the following properties.*

1. ***Invariance:*** *If* $A \downarrow_C B$ *and* f *is an automorphism of the monster model, then* $f(A) \downarrow_{f(C)} f(B)$.
2. ***Monotonicity:*** *If* $A \downarrow_B D$ *and* $B \subset C \subseteq D$ *then* $A \downarrow_C D$ *and* $A \downarrow_B C$.
3. ***Transitivity:*** *Let* $B \subseteq C \subseteq D$. *If* $A \downarrow_B C$ *and* $A \downarrow_C D$, *then* $A \downarrow_B D$.
4. ***Symmetry:*** $A \downarrow_C B$ *if and only if* $B \downarrow_C A$.
5. ***Extension:*** *For any* $\bar{a}$ *and* $C \subseteq B$ *there is* $\bar{b}$ *such that* $\mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C)$ *and* $\bar{b} \downarrow_C B$.
6. ***Finite character:*** $A \downarrow_C B$ *if and only if* $\bar{a} \downarrow_C \bar{b}$ *for every finite* $\bar{a} \in A$ *and* $\bar{b} \in B$.
7. ***Local character:*** *For any finite* $\bar{a}$ *and any* B *there exists a finite* $E \subseteq B$ *such that* $\bar{a} \downarrow_E B$.
8. ***Reflexivity:*** *If the weak type* $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A)$ *is not bounded, then* $\bar{a} \not\downarrow_A \bar{a}$.
9. ***Stationarity:*** *If* $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C)$, $\bar{a} \downarrow_C B$ *and* $\bar{b} \downarrow_C B$, *then* $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/B) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/B)$.
10. ***Superstability:*** *For any increasing sequence of finite sets* A_i, $i < \omega$, *and any finite sequence* $\bar{a}$, *there is* $n < \omega$ *with* $\bar{a} \downarrow_{A_n} A_{n+1}$.

We remark the following property given by superstability.

Lemma 2.3. *Let* Q *be some, possibly unbounded, set and let* $\bar{a}$ *be some finite tuple. There is a finite set* $D \subseteq Q$ *with*

$$\bar{a} \downarrow_D C$$

for any subset $C \subseteq Q$.

Proof: Assume there does not exist such D. We define an increasing sequence of finite sets $A_i \subseteq Q$, $i < \omega$ such that $\bar{a} \not\downarrow_{A_i} A_{i+1}$. This will contradict superstability. First, define $A_0 = \emptyset$. Assume we have defined A_i. However, the set A_i cannot be as required in lemma, and hence there is some $C \subseteq Q$ with $\bar{a} \not\downarrow_{A_i} C$. By finite character of $\downarrow$ we may assume C is finite, and hence take $A_{i+1} = C \cup A_i$. □

We also recall the following facts, which are proved in [11].

Fact 2.4. *The supremum for the number of Lascar strong types over any finite set is bounded.*

Fact 2.5. *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be simple and superstable.*

Let C *be a countable set and let* $\bar{a}$, $\bar{b}$ *be finite tuples such that* $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C)$. *Then there is* $f \in \mathrm{Aut}(\mathfrak{M}/C)$ *such that* $f(\bar{a}) = \bar{b}$.

Furthermore, if p_i, $i < \omega$, *are countably many Lascar types over subsets* $D_i \subseteq C$, *we can choose* f *such that* $f(p_i) = p_i$ *for all* $i < \omega$.

3. Regular types

For the rest of this paper let $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ be a simple, superstable, finitary AEC. From now on we will not use finite character or other details of the definition of the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. Essentially we need a class of structures with a monster model and a notion of independence as in section 2. We also need the notion of a Lascar strong type (or other notion of type) with a related notion of a strong automorphism and the properties listed in section 2, especially we need stationarity and results comparable to Fact 2.4 and Fact 2.5.

We fix a finite set A.

We assume that p is some unbounded Lascar strong type over A. That is, the set

$$\mathbf{P} = \{a \in \mathfrak{M} : \mathrm{Lstp}(a/A) = p\}$$

is unbounded. As notation, we write a, b, c etc to denote realizations of p, that is, elements in $\mathbf{P}$. The notation $\bar{a}, \bar{b}, \bar{c}$ refers to finite sequences of realizations of p. We note that $\mathbf{P}$ in general is not invariant under automorphisms fixing A pointwise. However if an automorphism $f \in \mathrm{Aut}(\mathfrak{M}/A)$ maps *some* element $a \in \mathbf{P}$ to $\mathbf{P}$, then f fixes $\mathbf{P}$ setwise, since $\mathrm{Lstp}(b/A) = p$ implies $\mathrm{Lstp}(f(b)/A) = \mathrm{Lstp}(f(a)/A) = p$.

When C is a bounded subset of $\mathfrak{M}$ and p' is a type, we define the following operator on the realizations of p':

$$\mathrm{cl}_{\mathrm{C}}(B) = \{a \models p' : \ a \not\downarrow_{A \cup C} B\}.$$

Furthermore, we assume that the type p is *regular*, that is, the closure operator $\mathrm{cl}_{\mathrm{A}}(-) = \mathrm{cl}(-)$ defines a *pregeometry* on $\mathbf{P}$. Hence we assume:

Assumption 3.1 (Regularity). *For any subsets $B \subseteq B' \subset \mathbf{P}$ and elements $a, b \in \mathbf{P}$*

(i) $B \subseteq \mathrm{cl}(B) \subseteq \mathrm{cl}(B')$,
(ii) $\mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$,
(iii) Exchange: *if $a \in \mathrm{cl}(B \cup \{b\}) \setminus \mathrm{cl}(B)$, then $b \in \mathrm{cl}(B \cup \{a\})$,*
(iv) Finite character: *if $a \in \mathrm{cl}(B)$, then $a \in \mathrm{cl}(B_0)$ for some finite subset B_0 of B.*

We prove that this definition of regularity is equivalent to the more traditional one based on *orthogonality.* This equivalence is proved exactly as the same result with forking in stable first order theories (see for example Pillay [15]), but we prove it as an exercise.

Lemma 3.2. *Let p be a Lascar strong type over a finite set A. The following are equivalent:*

1. $\mathrm{cl}_{\mathrm{A}}(-)$ *defines a pregeometry on the realizations of p.*
2. $\mathrm{cl}_{\mathrm{C}}(-)$ *defines a pregeometry on the realizations of p' for any set C containing A, where p' is the free extension of p to C.*

3. *Let C contain A and let p' be a free extension of p to C. For any B and any b realizing p' such that $b \not\downarrow_C B$, the types p' and* $\mathrm{Lstp}^{\mathrm{w}}(b/B)$ *are orthogonal, that is, for any $D \supseteq C \cup B$ and a, b' satisfying the free extensions of p' and* $\mathrm{Lstp}^{\mathrm{w}}(b/B)$ *to D respectively, we have*

$$a \downarrow_D b'.$$

Proof: Clearly 2. implies 1. We show that 3. implies 2. First we show the following claim: Assume that B, D contain realizations of p' and $a \models p'$ such that $a \not\downarrow_C D$ and for all $d \in D$, $d \not\downarrow_C B$. Then $a \not\downarrow_C B$.

To prove the claim, we assume the contrary that $a \downarrow_C B$. By item 3. we get for each $d \in D$ that $a \downarrow_{C \cup B} d$ and by transitivity, $a \downarrow_C B \cup d$. We show by induction on n that this holds for every finite $d_0, , , d_n \subseteq D$: on the $n+1$th step, we use item 3. and induction to show that $a \downarrow_{C \cup B \cup d_0, \dots, d_n} d_{n+1}$, and then get $a \downarrow_C B \cup d_0, \dots, d_n, d_{n+1}$ by transitivity. Hence finite character of $\downarrow$ gives that $a \downarrow_C D$, a contradiction.

Now we use this claim to show that $\mathrm{cl}_{\mathrm{C}}(\mathrm{cl}_{\mathrm{C}}(B)) \subseteq \mathrm{cl}_{\mathrm{C}}(B)$ with taking $D = \mathrm{cl}_{\mathrm{C}}(B)$. If a realizes p' and a is in $\mathrm{cl}_{\mathrm{C}}(\mathrm{cl}_{\mathrm{C}}(B))$, we have that $a \not\downarrow_C \mathrm{cl}_{\mathrm{C}}(B)$ and for all $d \in \mathrm{cl}_{\mathrm{C}}(B)$, $d \not\downarrow_C B$. Hence $a \not\downarrow_C B$, by the previous claim, that is, $a \in \mathrm{cl}_{\mathrm{C}}(B)$.

Then $(i), (iii)$ and (iv) follow from the properties of $\downarrow$. Since p' is unbounded, we get that $b \not\downarrow_C b$ for each b realizing p'. This and monotonicity imply (i). Item (iv) is given by finite character of $\downarrow$. To prove Exchange, let $a \downarrow_C B$ and $a \not\downarrow_C B \cup b$. By transitivity, $a \not\downarrow_{C \cup B} b$ and furthermore by symmetry, $b \not\downarrow_{C \cup B} a$. Monotonicity gives that $b \not\downarrow_C B \cup a$.

Then we show that 1. implies 3. First we prove the implication in the case where $C = A$ and for finite sets B and D. Let a, b realize extensions of p to $D \supseteq A \cup B$ such that $a \downarrow_A D$ and $b \not\downarrow_A B$. We want to show that

$$a \downarrow_D b.$$

We assume the contrary, that $a \not\downarrow_D b$. By Lemma 2.3 there is $\bar{e} \in \mathbf{P}$ such that

$$D \downarrow_{A \cup \bar{e}} B'$$

for any subset B' of $\mathbf{P}$.

We may assume that

$$\bar{e}, D \downarrow_A a :$$

by extension there is $\bar{e}'$ realizing $\mathrm{Lstp}^{\mathrm{w}}(\bar{e}/A \cup D)$ such that $\bar{e}' \downarrow_{A \cup D} a$. Then by transitivity and symmetry $\bar{e}', D \downarrow_A a$. Furthermore, since $A \cup D$ is finite, there is $f \in \mathrm{Saut}(\mathfrak{M}/A \cup D)$ with $f(\bar{e}) = \bar{e}'$. Then since f fixes $\mathbf{P}$ setwise, we can take $\bar{e}'$ as $\bar{e}$.

Then since $\bar{b} \downarrow_{A \cup \bar{e}} D$ by symmetry, transitivity implies that

$$b \not\downarrow_A \bar{e}.$$

Furthermore, we claim that

$$a \not\downarrow_A \bar{e}, b.$$

If not, then $a \downarrow_{A\cup\bar{e}} b$. The definition of $\bar{e}$ implies that $D \downarrow_{A\cup a} b$. Then by symmetry and transitivity, $a \cup D \downarrow_{A\cup\bar{e}} b$ and furthermore by monotonicity and transitivity, $a \downarrow_D b, \bar{e}$, which is a contradiction.

Hence we have that $a \in \mathrm{cl}_{\mathrm{A}}(b, \bar{e})$ and $b \in \mathrm{cl}_{\mathrm{A}}(\bar{e})$. Then by (i) and (ii) of the definition of a pregeometry,

$$a \in \mathrm{cl}_{\mathrm{A}}(\bar{e}, b) \subseteq \mathrm{cl}_{\mathrm{A}}(\mathrm{cl}_{\mathrm{A}}(\bar{e})) = \mathrm{cl}_{\mathrm{A}}(\bar{e}).$$

Hence $a \not\downarrow_A \bar{e}$, a contradiction.

Then finally we prove 3. for arbitrary C, B and D. Assume that p' is a free extension of p to C, let a, b realize p' and $D \supset C \cup B$ where $a \downarrow_C D$ and $b \not\downarrow_C B$. We want to show that $a \downarrow_D b$.

Since p' is a free extension of p we get by transitivity that $a \downarrow_A D$. By monotonicity $b \not\downarrow_A C \cup B$ and by finite character there is finite $B_0 \subset C \cup B$ containing A such that $b \not\downarrow_A B_0$. Then by the previous claim, for arbitrary finite $D_0 \subset D$ containing $A \cup B_0$, $a \downarrow_{D_0} b$. Furthermore by transitivity, $a \downarrow_A D_0 \cup b$. Since D_0 was arbitrary, finite character implies that $a \downarrow_D b$, and hence we have shown the claim. $\square$

By the previous result, Assumption 3.1 implies that for any $C \subset \mathfrak{M}$ and p' a free extension of p to C, the operator $\mathrm{cl}_{\mathrm{C}}(-)$ defines a pregeometry on the realizations of p'. Hence we can define a notion of dimension $dim(-/C)$ on the realizations of the free extension of p to $C \supseteq A$. There a sequence $a_1, ..., a_m$ is C-independent of a set B, if

$$a_i \downarrow_C B \cup \{a_0, ..., a_{i-1}, a_{i+1}, ..., a_m\} \text{ for each } i \in \{1, ..., m\}.$$

Equivalently,

$$a_i \downarrow_A C \cup B \cup \{a_0, ..., a_{i-1}, a_{i+1}, ..., a_m\} \text{ for each } i \in \{1, ..., m\}.$$

By independence calculus, it follows that

$$a_0, ..., a_m \downarrow_A C \cup B.$$

We write *independent* for A-independent.

Now we give our geometric assumption for the sets $\mathbf{P}$ and $\mathbf{Q}$, where p is regular in the sense of Assumption 3.1. This assumption is strenghtened in Assumption 5.7, which we need to gain the main theorem. However, this weaker assumption is enough for section 4.

Assumption 3.3. *Assume that A is finite, $\mathbf{Q}$ is an A-invariant set and that p is a regular unbounded Lascar strong type over A. Let $n < \omega$. Assume that*

1. *For any independent sequence $(a_1, ..., a_n)$ of realizations of p and any finite subset C of $\mathbf{Q}$ we have*

$$\dim(a_1, ..., a_n/A) = \dim(a_1, ..., a_n/A \cup C).$$

2. *For some independent sequence* $(a_1, ..., a_{n+1})$ *of realizations of* p *there is a finite subset* C *of* $\mathbf{Q}$ *such that*

$$\dim(a_1, ..., a_{n+1}/A) > \dim(a_1, ..., a_{n+1}/A \cup C).$$

We should interpret item 1. so that for any element a realizing p and any (finite) set $C \subseteq \mathbf{Q}$, $a \downarrow_A C$. Hence that gives that the dimension $dim(-/A \cup C)$ is well-defined on $\mathbf{P}$. We note that item 2. of the assumption actually implies that the set $\mathbf{Q}$ is unbounded. One property of our independence relation is that if $\mathrm{tp}^{\mathrm{w}}(c/A)$ is bounded, then $c \downarrow_A B$ for any subset B of the monster model.

Furthermore, we make $\mathbf{P}$ into a geometry $\mathbf{P}/E$ by considering the A-invariant equivalence relation

$$E(x, y), \text{ defined by } \mathrm{cl_A}(x) = \mathrm{cl_A}(y).$$

Then $\mathbf{P}/E$ is a geometry with universe consisting of elements $\mathrm{cl_A}(x)$, $x \in \mathbf{P}$. We use the notation $\mathrm{cl_A}$ also for the canonical closure operator on $\mathbf{P}/E$, that is

$$\mathrm{cl_A}(\{\mathrm{cl_A}(x) : x \in X\}) = \{\mathrm{cl_A}(y) : y \in \mathrm{cl_A}(X)\} = \{\mathrm{cl_A}(y) : y \models p \text{ and } y \not\downarrow_A X\}.$$

Any sequence $a_1, ..., a_k \in \mathbf{P}$ is independent of $X \subset \mathbf{P}$ if and only if $\mathrm{cl_A}(a_1)$,...,$\mathrm{cl_A}(a_k)$ in $\mathbf{P}/E$ is independent of $\{\mathrm{cl_A}(x) : x \in X\}$.

Since p is unbounded both $\mathbf{P}$ and $\mathbf{P}/E$ have infinite dimension. Also by simplicity, $\mathrm{cl_A}(\emptyset) = \emptyset$ in $\mathbf{P}$.

4. The group G of permutations of $\mathbf{P}/E$

Let E be the equivalence relation on $\mathbf{P}$ with

$$E(x, y) \text{ iff } \mathrm{cl_A}(x) = \mathrm{cl_A}(y).$$

We define $\mathbf{G}$, the group of permutations of $\mathbf{P}/E$ as follows.

Definition 4.1. *Let* $\mathbf{G}$ *be the the group of permutations* g *of* $\mathbf{P}/E$ *such that for each countable* $C \subset \mathbf{Q}$ *and finite* $X \subset \mathbf{P}$ *there is* $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ *fixing* $\mathbf{P}$ *setwise such that* $\sigma(a)/E = g(a/E)$ *for each* $a \in X$.

Then we will show that this group *n-acts* on P/E. We define:

Definition 4.2. *An action of* G *on a pregeometry* P *is an* n*-action if*

1. *The action has rank* n*: Whenever the tuples* $\bar{x}$ *and* $\bar{y}$ *are two* n*-tuples of elements of* P *such that* $\dim(\bar{x}\bar{y}) = 2n$*, then there is* $g \in G$ *such that* $g(\bar{x}) = \bar{y}$*. However, for some* $(n+1)$*-tuples* $\bar{x}$*,* $\bar{y}$ *with* $\dim(\bar{x}\bar{y}) = 2n+2$*, there is no* $g \in G$ *such that* $g(\bar{x}) = \bar{y}$*.*
2. *The action is* $(n+1)$*-determined: Whenever the action of* $g, h \in G$ *agree on a* $(n+1)$*-dimensional subset* X *of* P*, then* $g = h$*.*

4.1. Interpreting an n-action

First we use Fact 2.5 to show that our action has rank n.

Lemma 4.3. *Let $a_1, ..., a_m$ be a finite sequence in $\mathbf{P}$ and $C \subset \mathbf{Q}$ with*

$$\dim(a_1, ..., a_m/A \cup C) = m.$$

Then for any $k \leq m$ and $i_1 < ... < i_k, j_1 < ... < j_k \in \{1, ..., m\}$ we have that

$$\mathrm{Lstp}^{\mathrm{w}}(a_{i_1}, ..., a_{i_k}/A \cup C) = \mathrm{Lstp}^{\mathrm{w}}(a_{j_1}, ..., a_{j_k}/A \cup C).$$

Furthermore, if C is countable, for a given countable collection $\mathcal{S}$ of types over subsets of $A \cup C$ there is an automorphism $f \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ preserving $\mathcal{S}$ and mapping $a_{i_1}, ..., a_{i_k}$ to $a_{j_1}, ..., a_{j_k}$.

Proof: By Fact 2.5 it is enough to prove the first claim. Furthermore, we may assume that $j_1, ..., j_k = 1, ..., k$. We prove the claim by induction on k. If $k = 1$, we get the claim by stationarity of weak Lascar strong types, since for each $i \in \{1, ..., m\}$, $a_i \models p$ and $a_i \downarrow_A C$ by Assumption 3.3. Assume we have shown the claim for k.

To prove the claim for $k+1$, let $C_0 \subseteq C$ be finite. By induction,

$$\mathrm{Lstp}(a_{i_1}, ..., a_{i_k}/A \cup C_0) = \mathrm{Lstp}(a_1, ..., a_k/A \cup C_0).$$

Hence there is a strong automorphism $f \in \mathrm{Saut}(\mathfrak{M}/A \cup C_0)$ mapping $a_{i_1}, ..., a_{i_k}$ to $a_1, ..., a_k$. Using the fact that $\dim(a_1, ..., a_m/A \cup C) = m$ and invariance, we get that

$$a_{k+1} \downarrow_A C_0 \cup a_1, ..., a_k \text{ and}$$

$$f(a_{i_{k+1}}) \downarrow_A C_0 \cup a_1, ..., a_k.$$

Since both $f(a_{i_{k+1}})$ and a_{k+1} realize p, we can use stationarity to conclude that

$$\mathrm{Lstp}(f(a_{i_{k+1}})/A \cup C_0 \cup a_1, ..., a_k) = \mathrm{Lstp}(a_{k+1}/A \cup C_0 \cup a_1, ..., a_k).$$

Furthermore, we get that

$$\mathrm{Lstp}(a_{i_1}, ..., a_{i_{k+1}}/A \cup C_0) = \mathrm{Lstp}(a_1, ..., a_k, f(a_{i_{k+1}})/A \cup C_0)\, \mathrm{Lstp}(a_1, ..., a_{k+1}/A \cup C_0).$$

Since the same holds for all finite $C_0 \subseteq C$, the claim follows. □

Lemma 4.4. *Let $a_1, ..., a_n \in \mathbf{P}$ and $b_1, ..., b_n \in \mathbf{P}$ be two independent sequences and let $C \subset \mathbf{Q}$ be countable and let $\mathcal{S}$ be a countable collection of types over subsets of $A \cup C$. Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ preserving $\mathcal{S}$ and mapping a_i to b_i for each $i \in \{1, ..., n\}$.*

Futhermore, if C is finite, we can take $f \in \mathrm{Saut}(\mathfrak{M}/A \cup C)$.

Proof: By Assumption 3.3, we have that

$$\dim(a_1, ..., a_n/A \cup C_0) = \dim(b_1, ..., b_n/A \cup C_0) = n$$

for any finite subset C_0 of C. Hence by finite character, the sequences $a_1, ..., a_n$ and $b_1, ..., b_n$ are independent over C. By using a third sequence if necessary, we may assume that

$$\dim(a_1, ..., a_n, b_1, ..., b_n/A \cup C) = 2n.$$

The previous Lemma implies the claim. $\square$

As in [13], we define a notion of a *good pair* in order to show $n+1$-determinacy. However, since we have neither $\aleph_0$-stability or strong minimality, we have to define a different notion.

Definition 4.5 (Good pair). *We say that (X, C) is a good pair, if $X \subset \mathbf{P}$ is countable and infinite-dimensional and $C \subset \mathbf{Q}$ is countable and the following holds:*

For any $n+1$-tuple $\bar{a} \in X$ there is Morley-sequence $(C_i)_{i<\omega} \subseteq C$ of finite sets witnessing the dimension of $\bar{a}$ over $\mathbf{Q}$, that is

1. *Each $C_i \subset \mathbf{Q}$ is finite,*
2. $\mathrm{Lstp}(C_i/A \cup \bar{a}) = \mathrm{Lstp}(C_0/A \cup \bar{a})$,
3. $C_i \downarrow_{A \cup \bar{a}} \bigcup_{j<i} C_j$ *and*
4. $\dim(\bar{a}/A \cup C_i) = n$ *for each $i < \omega$*

Clearly by Assumption 3.3 and simplicity, for any countable $X' \subset \mathbf{P}$ there is a good pair (X, C) such that X contains X'.

Lemma 4.6. *Let (X, C) be a good pair. Suppose that $(a_1, ..., a_{n+1}) \subseteq X$ are independent and $\sigma(a_i)/E = a_i/E$, for $i = 1, ..., n+1$, for some $\sigma \in \mathrm{Aut}(\mathbf{P}/A \cup C)$. Then $\sigma(c/E) = c/E$ for any $c \in X$.*

Proof: We first prove the lemma for $c \in X$ with $c \downarrow_A a_1, ..., a_{n+1}$. First we claim that

$$\sigma(c) \not\downarrow_{\{a_1,...,a_{n+1}\} \setminus \{a_i\}} c, \text{ for each } i = 1, ..., n+1.$$

We only prove that

$$\sigma(c) \not\downarrow_{\{a_1,...,a_n\}} c.$$

Assume, for a contradiction, that this fails. Now $c, a_1, ..., a_n$ is an independent $n+1$-tuple in X, and hence by the definition of a good pair there is a Morley-sequence $(C_i)_{i<\omega} \subseteq C$ witnessing the dimension of $c, a_1, ..., a_n$ in $\mathbf{Q}$.

By extension, there is $e \in \mathbf{P}$ realizing $\mathrm{Lstp}(\sigma(c)/a_1, ..., a_n, c \cup A)$ such that

$$e \downarrow_{a_1,...,a_n \cup A} c \bigcup_{i<\omega} C_i.$$

We claim that there is finite $C' \subseteq C$ such that

$$\mathrm{Lstp}(\sigma(c), C'/c, a_1, ..., a_n \cup A) = \mathrm{Lstp}(e, C_0/c, a_1, ..., a_n \cup A).$$

To prove the claim, we first show that there is $p < \omega$ such that

$$\sigma(c) \downarrow_{a_1,...,a_n,c \cup A} C_p.$$

By superstability, there is some $i < \omega$ such that $\sigma(c) \downarrow_{a_1,...,a_n,c \cup A \cup C_i} C_{i+1}$. Then using symmetry, the fact that $C_{i+1} \downarrow_{a_1,...,a_n,c \cup A} C_i$ and transitivity, the get that

$$C_{i+1} \downarrow_{a_1,...,a_n,c \cup A} C_i \cup \sigma(c).$$

Hence we can choose C_{i+1} as C_p by monotonicity and symmetry. Now we can also take C_p as C', since by symmetry and stationarity of Lascar strong types,

$$\mathrm{Lstp}(\sigma(c), C_p/a_1, ..., a_n, c, A) =$$

$$\mathrm{Lstp}(e, C_p/a_1, ..., a_n, c, A) = \mathrm{Lstp}(e, C_0/a_1, ..., a_n, c, A).$$

Now let $f \in \mathrm{Saut}(\mathfrak{M}/A \cup a_1, ..., a_n, c)$ map (e, C_0) to $(\sigma(c), C')$. By invariance,

$$\dim(c, a_1, ..., a_n/A \cup C') = n.$$

Since $\dim(a_1, ..., a_n/A \cup C') = n$ by Assumption 3.3, we must have that

$$c \in \mathrm{cl}_{A \cup C'}(a_1, ..., a_n). \tag{1}$$

Furthermore, $e \downarrow_{a_1,...,a_n, \cup A} c \cup C_0$ implies

$$\sigma(c) \downarrow_{a_1,...,a_n \cup A} c \cup C'.$$

Furthermore, $\dim(c, a_1, ..., a_n/A) = n + 1$ implies $\sigma(c) \downarrow_A a_1, ..., a_n$, and hence by transitivity,

$$\sigma(c) \downarrow_A (a_1, ..., a_n \cup C').$$

This is, $\sigma(c) \notin \mathrm{cl}_{A \cup C'}(a_1, ..., a_n)$. But we have that σ fixes each a_i/E and hence $\mathrm{cl}_{A \cup C'}(a_1, ..., a_n) = \mathrm{cl}_{A \cup C'}(\sigma(a_1), ..., \sigma(a_n))$, giving

$$\sigma(c) \notin \mathrm{cl}_{A \cup C'}(\sigma(a_1), ..., \sigma(a_n)).$$

But then 1 implies that

$$\sigma(c) \in \mathrm{cl}_{A \cup C'}(\sigma(a_1), ..., \sigma(a_n)),$$

a contradiction.

Then we show that $\sigma(c/E) = c/E$. Again we assume the contrary, that

$$c \downarrow_A \sigma(c).$$

The previous claim and symmetry give that $c \in \mathrm{cl}_{\mathrm{A}}(\sigma(c), a_1, ..., a_n)$. By exchange, there is $i \in \{1, ..., n\}$ such that

$$a_i \in \mathrm{cl}_{\mathrm{A}}(c \cup \sigma(c) \cup \{a_1, ..., a_n\} \setminus \{a_i\}).$$

By the previous claim, $\sigma(c) \in \mathrm{cl}_{\mathrm{A}}(c \cup \{a_1, ..., a_{n+1}\} \setminus \{a_i\})$ and we get that

$$\dim(c, \sigma(c), a_1, ..., a_{n+1}/A) = n + 1.$$

But we assumed $c, \sigma(c) \notin \mathrm{cl}_{\mathrm{A}}(a_1, ..., a_{n+1})$, a contradiction.

We still need to prove the lemma for $c \in X$ with $c \not\downarrow_A a_1, ..., a_n$. For this, let $b_1, ..., b_n \in X$ be independent of $(c, a_1, ..., a_n)$. These can be found in X since X is infinite-dimensional. By the first case, we must have that

$$\sigma(b_i/E) = b_i/E \text{ for each } i = 1, ..., n.$$

Now $c \not\downarrow_A b_1, ..., b_n$ and we get $\sigma(c/E) = c/E$ by the first case. □

We deduce the next proposition.

Proposition 4.7. *Let $a_1, ..., a_{n+1} \in \mathbf{P}$ be independent. Let $c \in \mathbf{P}$. The exists a countable $C_c \subset \mathbf{Q}$ such that if $\sigma, \tau \in \mathrm{Aut}(\mathfrak{M}/A \cup C_c)$ fix $\mathbf{P}$ setwise and*

$$\sigma(a_i)/E = \tau(a_i)/E, \textit{ for each } i = 1, ..., n + 1,$$

then $\sigma(c)/E = \tau(c)/E$.

Proof: Let (X, C) be a good pair with X containing $a_1, ..., a_{n+1}, b_1, ..., b_{n+1}, c$. We let C_c be C. Then, for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{M}/A \cup C_c)$ fixing $\mathbf{P}$ setwise with $\sigma(a_i)/E = \tau(a_i)/E$, for each $i = 1, ..., n$, we have that $\tau^{-1} \circ \sigma(a_i)/E = a_i/E$ for each $i = 1, ..., n+1$. Hence by the previous lemma, we have that $\tau^{-1} \circ \sigma(c)/E = c/E$. This implies that $\sigma(c)/E = \tau(c)$. □

Proposition 4.8. *The action of $\mathbf{G}$ on $\mathbf{P}/E$ is an n-action.*

Proof: The $(n+1)$-determinacy of the action of $\mathbf{G}$ on $\mathbf{P}$ follows from the previous proposition. Now we have to show that the action has rank n.

First we prove the following claim: Assume that $\bar{a} = a_1, ..., a_n$ and $\bar{b} = b_1, ..., b_n$ are two independent sequences and let $c \downarrow_A \bar{a}\bar{b}$. Then there is $d \in \mathbf{P}$ such that for each countable $C \subset \mathbf{Q}$ there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ preserving p with $\sigma(c) = d$ and $\sigma(a_i) = b_i$ for each $i = 1, ..., n$.

By Lemma 2.3 there is a finite set $D \subseteq A \cup \mathbf{Q}$ such that $\bar{a}, c \downarrow_D C$ for any set $C \subseteq A \cup \mathbf{Q}$. By Lemma 4.4 there is a strong automorphism $f \in \mathrm{Saut}(\mathfrak{M}/A \cup D)$ such that $f(\bar{a}) = \bar{b}$. We take $d = f(c)$ and claim this is a s required. Let $C \subset \mathbf{Q}$ be countable. By the choice of D, we have that $\bar{a}, c \downarrow_{A \cup D} C$, and since $f^{-1}(C) \subseteq A \cup \mathbf{Q}$, also $\bar{a}, c \downarrow_{A \cup D} f^{-1}(C)$. Invariance gives that $\bar{b}, d \downarrow_{A \cup D} C$. Now the claim follows by stationarity of weak Lascar strong types and Fact 2.5.

We can now show the action has rank n. Assume that $\bar{a}$ and $\bar{b}$ are independent n-tuples of realizations of p. We must find $g \in \mathbf{G}$ such that $g(\bar{a}/E) = \bar{b}/E$. Let c be in $\mathbf{P}$ be such that $c \downarrow_A \bar{a}\bar{b}$ and choose d as in the previous claim. We now define the following function $g : \mathbf{P}/E \to \mathbf{P}/E$. For each $e \in \mathbf{P}$, choose C_e as in Proposition 4.7, i.e. for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{M}/A \cup C_e)$ fixing $\mathbf{P}$ setwise such that $\sigma(\bar{a}/E) = \bar{b}/E = \tau(\bar{a}/E)$ and $\sigma(c)/E = d/E = \tau(c)/E$, we have $\sigma(e)/E = \tau(e)/E$.

By the choice of d, there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C_e)$ preserving p and sending the $n+1$-tuple $(\bar{a}, c)$ to the $n+1$-tuple $(\bar{b}, d)$. Define

$$g(e/E) = \sigma(e)/E.$$

The choice of C_e guarantees that it is well-defined.

We can also see that g is a permutation of P/E: We see that $g(e)$ does not depend on the choice of the set C_e. Let C_e and D_e be given by Proposition 4.7, and $\tau \in \mathrm{Aut}(\mathfrak{M}/C_e)$ and $\tau' \in \mathrm{Aut}(\mathfrak{M}/D_e)$ are as in the definition of g. Again by the choice of d there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/C_e \cup D_e)$ mapping $(\bar{a}, c)$ to $(\bar{b}, d)$. Then by the choice of C_e and D_e, $\tau'(e)/E = \sigma(e)/E = \tau(e)/E$. Furthermore, studying the argument in Proposition 4.7, if $\tau \in \mathrm{Aut}(\mathfrak{M}/C_e)$ maps $(\bar{a}, c)$ to $(\bar{b}, d)$, we can choose C_e as $C_{\tau(e)}$. Then we see that $g \circ g(e/E) = e/E$ for e outside the E-classes of $(\bar{a}, c, \bar{b}, d)$ and hence g is bijective.

Further, suppose countable $C \subset \mathbf{Q}$ and finite $X \subset \mathbf{P}$ are given. By the choice of d, there is

$$\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C \cup \bigcup_{e \in X} C_e)$$

preserving p and sending $(\bar{a}, c)$ to $(\bar{b}, d)$. By definition, we have $\sigma(e)/E = g(e/E)$. This implies that $g \in \mathbf{G}$. Since this fails for independent $n+1$-tuples by Assumption 3.3, the action of $\mathbf{G}$ on $\mathbf{P}$ has rank n. $\square$

Definition 4.9. *A group $(G, \cdot)$ is* interpretable *in $\mathfrak{M}$ if there is a (bounded) subset $B \subseteq \mathfrak{M}$ and an unbounded set $U \subseteq \mathfrak{M}^k$ (for some $k < \omega$), an equivalence relation E on U, and a binary relation $*$ on U/E which are B-invariant and such that $(G, \cdot)$ is isomorphic to $(U/E, *)$.*

As in Hyttinen, Lessmann and Shelah [13], we can now prove:

Proposition 4.10. *The group $\mathbf{G}$ is interpretable in $\mathfrak{M}$ (over a finite set).*

Proof: This follows from the $(n+1)$-determinacy of the group action. Fix $\bar{a}$ an independent $(n+1)$-tuple of elements of P/E. Let $B = A \cup \bar{a}$.

We let $U/E \subseteq P^{(n+1)}/E$ consist of those $b \in P^{(n+1)}/E$ such that $g(\bar{a}) = \bar{b}$ for some $g \in \mathbf{G}$.

We show that U/E is B-invariant: Let $\tau \in \mathrm{Aut}(\mathfrak{M}/B)$. Since τ fixes $A \cup \bar{a}$ pointwise, it fixes $\mathbf{P}/E$ setwise. Also τ induces an automorphism of $\mathbf{G}$, where $\tau(g) \in G$ maps $\bar{a}$ to $\tau(\bar{b})$.

We now define $\bar{b}_1 * \bar{b}_2 = \bar{b}_3$ on U/E, if whenever $g_l \in \mathbf{G}$ such that $g_l(\bar{a}) = \bar{b}_l$, then $g_1 \circ g_2 = g_3$. This is well-defined by $(n+1)$-determinacy and the definition of U/E. Furthermore, the binary function $*$ is B-invariant. Also $(n+1)$-determinacy implies that the map $g \mapsto g(\bar{a})$ defines an isomorphism between $(\mathbf{G}, \circ)$ and $(U/E, *)$. $\square$

5. Stationarity and unique generics

Following Hyttinen, Lessmann and Shelah [13], we choose a group Σ of automorphisms of the group action and show that the group $(\mathbf{G}, \circ)$ (Σ, n)-acts on a pregeometry (P, cl). That is, the group $\mathbf{G}$ n-acts on the universe P of the pregeometry in a way which respects the closure operator and which is ω-homogeneous with respect to Σ: for any finite $X \subseteq \mathbf{P}/E$ and $x, y \notin \mathrm{cl}_{\mathrm{A}}(X)$ there is $\tau \in \Sigma$ fixing X pointwise and mapping x to y. Although [13] studies an arbitrary infinite-dimensional pregeometry (P, cl) with $\mathrm{cl}(\emptyset) = \emptyset$, we will only study the geometry $(\mathbf{P}/E, \mathrm{cl}_{\mathrm{A}})$.

Let $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ be a strong automorphism. Then τ induces an automorphism τ' of the group action as follows: τ' maps the equivalence class a/E in $\mathbf{P}/E$ to the class $\tau(a)/E = \tau(a/E)$ and for $g \in \mathbf{G}$, $\tau'(g)(a/E) = \tau(g(\tau^{-1}(a/E)))$. It is easy to verify that

$$\tau' : \mathbf{G} \to \mathbf{G}$$

is an automorphism of $\mathbf{G}$ and preserves the action.

We let Σ be the group of automorphisms of the action induced by strong automorphisms of the Monster model over the finite set A:

$$\Sigma = \{\tau' : \tau \in \mathrm{Saut}(\mathfrak{M}/A)\}.$$

We denote by Σ_X the subgroup consisting of those $\tau \in \Sigma$ which fix $X \subset \mathbf{P}/E$ pointwise.

Then we remark that the n-action defined in the previous section is ω-homogeneous with respect to this Σ.

Lemma 5.1. *If $X \subseteq \mathbf{P}/E$ is finite and $x, y \in \mathbf{P}/E$ are outside $\mathrm{cl}_{\mathrm{A}}(X)$, then there is a strong automorphism $\tau \in \Sigma$ of the group action sending x to y which is the identity on X.*

Proof: Choose a, b elements and $\bar{d}$ a finite subset of $\mathbf{P}$ such that $x = a/E$, $y = b/E$ and $X = \bar{d}/E$. That is, $a, b, \bar{d}$ are chosen as representatives of the E-classes of x, y, X. Then $a \downarrow_A \bar{d}$ and $b \downarrow_A \bar{d}$. By stationarity, we have that $\mathrm{Lstp}^{\mathrm{w}}(a/A \cup \bar{d}) = \mathrm{Lstp}^{\mathrm{w}}(b/A \cup \bar{d})$. Since $A \cup \bar{d} \subseteq \mathfrak{M}$ is finite, we get a strong automorphism $\tau \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{d})$ mapping a to b. Then, since τ preserves all E-classes, τ maps x to y and maps each element of $X \subseteq \mathbf{P}/E$ to itself. We can take $\tau' \in \Sigma$ to be the automorphism of the group action induced by τ. $\square$

Definition 5.2. *We say that* $g \in \mathbf{G}$ *is* generic *over* $X \subseteq \mathbf{P}/E$*, if there exists an independent* n*-tuple* $\bar{x}$ *of* $\mathbf{P}$ *such that*

$$\dim(\bar{x}g(\bar{x})/X) = 2n.$$

Since $\mathbf{P}/E$ has infinite dimension and the action has rank n, for a given finite set $X \subset \mathbf{P}/E$, there is $g \in \mathbf{G}$ generic over X.

For $\tau \in \Sigma_X$, g is generic over X if and only if $\tau(g)$ is generic over X. Hence we can talk about *generic types* over X, which are orbits of generic elements $g \in \mathbf{G}$ under automorphisms in Σ_X, written $\mathrm{tp}(g/X)$.

Remark 5.3. *For any independent* $n+1$*-tuple* $\bar{x}$ *in* $\mathbf{P}/E$ *and any* $g \in \mathbf{G}$*, always* $\dim(\bar{x}g(\bar{x})/A) \leq 2n+1$.

Proof: Assume to the contrary, that $\dim(\bar{x}g(\bar{x})/A) = 2n+2$. Since the action has rank n, there are some $n+1$-tuples $\bar{x}'$ and $\bar{y}'$ with $\dim(\bar{x}'\bar{y}'/A) = 2n+2$ such that there do not exist $h \in \mathbf{G}$ with $h(\bar{x}') = \bar{y}'$. Since $\bar{x}g(\bar{x})$ and $\bar{x}'\bar{y}'$ are two independent tuples of the same length, by ω-homogeneity there is $\sigma \in \Sigma$ mapping $\bar{x}g(\bar{x})$ to $\bar{x}'\bar{y}'$. Then $\sigma'(g) \in \mathbf{G}$ and

$$\sigma'(g)(\bar{x}') = \sigma'(g)(\sigma(\bar{x})) = \sigma(g(\bar{x})) = \bar{y}',$$

a contradiction. □

We can now define stationarity of $\mathbf{G}$ with respect to Σ. Notice that the extra condition on the number of types follows from Fact 2.4.

Definition 5.4. *We say that* $\mathbf{G}$ *is* stationary *if whenever* $g, h \in \mathbf{G}$ *with* $\mathrm{tp}(g/\emptyset) = \mathrm{tp}(h/\emptyset)$ *and* $X \subset \mathbf{P}/E$ *is finite and both* g *and* h *are generic over* X*, then* $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$*. Furthermore, we assume that the number of types over each finite set is bounded.*

The following is a strengthening of stationarity.

Definition 5.5. *We say that a subgroup* G *of* $\mathbf{G}$ *has* unique generics *if for all finite* $X \subset \mathbf{P}/E$ *and* $g, h \in G$ *generic over* X *we have* $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$.

In [13] the following fact is proved for any group $(G, \cdot)$ (Σ, n)-acting an an infinite-dimensional pregeometry (P, cl) as Proposition 2.8. The proof also refers to Lemma 3.2 of Hyttinen [8].

Fact 5.6. *The* connected component $\mathbf{G}^0$ *is the intersection of all invariant, normal subgroups with bounded index.*

If $\mathbf{G}$ *is stationary, then* $\mathbf{G}^0$ *is a normal invariant subgroup of* $\mathbf{G}$ *of bounded index and* $\mathbf{G}^0$ (Σ^0, n)*-acts on the pregeometry* $(\mathbf{P}/E, \mathrm{cl}_\mathrm{A})$ *by restriction, where* Σ^0 *is obtained from* Σ *by restriction to* $\mathbf{G}^0$*. Also the stationarity of* $\mathbf{G}$ *implies that* $\mathbf{G}^0$ *has unique generics.*

In [13], stationarity of Lascar strong types is used to show stationarity for $\mathbf{G}$. The proof also uses quasiminimality of p. For regular types we can do something similar, but we need the additional assumption 5.7. This assumption is analogous to a condition holding in Hrushovski [6], where $\mathbf{P}$ and $\mathbf{Q}$ are slightly modified using the techniques available with $\mathfrak{M}^{eq}$. This assumption is a strengthening of Assumption 3.3(2).

Assumption 5.7. *For some independent sequence $a_1, ..., a_{n+1}$ of realizations of p there is finite $C \subset \mathbf{Q}$ such that $(a_1, ..., a_n)$ dominates $(a_1, ..., a_{n+1})$ over $A \cup C$, written*

$$(a_1, ..., a_n) \rhd_{CA} (a_1, ..., a_{n+1}).$$

That is, whenever $\bar{d}$ is some finite tuple in the monster model, $\bar{d} \downarrow_{A \cup C} a_1, ..., a_n$ implies $\bar{d} \downarrow_{A \cup C} a_1, ..., a_{n+1}$.

We remark that equivalently the same holds for *all* independent sequences $a_1, ..., a_{n+1}$ of realizations of p.

We also have be careful when we want to apply results about Lascar strong types in $\mathbf{P}$ to $\mathbf{P}/E$, since for an element $a \in \mathbf{P}$, the closure $\mathrm{cl}_{\mathrm{A}}(a)$ can be unbounded. For a generic element $g \in \mathbf{G}$, we introduce a concept of *generic witnesses* in $\mathbf{P}$. Especially, we use Assumption 5.7 to get item 4.

Definition 5.8. *Assume that $g \in \mathbf{G}$ is generic over finite $X \subset \mathbf{P}/E$, where $\bar{d} \in \mathbf{P}$ such that $X = \bar{d}/E$. We say that two $(n+1)$-tuples $\bar{a} = a_1, ..., a_{n+1}$ and $\bar{b} = b_1, ..., b_{n+1}$ are* generic witnesses *for g over $\bar{d}$, if*

1. *$g(\bar{a}/E) = \bar{b}/E$ and*
2. $\dim(\bar{a}, b_1, ..., b_n/A \cup \bar{d}) = 2n + 1$.
3. *There are $n+1$-tuples $\bar{a}'$, $\bar{b}'$ such that $g(\bar{a}'/E) = \bar{b}'/E$, $\dim(\bar{a}', \bar{b}'/A \cup \bar{d}) = 2n + 1$ and $\bar{a} \downarrow_A \bar{a}'\bar{b}'\bar{d}$.*
4. *The $2n + 1$-tuple $a_1, ..., a_{n+1}b_1, ..., b_n$ dominates $\bar{a}\bar{b}$ over A.*

Note that if $\bar{a}$, $\bar{b}$ are generic witnesses for g over $\bar{d}$ and $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$, then $\tau(\bar{a})$ and $\tau(\bar{b})$ are generic witnesses for $\tau'(g)$ over $\tau(\bar{d})$.

Lemma 5.9. *Let $g \in \mathbf{G}$ be generic over finite $X \subseteq \mathbf{P}/E$, where $\bar{d} \in \mathbf{P}$ such that $X = \bar{d}/E$. There are $\bar{a}$ and $\bar{b}$ such that they are generic witnesses for g over $\bar{d}$.*

Proof: By the definition of genericity, there are $n + 1$-tuples $\bar{a}'$ and $\bar{b}' = b'_1, ..., b'_{n+1}$ such that $g(\bar{a}'/E) = \bar{b}'/E$ and $dim(\bar{a}', b'_1, ..., b'_n/A \cup \bar{d}) = 2n + 1$.

By extension, there is $\bar{a}$ realizing $\mathrm{Lstp}(\bar{a}'/A \cup \bar{d})$ such that

$$\bar{a} \downarrow_{A \cup \bar{d}} \bar{a}'\bar{b}'.$$

Then by transitivity, also $\bar{a} \downarrow_A \bar{a}'\bar{b}'\bar{d}$. Furthermore, we get that

$$dim(\bar{a}, \bar{a}', b'_1, ..., b'_n/A \cup \bar{d}) = 3n + 2.$$

By Assumption 5.7, there is a finite set $C' \subset \mathbf{Q}$ such that

$$a_1, ..., a_n \rhd_{A\cup C'} a_1, ..., a_{n+1}.$$

Futhermore by extension there are finite sets C_i realizing $\mathrm{Lstp}(C'/A \cup \bar{a})$ such that

$$C_i \downarrow_{A\cup\bar{a}} \bigcup_{j<i} C_j \cup \bar{d}.$$

Then we choose $\bar{b}$ such that $g(\bar{a}/E) = \bar{b}/E$ and there exists $\tau \in \mathrm{Aut}(\mathfrak{M}/A \cup \bigcup_{i<\omega} C_i)$ mapping $\bar{a}$ to $\bar{b}$. This is possible by the definition of $\mathbf{G}$. Then for each $i < \omega$, $b_1, ..., b_n$ dominates $\bar{b}$ over $A \cup C_i$.

Also we must have that $\dim(\bar{a}, b_1, ..., b_n/A \cup \bar{d}) = 2n + 1$. This holds, since by Remark 5.3, for each $i \in 1, ..., n+1$, $b_i \in \mathrm{cl}_{\mathrm{A}}(a_i, a'_1, ..., a'_n, b'_1, ..., b'_n)$ and $b'_i \in \mathrm{cl}_{\mathrm{A}}(a'_i, a_1, ..., a_n, b_1, ..., b_n)$ and hence

$$3n + 2 = \dim(\bar{a}, \bar{a}', b'_1, ..., b'_n/A \cup \bar{d}) = \dim(\bar{a}, \bar{a}', b'_1, ..., b'_n, b_1, ..., b_n/A \cup \bar{d})$$

$$= \dim(\bar{a}, \bar{a}', b_1, ..., b_n/A \cup \bar{d}).$$

Now it is left to show item 4. For this, let $\bar{d}'$ be arbitrary such that

$$\bar{d}' \downarrow_A \bar{a}, b_1, ..., b_n.$$

We need to show that $\bar{d}' \downarrow_A \bar{a}\bar{b}$.

Since $\dim(\bar{a}, b_1, ..., b_n/A) = 2n + 1$ implies that $\bar{a} \downarrow_A b_1, ..., b_n$, we get by transitivity that

$$\bar{a} \downarrow_A b_1, ..., b_n, \bar{d}'. \tag{2}$$

By superstability, there is some $i < \omega$ such that

$$C_{i+1} \downarrow_{A\cup\bar{a}\cup C_i} \bar{b}\bar{d}'.$$

We denote $C = C_{i+1}$. Since $C_{i+1} \downarrow_{A\cup\bar{a}} C_i$, we get by transitivity that

$$C \downarrow_{A\cup\bar{a}} \bar{b}\bar{d}'. \tag{3}$$

Since $\bar{d}' \downarrow_A \bar{a}, b_1, ..., b_n$, 3, symmetry and transitivity imply that

$$\bar{d}' \downarrow_A \bar{a}, b_1, ..., b_n, C. \tag{4}$$

furthermore, 2,3 and transitivity imply that $C\bar{a} \downarrow_A b_1, ..., b_n, \bar{d}'$ and hence by 4, $C\bar{a}\bar{d}' \downarrow_A \bar{b}_n$. Furthermore by monotonicity, $\bar{a}\bar{d}' \downarrow_{A\cup C} b_1, ..., b_n$ and then since $b_1, ..., b_n$ dominates $\bar{b}$ over $(A \cup C)$,

$$\bar{a}\bar{d}' \downarrow_{A\cup C} \bar{b}. \tag{5}$$

Then 4, 5 and transitivity give that $\bar{d}' \downarrow_A \bar{a}, \bar{b}, C$. This proves the claim. □

To prove stationarity, we need one more lemma about Lascar strong types.

Lemma 5.10. *Assume that $\bar{a}, \bar{b}$ and $\bar{c}, \bar{d}$ are both witnesses for a generic $g \in \mathbf{G}$ over $\bar{d}$ in $\mathbf{P}$. Then there are $\bar{c}'\bar{d}'$, generic witnesses for g over $\bar{d}$ such that $\bar{c}'/E = \bar{c}/E$, $\bar{d}'/E = \bar{d}/E$ and $\bar{c}', \bar{d}'$ realizes the Lascar strong type* $\mathrm{Lstp}(\bar{a}, \bar{b}/A \cup \bar{d})$.

Proof: Since $\bar{a}$ and $\bar{b}$ are generic witnesses, there are $n+1$-tuples $\bar{e}$, $\bar{f}$ such that $g(\bar{e}/E) = \bar{f}/E$ and $\bar{a} \downarrow_A \bar{e}, \bar{f}, \bar{d}$. Similarly, there are such $n+1$-tuples $\bar{e}', \bar{f}'$ for $\bar{c}$.

First, let $\bar{a}'$ realize $\mathrm{Lstp}(\bar{a}/A \cup \bar{e} \cup \bar{f} \cup \bar{d})$ such that $\bar{a}' \downarrow_{A \cup \bar{e} \cup \bar{f} \cup \bar{d}} \bar{e}' \cup \bar{f}'$. By transitivity, $\bar{a}' \downarrow_A \bar{e}', \bar{f}', \bar{d}$. As independent $n+1$-tuples, $\bar{a}'$ and $\bar{c}$ realize the same Lascar strong type over A. Then by stationarity, $\bar{a}'$ and $\bar{c}$ realize the same Lascar strong type over $A \cup \bar{e}' \cup \bar{f}' \cup \bar{d}$.

We get two strong automorphisms $\tau_1 \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{e} \cup \bar{f} \cup \bar{d})$ and $\tau_2 \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{e}' \cup \bar{f}' \cup \bar{d})$ such that $\tau_1(\bar{a}) = \bar{a}'$ and $\tau_2(\bar{a}') = \bar{c}$. By $n+1$-determinacy we get that $\tau_1'(g) = g$ and $\tau_2'(g) = g$.

We write $\sigma = \tau_2 \circ \tau_1 \in \mathrm{Saut}(\mathfrak{M}/A)$ and $\bar{d}' = \sigma(\bar{b})$. Then $\bar{c}, \bar{d}' = \sigma(\bar{a}, \bar{b})$ realize $\mathrm{Lstp}(\bar{a}, \bar{b}/A)$ and are generic witnesses for $\sigma(g) = g$ over $\bar{d}$. Hence $\bar{c}, \bar{d}'$ are as needed for the claim. $\square$

Finally we prove stationarity.

Proposition 5.11. $\mathbf{G}$ *is stationary with respect to* Σ.

Proof: First, notice that the number of Lascar strong types of $2n+2$-sequences over A is bounded by Fact 2.4. Since by $n+1$-determinacy the type of any $g \in \mathbf{G}$ is determined by the Lascar strong type of any $2n+2$-tuple $\bar{a}, \bar{b}$ such that $g(\bar{a}/E) = \bar{b}/E$, we get that the number of types $\mathrm{tp}(g/A)$ for $g \in \mathbf{G}$ is bounded.

Now assume that both g and h in $\mathbf{G}$ are generic over some finite $X \subset \mathbf{P}/E$ such that $\mathrm{tp}(g/A) = \mathrm{tp}(h/A)$. We want to show that $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$.

Let $\bar{e} \in \mathbf{P}$ be finite such that $\bar{e}/E = X$. By lemma 5.9 there are generic witnesses $\bar{a}, \bar{b}$ for g over $\bar{e}$ and generic witnesses $\bar{c}, \bar{d}$ for h over $\bar{e}$.

Since $\mathrm{tp}(g/A) = \mathrm{tp}(h/A)$, there is $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ such that $\tau(g) = h$. We have that $\tau(\bar{a}), \tau(\bar{b})$ are generic witnesses for h over $\emptyset$. Then by Lemma 5.10 there are $\bar{c}'$, $\bar{d}'$ generic witnesses for h over $\emptyset$ realizing $\mathrm{Lstp}(\tau(\bar{a}), \tau(\bar{b})/A) = \mathrm{Lstp}(\bar{a}, \bar{b}/A)$ such that $\bar{c}'/E = \bar{c}/E$ and $\bar{d}'/E = \bar{d}/E$.

We claim that $\bar{c}'\bar{d}' \downarrow_A \bar{e}$. By domination, it is enough to show that

$$\bar{c}', d_1', ..., d_n' \downarrow_A \bar{e}.$$

But hence $\bar{c}' \subset \mathrm{cl_A}(\bar{c})$ and $d_1', ..., d_n' \subset \mathrm{cl_A}(d_1, ..., d_n)$ and vice versa, we have that

$$\dim(\bar{c}', d_1', ..., d_n'/A \cup \bar{e}) =$$

$$\dim(\bar{c}, \bar{c}', d_1, ..., d_n, d_1', ..., d_n'/A \cup \bar{e}) = \dim(\bar{c}, d_1, ..., d_n/A \cup \bar{e}) = 2n+1.$$

Since then $\bar{c}', d_1', ..., d_n'$ is independent over $\bar{e}$, we get the claim.

Similarly by domination, $\bar{a}, \bar{b} \downarrow_A \bar{e}$. Now $\bar{c}'\bar{d}' \downarrow_A \bar{e}$, $\bar{a}, \bar{b} \downarrow_A \bar{e}$ and the sequences $\bar{c}'\bar{d}'$ and $\bar{a}, \bar{b}$ realize the same Lascar strong type over A. By stationarity, they realize the same Lascar strong type over $A \cup \bar{e}$. Hence there is $\tau \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{e})$

mapping $\bar{a}\bar{b}$ to $\bar{c}'\bar{d}'$. This τ also fixes $X \subset \mathbf{P}/E$ pointwise. By $n+1$-determinacy, $\tau'(g) = h$, and hence we are done with the proof. $\square$

The following corollary follows from Fact 5.6

Corollary 5.12. *The connected component $\mathbf{G}^0$ has unique generics with respect to Σ.*

5.1. Localization and hereditarily unique generics

The following definitions of a localised group action and hereditarily unique generics are from [13] and are the same for any group G (Σ, n)-acting on a pregeometry (P, cl), where G is ω-homogeneous with respect to some group Σ of automorphism of the group action.

When $B \subseteq P$ is an independent set of size k with $k < n$, we can form a new ω-homogeneous group action by *localizing* at B: The group G_B is the pointwise stabilizer of B, $G_B = \{g \in G : g \restriction B = Id\}$, which is a subgroup of G. The pregeometry P_B is obtained from P by considering the new closure operator

$$\mathrm{cl}_B(X) = \mathrm{cl}(B \cup X) \setminus \mathrm{cl}(B)$$

on the set $P \setminus \mathrm{cl}(B)$; then G_B acts on P_B by restriction; and let Σ_B be the group of automorphisms in σ fixing B pointwise. Then the group G_B $(\Sigma_B, n-k)$-acts on the pregeometry P_B.

If P is a *geometry*, it is not necessarily true that P_B would be a geometry, since for the elements $b \in P \setminus \mathrm{cl}(B)$, the closure $\mathrm{cl}(B \cup \{b\})$ is not necessarily contained in $\mathrm{cl}(b) \cup \mathrm{cl}(B)$. However, ω-homogeneity (now with respect to Σ_B), infinite-dimensionality and empty closure of the empty set are inherited.

Definition 5.13. *Assume that a group G (Σ, n)-acts on a pregeometry (P, cl). We say that G* admits hereditarily unique generics *if G has unique generics and for any independent k-set $B \subseteq P$ with $k < n$ there is a normal subgroup G' of G_B such that G' $(\Sigma', n-k)$-acts on the pregeometry P_B (for some subgroup Σ' of Σ), which has unique generics with respect to Σ.*

We claim that $\mathbf{G}^0$ admits hereditarily unique generics. For any independent k-tuple $\bar{x}$ in $\mathbf{P}/E$, we should consider the $(\Sigma_{\bar{x}}, n-k)$-action $(\mathbf{G}^0)_{\bar{x}}$ on $(\mathbf{P}/E)_{\bar{x}}$, where the connected component is defined with $\Sigma_{\bar{x}}$ and hence is $\Sigma_{\bar{x}}$-invariant. To prove that this action has unique generics it is enough to show that any g generic in $(\mathbf{G}^0)_{\bar{x}}$ is also generic in $\mathbf{G}^0$. Then, since $\mathbf{G}^0$ has unique generics, for any two such generics g, h there is $\sigma \in \Sigma$ mapping g to h. Note that by definition it is enough that $(\mathbf{G}^0)_{\bar{x}}$ has unique generics with respect to Σ.

To simplify notation we write $(\mathbf{G}^0)_{\bar{a}}$ for $(\mathbf{G}^0)_{\bar{x}}$, where $\bar{x} = \bar{a}/E$.

Proposition 5.14. *Let $\bar{a} = a_1, ..., a_k$ be an independent k-tuple for $0 < k < n$. Assume that g generic in $(\mathbf{G}^0)_{\bar{a}}$. Then it is also generic in $\mathbf{G}^0$.*

Proof: Since g is generic in $(\mathbf{G}^0)_{\bar{a}}$, there are $a_{k+1}, ..., a_n$, $b_{k+1}, ..., b_n$in $\mathbf{P}$ such that $g(a_i/E) = b_i/E$ for each $i \in \{k+1, ..., n\}$ and

$$\dim(a_{k+1}, ..., a_n, b_{k+1}, ..., b_n/A \cup \bar{a}) = 2(n-k).$$

Denote $\bar{a}' = a_1, ..., a_k, a_{k+1}, ..., a_n$ and $\bar{b} = b_1, ..., b_n = a_1, ..., a_k, b_{k+1}, ..., b_n$. Then we have that $g(\bar{a}'/E) = \bar{b}/E$ and

$$\dim(\bar{a}'\bar{b}/A) = n + (n-k).$$

We choose a_{n+1}, b_{n+1} such that

$$a_{n+1} \downarrow_A \bar{a}'\bar{b},$$

and $g(a_{n+1}/E) = b_{n+1}$. Then we choose a_{n+2}, b_{n+2} respectively such that

$$a_{n+2} \downarrow_A \bar{a}', \bar{b}, a_{n+1}, b_{n+1}$$

and $g(a_{n+2}/E) = b_{n+2}/E$. It follows that

$$\dim(\bar{a}', a_{n+1}, a_{n+2}, \bar{b}, b_{n+1}, b_{n+2}/A) = n + (n-k) + 2.$$

Denote $\bar{c} = a_2, ..., a_{n+2}, b_2, ..., b_{n+2}$. We claim that $\dim(\bar{c}/A) = n + (n-k) + 2$. Since $a_1 = b_1$, it is enough to show that

$$a_1 \not\downarrow_A \bar{c}.$$

We assume to the contrary, that $a_1 \downarrow_A \bar{c}$. The by extension we can choose d_p realizing $\mathrm{Lstp}(a_1/A \cup \bar{c})$ for $p = 1, ..., n-k+1$ such that $d_p \downarrow_A \bar{c}, a_1, d_1, ..., d_{p-1}$. Since the $2(n+1)$-sequence $\bar{c}$ determines g, we must have that $g(d_p/E) = d_p/E$ for each $p = 1, ..., n-k+1$. But then g fixes the $n+1$- sequence $d_1/E, ..., d_{n-k+1}/E, a_1/E, ..., a_k/E$ and hence by $n+1$-determinacy we must have that $g = \mathrm{Id}_{(\mathbf{P}/E)}$. On the other hand $g(a_{k+1}/E) = b_{k+1}/E \neq a_{k+1}/E$. This contradiction proves the claim, that is

$$\dim(a_2, ..., a_{n+2}, b_2, ..., b_{n+2}/A) = n + (n-k) + 2.$$

Furthermore, for each $m \in \{1, ..., k-1\}$ we choose a_{n+2+m} and b_{n+2+m} such that $g(a_{n+2+m}/E) = b_{n+2+m}$ and

$$a_{n+2+m} \downarrow_A \bar{a}'\bar{b}, a_{n+1}, ..., a_{n+1+m}, b_{n+1}, ..., b_{n+1+m}.$$

As in the previous claim, we conclude that

$$\dim(a_{2+m}, ..., a_{n+2+m}, b_{2+m}, ..., b_{n+2+m}/A) = n + (n-k) + 2 + m.$$

Then finally when $m = k-1$ we get that

$$\dim(a_{k+1}, ..., a_{n+k+1}, b_{k+1}, ..., b_{n+k+1}/A) = 2n + 1.$$

Now we have shown that g is generic in $\mathbf{G}^0$. $\square$

As explained below Definition 5.13, we get the following corollary.

Corollary 5.15. $\mathbf{G}^0$ *admits hereditarily unique generics.*

We mention another corollary.

Corollary 5.16. *Assume that $\bar{x} \in \mathbf{P}/E$ is an independent k-tuple for $k < n$. Let $g \in (\mathbf{G}^0)_{\bar{x}}$. Then g is generic in $\mathbf{G}^0$ if and only if g is generic in $(\mathbf{G}^0)_{\bar{x}}$*

Proof: The other direction follows from Proposition 5.14. Then assume that g is generic in $\mathbf{G}^0$ and fixes $\bar{x}$. Hence there is an independent n-sequence $\bar{y} = y_1, ..., y_n$ such that $\dim(\bar{y}, g(\bar{y})/A) = 2n$. Since at most k elements of the independent sequence $\bar{y}, g(\bar{y})$ can belong to $\mathrm{cl}(\bar{x})$, we may assume that $y_{k+1}, ..., y_n, g(y_{k+1}), ..., g(y_n)$ are outside $\mathrm{cl}(\bar{x})$. Since

$$\dim(y_{k+1}, ..., y_n, g(y_{k+1}), ..., g(y_n)/A) = 2(n-k),$$

these elements witness that g is generic in $(\mathbf{G}^0)_{\bar{x}}$. □

Hereditarily unique generics gives us either a non-classical group or n-determinacy and furthermore that $n \in \{1, 2, 3\}$. These are Definitions 1.1 and 1.11 of Hyttinen, Lessman and Shelah [13] and Facts 2.10 and 2.12 of [13] referring to Theorem 2.7 and Lemma 2.8 of Hyttinen [8]. It is a open question whether non-classical groups exist.

Definition 5.17. *We say that a group G* carries an ω-homogeneous pregeometry *if there exists a closure operator* cl *on the subsets of G satisfying the axioms of a pregeometry with $\dim(G) = |G|$, and such that whenever $A \subseteq G$ is finite and $a, b \notin \mathrm{cl}(A)$, then there is an automorphism of G, preserving* cl *and fixing A pointwise and sending a to b.*

We say that a group G is non-classical *if it is nonabelian and carries an ω-homogeneous pregeometry.*

In the following facts we assume that the pregeometry (P, cl) is infinite-dimensional and that $\mathrm{cl}(\emptyset) = \emptyset$.

Fact 5.18. *Assume that G (Σ, n)-acts on a pregeometry (P, cl). Assume that G admits hereditarily unique generics. Then either $(G_B)^0$ is non-classical, for some independent $(n-1)$-subset $B \subseteq P$ or the action of G on P is n-determined.*

Fact 5.19. *Assume that the (Σ, n)-action of G on a pregeometry (P, cl) is n-determined. Then $n \in \{1, 2, 3\}$.*

We prove a small Lemma which will be used several times in the proof of the main theorem. A similar Lemma is used to prove Fact 5.18, but the proof if simpler due to 1-determinacy.

Lemma 5.20. *Assume that G $(\Sigma, 1)$-acts on an infinite-dimensional pregeomerty (P, cl), where $\mathrm{cl}(\emptyset) = \emptyset$. Assume that the action is 1-determined. Then G admits an ω-homogeneous pregeometry.*

Proof: We define a closure operator cl on the subsets of G as follows: for $g \in G$ and $g_1, ..., g_k \in G$ we let

$$g \in \mathrm{cl}(g_1, ..., g_k),$$

if for some element $y \in P$ and $x \in P \setminus \mathrm{cl}(y, g(y), g_1(y), ..., g_k(y))$ we have that

$$g(x) \in \mathrm{cl}(x, g_1(x), ..., g_k(x)).$$

We note that then the same holds for all such x and y: let

$$x' \notin \mathrm{cl}(y', g(y'), g_1(y'), ..., g_k(y')).$$

Let z be such that

$$z \notin \mathrm{cl}(y, g(y), g_1(y), ..., g_k(y), y', g(y'), g_1(y'), ..., g_k(y')).$$

Then since the action is ω-homogeneous with respect to Σ, there are $\tau, \tau' \in \Sigma$ such that $\tau(x) = z, \tau'(z) = x'$,

$$\tau \restriction \{y, g(y), g_1(y), ..., g_k(y)\} = \mathrm{Id} \text{ and}$$

$$\tau' \restriction \{y', g(y'), g_1(y'), ..., g_k(y')\} = \mathrm{Id}\,.$$

But then by 1-determinacy, $\tau'(g) = \tau(g) = g$ and $\tau'(g_i) = \tau(g_i) = g_i$ for each $i \in \{1, ..., k\}$. Hence $g(x) \in \mathrm{cl}(x, g_1(x), ..., g_k(x))$ if and only if $g(x') \in \mathrm{cl}(x', g_1(x'), ..., g_k(x'))$ by applying $\tau \circ \tau'$.

For an arbitrary subset $A \subseteq G$ we define that $g \in \mathrm{cl}(A)$ if there are $k < \omega$ and $g_1, ..., g_k \in A$ such that $g \in \mathrm{cl}(g_1, ..., g_k)$. It is not difficult to check that this induces a pregeometry on G with the same infinite dimension as P. Notice however, that even though the closure of the empty set is empty in P by assumption, the induced closure on G contains the identity element of G.

Also since the action is ω-homogeneous with respect to Σ, the induced pregeometry in ω-homogeneous with respect to Σ: suppose that $g, h \notin \mathrm{cl}(A)$ for some finite subset $A \subseteq G$. Then for some element $y \in P$ define $A(y) = \{f(y) : f \in A\}$ and let

$$x \in P \setminus \mathrm{cl}(y, g(y), h(y), A(y)\}.$$

Then by the definition of closure,

$$g(x), h(x) \notin \mathrm{cl}(x, A(x)).$$

There is $\tau \in \Sigma_{\{x, A(x)\}}$ mapping $g(x)$ to $h(x)$. Again by 1-determinacy, $\tau(g) = h$ and $\tau(f) = f$ for each $f \in A$. $\square$

6. The main result

We want to use Theorem 2.32 of Hyttinen, Lessmann and Shelah [13] to conclude the main result of this paper. There is one more obstacle we have to be aware of. In [13] it is assumed that $\dim(P) > 2^{|\operatorname{cl}(B)|}$ for any finite subset B of the pregeometry P. There it is a minor assumption, since in the quasiminimal case the closure $\operatorname{cl}(B)$ of a finite set B is bounded. Here we cannot assume such a thing. However, we are able to copy the proofs of [13] only replacing the parts where this assumption is used and conclude our main result. More specifically, this assumption is used in Lemmas 2.17 and 2.28 of [13], which are needed to prove Proposition 2.29. We will reprove these, but only in our context, not in the context of a group acting on an arbitrary pregeometry. Note that the assumption $\dim(P) > 2^{|\operatorname{cl}(B)|}$ is not used in the paper Hyttinen [8].

6.1. The pregeometry $(\mathbf{P}/E)_x$ is a geometry

In this section we prove the following proposition, which replaces Proposition 2.29 of [13].

Proposition 6.1. *Assume that G^0 $(\Sigma, 3)$-acts on the geometry $\mathbf{P}/E$. Let x be an element in $\mathbf{P}/E$. Then the pregeometry $(\mathbf{P}/E)_x$ is a geometry.*

First we prove the following Lemma, replacing Lemma 2.17 of [13].

Lemma 6.2. *Assume that G^0 $(\Sigma, 3)$-acts on $\mathbf{P}/E$ and the action is n-determined. Let x, y be independent elements in $\mathbf{P}/E$ and $g \in (G^0)_x$ generic such that that $g(y) = y$. Then g fixes $\operatorname{cl}(a, b)$ pointwise in $\mathbf{P}/E$.*

Proof: Note that it is impossible for g to fix anything pointwise in the pregeometry $\mathbf{P}$, since g is only defined for the equivalence classes in $\mathbf{P}/E$, not for the elements in $\mathbf{P}$. Note that it is equivalent to say that $x, y \in \mathbf{P}/E$ are independent and that y is an element in $(\mathbf{P}/E)_x$.

Notice also that by Corollary 5.16, if $g \in G^0$ fixes x it is equivalent to say that g is generic in G^0 or g is generic in $(G^0)_x$.

Since g is generic (in $(\mathbf{G}^0)_{x,y}$), there is z independent of x and y such that

$$\dim(x, y, z, g(z)/A) = 4.$$

Now it suffices to find *some* generic $g' \in G^0$ such that $g' \restriction cl(x, y) = \operatorname{Id}$. Since then there is z' such that $\dim(x, y, z', g(z')/A) = 4$ and hence there is a strong automorphism $\tau \in \operatorname{Saut}(\mathfrak{M}/A)$ mapping $x, y, z', g(z')$ to $x, y, z, g(z)$ by Lemma 4.4. But now by 3-determinacy, $\tau'(g') = g$. Then since g' fixes $\operatorname{cl}(x, y)$ pointwise, also g fixes pointwise the set $\tau(\operatorname{cl}(x, y)) = \operatorname{cl}(x, y)$.

Let us write $x = a/E, y = b/E, z = c/E$ and $g(z) = d/E$, where a, b, c and d are elements in $\mathbf{P}$. We have that $\dim(a, b, c, d/A) = 4$. Although $\operatorname{cl}(a, b)$ might be unbounded, by Lemma 2.3 there exists a finite $D \subseteq \operatorname{cl}(a, b)$ such that

$$c, d \downarrow_{A \cup D \cup a, b} \operatorname{cl}(a, b).$$

By extension, there are c', d' realizing $\mathrm{Lstp}(c, d/A \cup D \cup a, b)$ such that

$$c', d' \downarrow_{A \cup D \cup a,b} c, d,$$

and furthermore a strong automorphism $\tau \in \mathrm{Saut}(\mathfrak{M}/A \cup D \cup a, b)$ mapping c, d to c', d'. Denote $h = \tau(g)$. Then $h(x, y) = x, y$ and $h(c'/E) = d'/E$. Since $\mathbf{G}^0$ is Σ-invariant, we have that $h \in \mathbf{G}^0$.

Furthermore, since τ fixes $\mathrm{cl}(a, b)$ as a set, we have that

$$c'd' \downarrow_{A \cup D \cup a,b} \mathrm{cl}(a, b).$$

By stationarity, for each finite $\bar{e} \in \mathrm{cl}(a, b)$ there is $\tau_{\bar{e}} \in \mathrm{Saut}(\mathfrak{M}/A \cup, a, b, \bar{e})$ mapping c, d to c', d'. By 3-determinacy, $\tau_{\bar{e}}(g) = h$.

Now let $a_1 \in \mathrm{cl}(a, b)$ and a_2 be such that $g(a_1/E) = a_2/E$. Then since g fixes $x, y = a, b/E$, also a_2 is in $\mathrm{cl}(a, b)$. Now let $\bar{e}$ in $\mathrm{cl}(a, b)$ contain a_1 and a_2. Then $h(a_1/E) = \tau_{\bar{e}}(g)((a_1/E)) = \tau_{\bar{e}}(g(\tau_{\bar{e}}^{-1}(a_1/E))) = a_2/E = g(a_1/E)$. This implies that

$$g \restriction \mathrm{cl}(x, y) = h \restriction \mathrm{cl}(x, y).$$

Hence $h^{-1} \circ g \in \mathbf{G}^0$ fixes $\mathrm{cl}(x, y)$. We need to show that $h^{-1} \circ g$ is generic. It is enough to show that it is generic in $(G^0)_{x,y}$.

Let us write $z' = c'/E$. Then $h(z') = d'/E$ and

$$\dim(z, g(z), z', h(z')/A \cup x, y) = 4.$$

Let $e \in \mathbf{P}/E$ be independent of $z, g(z), z', h(z'), x$ and y. By 3-determinacy, any $\tau \in \Sigma$ fixing $x, y, z, g(z), e$ must fix g^{-1} and hence also $g^{-1}(e)$. This implies that $g^{-1}(e) \in \mathrm{cl}_{\mathrm{A}}(x, y, z, g(z), e)$. Similarly, $h^{-1}(e) \in \mathrm{cl}_{\mathrm{A}}(x, y, z', h(z'), e)$. Hence

$$\dim(z, g(z), z', h(z'), e, g^{-1}(e), h^{-1}(e)/A \cup x, y) =$$

$$\dim(z, g(z), z', h(z'), e/A \cup x, y) = 5.$$

By the same argument $g(z) \in \mathrm{cl}_{\mathrm{A}}(x, y, z, e, g^{-1}(e))$, $h(z') \in \mathrm{cl}_{\mathrm{A}}(x, y, z', e, h^{-1}(e))$ and hence

$$\dim(z, z', e, h^{-1}(e), g^{-1}(e)/A \cup x, y) = 5.$$

Thus $\dim(h^{-1}(e), g^{-1}(e)/A \cup x, y) = 2$, where $(h^{-1} \circ g)(g^{-1}(e)) = h^{-1}(e)$. This proves that $h^{-1} \circ g$ is generic in $(G^0)_{x,y}$. □

The proof of Lemma 2.28 of [13] uses again that $\mathrm{cl}(A)$ is bounded for a finite set A, but this is not really needed. We reprove a part of Lemma 2.28.

Lemma 6.3. *Assume that the $(\Sigma, 2)$-action of $(G^0)_x$ on the pregeometry $\mathbf{P}/E_x$ is 2-determined. Let $y, z \in \mathbf{P}/E_x$ be independent and $f \in (G^0)_x$ be such that for all $g \in (\mathbf{G}^0)_x$, $gfg^{-1}(y) \in \mathrm{cl}_x(y)$ and $gfg^{-1}(z) \in \mathrm{cl}_x(z)$.*

Then there are $k, l \in (\mathbf{G}^0)_x$ such that $kfk^{-1} = lfl^{-1}$ and

$$\dim(y, z, k(y), k(z), l(y), l(z)/A \cup x) = 6.$$

Proof: By simplicity and extension, there are independent $y' = a/E$ and $z' = b/E$ such that

$$a, b \downarrow_A x, y, z, f(y), f(z)$$

and hence $\dim(y', z', y, z/A \cup x) = 4$. Here we abuse the notation to mean that a, b are free of some representatives of the equivalence classes of x, y etc in $\mathbf{P}$. Since $(G^0)_x$ has rank 2, there is $k \in (G^0)_x$ such that $k(y, z) = y', z'$. Now since $kfk^{-1}(y) \in \mathrm{cl}_x(y)$ and $kfk^{-1}(z) \in \mathrm{cl}_x(z)$, we have that

$$a, b \downarrow_A x, y, z, f(y), f(z), kfk^{-1}(y), kfk^{-1}(z).$$

Then let $y'' = c/E, z'' = d/E$ be such that $\dim(y, z, y', z', y'', z''/A \cup x) = 6$, $c, d \models \mathrm{Lstp}(a, b/A)$ and

$$c, d \downarrow_A x, y, z, f(y), f(z), kfk^{-1}(y), kfk^{-1}(z).$$

Hence by stationarity, there is $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ fixing x, y, z, $f(y)$, $f(z)$, $kfk^{-1}(y)$, $kfk^{-1}(z)$ and mapping y', z' to y'', z''. Then by 2-determinacy, $\tau'(f) = f$ and $\tau'(kfk^{-1}) = kfk^{-1}$. We choose $l = \tau'(k)$. Then $l^{-1} = \tau'(k^{-1})$.

Now we claim that these k, l will do. We have that $l(x, y, z) = \tau(k(\tau(x, y, z))) = x, y'', z''$ and hence $l \in (\mathbf{G}^0)_x$ and $\dim(y, z, k(y), k(z), l(y), l(z)/A \cup x) = 6$. Furthermore, for any element $w \in \mathbf{P}/E$,

$$kfk^{-1}(w) = \tau'(kfk^{-1})(w) = \tau'(k)\tau'(f)\tau'(k^{-1})(w) = lfl^{-1}(w),$$

since $\tau(kfk^{-1})\tau^{-1}(w) = (\tau k \tau^{-1})(\tau f \tau^{-1})(\tau k^{-1} \tau^{-1})(w)$. □

With these Lemmas the proof of Proposition 6.1 is identical to the proof of Proposition 2.29 of [13]. Note that this implies that the pregeometry cl_{A} on $\mathbf{P}$ is 2-trivial: since for any pair $x, y \in \mathbf{P}/E$, $\mathrm{cl}_{\mathrm{A}}(x, y) = \mathrm{cl}_{\mathrm{A}}(x) \cup \mathrm{cl}_x(y) = \{x, y\}$, we get that for any $a, b \in \mathbf{P}$, $\mathrm{cl}_{\mathrm{A}}(a, b) = \mathrm{cl}_{\mathrm{A}}(a) \cup \mathrm{cl}_{\mathrm{A}}(b)$.

6.2. The main result

Now our main result follows as Theorem 2.32 of Hyttinen, Lessmann and Shelah [13]. We recall the main ingredients, but the proofs are identical. We define

$$I = \{g \in G : g^2 = 1\} \text{ and}$$

$$N_x = \{g \in G : \text{ the set } \{h(x) : h \in I, gh \notin I\} \text{ has bounded dimension}\}.$$

Several properties of N_x are shown in [13]. We list here those that are needed for the proof of our main theorem.

Fact 6.4. *Assume that a group G $(\Sigma, 2)$-acts on an infinite-dimensional geometry (P, cl) with $\mathrm{cl}(\emptyset) = \emptyset$. Then for each $x \in P$, $N_x \subseteq G$ is an invariant normal subgroup and $G = N_x \rtimes G_x$. Also the group N_x $(\Sigma', 1)$-acts on (P, cl), where the action is n-determined and Σ' is obtained from Σ by restriction.*

Furthermore, if G_x and N_x are abelian, then P can be given the structure of an algebraically closed field $(K, +, \times, 0, 1)$ and the action of $\mathbf{G}$ on P is isomorphic to the affine action of $K^+ \rtimes K^*$, $x \mapsto l + kx$, on K. Moreover, the field structure on P and the isomorphism of the group action are invariant once the identities of the field $0, 1$ are chosen.

The proof is as the proof of Propositions 2.27 and 2.31 of [13].

Theorem 6.5. *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, superstable finitary AEC and let* $\mathfrak{M}$ *be the monster model for* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$*. Assume that* A *is a finite set,* p *is an unbounded and regular Lascar strong type over* A *and* $\mathbf{Q}$ *is an* A*-invariant subset of* $\mathfrak{M}$*. Assume that there exists an integer* $0 < n < \omega$ *such that*

1. *For any independent sequence* $(a_1, ..., a_n)$ *of realizations of* p *and any finite subset* C *of* $\mathbf{Q}$ *we have*
$$\dim(a_1, ..., a_n/A) = \dim(a_1, ..., a_n/A \cup C).$$
2. *For some independent sequence* $a_1, ..., a_{n+1}$ *of realizations of* p *there is* C *a finite subset of* $\mathbf{Q}$ *such that* $(a_1, ..., a_n)$ *dominates* $(a_1, ..., a_{n+1})$ *over* $A \cup C$*.*

Then $\mathfrak{M}$ *interprets a group* G *which acts on the geometry* $\mathbf{P}/E$ *induced on the set* $\mathbf{P}$ *of realizations of* p*. Furthermore, either* $\mathfrak{M}$ *interprets a nonclassical group or* $n \in \{1, 2, 3\}$ *and*

- *If* $n = 1$*, then* G *is abelian and acts regularly on* $\mathbf{P}/E$*.*
- *If* $n = 2$*, the action of* G *on* $\mathbf{P}/E$ *is isomorphic to the affine action of* $K^+ \rtimes K^*$ *on the algebrically closed field* K*.*
- *If* $n = 3$*, the action of* G *on* $\mathbf{P}/E$ *is isomorphic to the action of* $PGL_2(K)$ *on the projective line* $\mathbb{P}^1(K)$ *of the algebraically closed field* K*.*

Proof: The group $\mathbf{G}$ is interpretable in $\mathfrak{M}$ by proposition 4.10. This group acts on the geometry $\mathbf{P}/E$; the action has rank n and is $n+1$-determined. Futhermore, $\mathbf{G}^0$ admits hereditarily unique generics with respect to the set of automorphisms induced by strong automorphisms of $\mathfrak{M}$. $\mathbf{G}^0$ is an invariant subgroup of of $\mathbf{G}$ and therefore interpretable. But, $\mathbf{G}^0$ (Σ^0, n)-acts on the geometry $\mathbf{P}/E$ (Σ^0 is simply obtained from Σ by restriction) and has hereditarily unique generics. Hence, we let $G = \mathbf{G}^0$. We also write only Σ for Σ^0.

Assume that $\mathfrak{M}$ does not interpret a nonclassical group. Then the action of G on $\mathbf{P}/E$ is n-determined by Fact 5.18, since groups of the form $((G^0)_B)^0$ are interpretable in $\mathfrak{M}$. Furthermore, then $n \in \{1, 2, 3\}$ by Fact 5.19.

Let $n = 1$. Since the $(\Sigma, 1)$-action of G on $\mathbf{P}/E$ is 1-determined, it is regular. Moreover, G carries a homogeneous pregeometry by Lemma 5.20. Since it cannot be nonclassical, it must be abelian.

Let $n = 2$. By Fact 6.4, N_x $(\Sigma', 1)$ acts on $\mathbf{P}/E$, where the action is 1-determined. Also G_x acts on the pregeometry $(\mathbf{P}/E)_x$ with an 1-determined action. The groups N_x and G_x are interpretable in $\mathfrak{M}$ and hence it follows from Lemma 5.20 that N_x and G_x must be abelian. Now the result follows from Fact 6.4.

Let $n = 3$. Choose a point $y \in \mathbf{P}/E$ and call it ∞. Then the $(\Sigma_\infty, 2)$-action of G_∞ on $(\mathbf{P}/E)_\infty$ is 2-determined. By Proposition 6.1, $(\mathbf{P}/E)_\infty$ is a geometry.

Choose $x \in (\mathbf{P}/E)_\infty$ and call it 0. Call $N_{\infty,0}$ the group N_x defined for $(\mathbf{P}/E)_\infty$ and let $G_{\infty,0}$ the group of elements in G_∞ fixing also 0. Then the

1-actions of $G_{\infty,0}$ on $(\mathbf{P}/E)_{\infty,0}$ and $N_{\infty,0}$ on $(\mathbf{P}/E)_{\infty}$ are 1-determined and the groups are interpretable in $\mathfrak{M}$. Again they must be abelian by Lemma 5.20.

By Proposition 6.4, the action of $G_{\infty} = N_{\infty,0} \rtimes G_{\infty,0}$ on $(\mathbf{P}/E)_{\infty}$ is isomorphic to the affine action of $K^{+} \rtimes K^{*}$ on the algebraically closed field K (notice that $0 \in (\mathbf{P}/E)_{\infty}$ chosen above is the 0 of the field). Let $1 \in (\mathbf{P}/E)_{\infty}$ be the identity element for the multiplicative structure of the field K. Since $(\mathbf{P}/E)_{\infty}$ is a geometry, the set $\{0, 1, \infty\} \subset \mathbf{P}/E$ is 3-dimensional.

Since the $(\Sigma, 3)$-action of G on $\mathbf{P}/E$ is 3-determined, there is a unique $\alpha \in G$ such that $\alpha(0) = \infty$, $\alpha(\infty) = 0$ and $\alpha(1) = 1$. Notice that $\alpha^2 = 1$ by 3-determinacy. Exactly as in the proof of Theorem 2.32 in [13] we see that conjugation by α induces an idempotent automorphism τ of $G_{\infty,0}$, which is not the identity and furthermore, $\tau(g) = g^{-1}$ for each $g \in G_{\infty,0}$.

We can now complete the proof as in [13]: the geometry $\mathbf{P}/E$ is isomorphic to the projective line $\mathbb{P}^1(K)$, with ∞ being the point at infinity. Given $x \in K^{*}$, choose $h \in G_{\infty,0}$ such that $h1 = x$. Then $\alpha x = \alpha h x = h^{-1}\alpha 1 = h^{-1}1 = x^{-1}$. Also α permutes 0 and ∞, so α acts like an inversion on $\mathbb{P}^1(K)$. It follows that G contains the group of automorphisms of $\mathbb{P}^1(K)$ generated by the affine transformations and inversion. Hence $PLG_2(K)$ embeds in G. The actions of $PLG_2(K)$ and G are both sharply 3-transitive: any three elements $x, y, z \in \mathbf{P}/E$ are independent, since $\mathbf{P}/E$ is 2-trivial (see the note in the end of section 6.1), and hence there is exactly one $g \in G$ mapping a triple x, y, z to another triple x', y', z'. Hence the embedding of $PLG_2(K)$ on G is surjective.

The projective line structure and the isomorphism of the group action are invariant over the points $0, 1, \infty \in \mathbf{P}/E$. □

Finally we study the relation of our theorem to the results in the paper [13]. The following lemma remarks that moving from quasiminimal types to regular Lascar strong types is a real generalization.

Lemma 6.6. *Assume that p is an unbounded quasiminimal type over a finite set A. Then p is a Lascar strong type. Furthermore, p is regular.*

Proof: We recall that a type q is said to be bounded is the set of realizations of q in $\mathfrak{M}$ is bounded. We recall that a type q over a set A is *quasiminimal* if for every bounded B, every $A \cup B$-invariant subset of the set of realizations of q is bounded or co-bounded. We often identify q with the set of realizations.

First we show that p is stationary as a weak type: let B be a bounded set and $a, b \models p$ such that $\mathrm{tp}^{\mathrm{w}}(a/B) \neq \mathrm{tp}^{\mathrm{w}}(b/B)$. By quasiminimality, either $\mathrm{tp}^{\mathrm{w}}(a/B)$ or $\mathrm{tp}(b/B)$ must be bounded. If $\mathrm{tp}^{\mathrm{w}}(a/B)$ is bounded, we have that $a \not\downarrow_A B$. Hence both a and b cannot realize free extensions of p.

To see that p is a Lascar strong type, let again $a, b \models p$. We want to show that $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$. For this, we use extension to find $c \models \mathrm{Lstp}(a/A)$ such that $c \downarrow_A a, b$. By symmetry and since p is stationary, we have that $\mathrm{tp}^{\mathrm{w}}(a/A \cup c) = \mathrm{tp}^{\mathrm{w}}(b/A \cup c)$. Since A is finite, there is $f \in \mathrm{Aut}(\mathfrak{M}/A \cup c)$ mapping a to b. Then since $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(c/A)$, the automorphism gives that $\mathrm{Lstp}(f(a)/A) = \mathrm{Lstp}(f(c)/A)$, which gives that $\mathrm{Lstp}(b/A) = \mathrm{Lstp}(c/A) = \mathrm{Lstp}(a/A)$.

Finally we show that p is regular. We notice that the closure

$$\mathrm{cl_A}(B) = \{b \models p : b \not\downarrow_A B\}$$

agrees with the following *bounded closure*:

$$\mathrm{bcl_A}(B) = \{b \models p : \mathrm{tp^w}(b/A \cup B) \text{ is bounded}\}.$$

The direction $\mathrm{bcl_A}(B) \subseteq \mathrm{cl_A}(B)$ is given by quasiminimality: since the free extension of p to B is unbounded, any 'non-free' extension is included in the complement and hence must be bounded.

We know that the properties of $\downarrow$ imply all other properties of a pregeometry for $\mathrm{cl_A}(\cdot)$, except maybe the property (ii), that $\mathrm{cl_A}(\mathrm{cl_A}(B)) = \mathrm{cl_A}(B)$. By finite character, it would be enough to show this for finite sets B. By the previous equivalence, also bounded closure satisfies finite character. But now, the bounded closure can be easily seen to satisfy the property (ii) for finite sets: Assume that $a \in \mathrm{bcl}(\mathrm{bcl}(B))$ for B finite. We want to show that $\mathrm{tp^w}(a/B)$ is bounded. Assume towards a contradiction, that $(a_i)_{i<\lambda}$ are unboundedly many realizations of $\mathrm{tp^w}(a/A \cup B)$. Since B is finite, for each i there is $f_i \in \mathrm{Aut}(\mathfrak{M}/A \cup B)$ mapping a_0 to a_i. By finite character, we have that $\mathrm{tp^w}(a/A \cup B \cup C)$ is bounded for some finite $C \subset \mathrm{bcl}(B)$. Then since $\mathrm{tp^w}(c/A \cup B)$ is bounded for each $c \in C$, we have that unboundedly many f_i must map each $c \in C$ similarly, say f_i maps C to C' for each $i \in I \subset \lambda$. Let $g \in \mathrm{Aut}(\mathfrak{M}/A \cup B)$ map C' to C. Now $g(f_i(a_0))$, $i \in I$, are different realizations of $\mathrm{tp^w}(a/A \cup B \cup C)$, a contradiction. □

The following theorem compares the group of our main theorem and the groups of the theorems in [13]. For the details of the definitions, see [13].

Theorem 6.7. *Suppose* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$*,* p*,* Q*,* A *and* n *satisfy the assumptions of Theorem 6.5 and in addition those in [13] Theorem 3.19 or 4.19. Let* $\mathbf{G}$ *be the group constructed in the proof of Theorem 6.5 and* H *the group constructed in the proof in [13]. Then it holds that* $H^0 = \mathbf{G}^0$*.*

Proof: It is immediate by the definitions that H is a subgroup of $\mathbf{G}$ with the same action. Also H is preserved under all automorphisms of $\mathfrak{M}$ fixing A pointwise and since the action is $n+1$-determined, H is a normal subgroup of $\mathbf{G}$ ($fgf^{-1} = F(g)$ for suitable $F \in \mathrm{Aut}(\mathfrak{M}/A)$). But then the claim follows if in addition H is of bounded index. For this, let us look at f/H for some $f \in \mathbf{G}$. Take and $x_1, ..., x_{n+1}, y_1, ..., y_n \in \mathbf{P}/E$ so that $dim(x_1, ..., x_{n+1}, y_1, ..., y_n/A) = 2n+1$. Since the action of H is an n-action we can choose the reprecentative f of f/H so that for all $1 \leq i \leq n$, $f(x_i) = y_i$. Since the action of $\mathbf{G}$ does not have rank $\geq n+1$, $f(x_{n+1}) \subseteq \mathrm{cl_A}(x_{n+1} \cup \bigcup_{1 \leq i \leq n}(x_i \cup y_i))$. Since p is quasi-minimal, there is only boundedly many choises for $f(x_{n+1})$. Since the action of $\mathbf{G}$ is $n+1$-determined, the index is bounded. □

Notice that our argument above for this is very general. There are also other natural ways of defining the group than those in [13] and in this paper and it is possible to construct examples showing that the way of defining the group may affect on what the group itself is. However, the real target group i.e. the connected component remains the same.

7. Examples

In this section we give two examples of situations as in the main theorem. The first example is a vector space with an affine copy of itself and the second example is a class of Pappian projective planes.

7.1. Vector space with an affine copy

The first example is actually an elementary class, i.e. a class definable in first order logic, but we find it quite illustrative. We have not found many such examples in the literature. Furthermore, assuming that there are no non-classical groups, our theorem says that in the context of simple and superstable finitary AECs nothing happens that does not already happen with first order logic. Since our technique is somewhat different from the proof in elementary classes, this example demonstrates why all the steps are needed in our proof, especially why we have to move to the connected component.

Example 7.1. *Let κ be an infinite regular cardinal such that $\kappa^\omega = \kappa$. Let $r > 2$ be a prime number. Let G be the group of finite support functions*

$$f : \kappa \to (\mathbb{Z}/r\mathbb{Z})$$

i.e. the κ-dimensional vector space over the field $\mathbb{Z}/r\mathbb{Z}$. Then define

$$M = (G \times \kappa, (Q, +), R, E)$$

be a structure with the universe $G \times \kappa$ and with Q a unary predicate and E binary and R ternary relations on the universe. We define the predicate Q to be $G \times \{0\}$. The addition $+$ is defined on Q making it an isomorphic copy of G. Then we define E to be the following equivalence relation:

$$\begin{aligned} (a,\alpha)E(b,\beta) \quad \textit{iff} \quad & (a,\alpha) = (b,\beta) \textit{ or} \\ & \alpha,\beta > 0 \textit{ and } a = b, \end{aligned}$$

and define R as the relation:

$$\begin{aligned} ((a,\alpha),(b,\beta),(c,\gamma)) \in R \quad \textit{iff} \quad & \alpha = 0,\ \beta,\gamma > 0 \textit{ and} \\ & (b + a = c \textit{ or } b - a = c). \end{aligned}$$

Denote $P = M \setminus (G \times \{0\}) = \{(a,\alpha) \in M : \alpha > 0\}$. Furthermore, denote

$H = \{g \in Sym(P/E) : g$ is induced by an automorphism $f \in \mathrm{Aut}(M/G\times\{0\})\}$.

We find that for $\alpha > 0$, $\mathrm{tp}^{\mathrm{w}}((a,\alpha)/\emptyset)$ is an unbounded regular Lascar strong type and $P/E = \{x \in M : x \models p\}/E$ is a geometry with $\mathrm{cl}_\emptyset(\cdot)$, H is the group defined as in Definition 4.1, which is not abelian and the connected component H^0 is G.

We prove a few lemmas towards showing the claims of the example.

Lemma 7.2. *M is saturated with respect to first order types.*

Proof: Let N be an ω_1-saturated model of $Th(M)$ of power κ such that for all $a \in N$, the E^N-equivalence class of a has either the size 1 or κ and Q^N has power κ. We will construct an isomorphism $f : N \to M$. Since $Th(G)$ is totally categorical, we may assume that $G^N = (Q,+)^N = (Q,+)$ and let $f \restriction Q^N = \mathrm{Id}$.

Let us choose $x, y \in N \backslash G^N$ and $a \in G \times \{0\} \backslash \{(0,0)\}$ such that $(a,x,y) \in R^N$. Then choose $f(x), f(y)$ such that $(a, f(x), f(y)) \in R$.

We notice that $Th(M)$ implies the following statements:

1. $(R(a,x,y) \wedge R(b,y,z)) \to (R(b-a,x,z) \vee R(b+a,x,z))$
2. $R(a,x,y) \to (R(a,y,x) \wedge R(-a,x,y))$
3. $(R(a,x,y) \wedge R(b,y,z) \wedge R(b,y,z') \wedge (z \neq z')) \to$
   $((R(b-a,x,z) \wedge R(b+a,x,z')) \vee (R(b-a,x,z') \wedge R(b+a,x,z))$
4. $(R(a,x,y) \wedge xEx' \wedge yEy') \to R(a,x',y')$
5. $\forall x \forall y \exists a ((\neg Q(x) \wedge \neg Q(y)) \to R(a,x,y))$

Then for every $z \in N \setminus G^N$ we choose the equivalence class of $f(z)$ so that

$$R^N(b,y,z) \to R(b, f(y), f(z)) \text{ and}$$
$$R^N(b-a,x,z) \leftrightarrow R(b-a, f(x), f(z)).$$

By the choise of N, f can be chosen so that in addition for all $z \in N \setminus G^N$, $f \restriction (z/E^N)$ is a one to one function onto $f(z)/E$. Now a straight forward calculation shows that f is an isomorphism.

By looking the automorphisms of M, one can see that over any countable set, only countably many Galois-types are realized in M. Since N is ω_1-saturated, it follows that $Th(M)$ is ω-stable. Thus we could have chosen N to be saturated. $\square$

Hence M is a monster model for the complete first order theory $Th(M)$. We have that weak types agree with first order types and hence also with Galois types over all sets.

Lemma 7.3. *For $\alpha > 0$, $p = \mathrm{tp}^{\mathrm{w}}((a,\alpha)/\emptyset)$ is a regular Lascar strong type and $\mathrm{cl}_{\emptyset}(\{(a,\alpha)\}) = (a,\alpha)/E$.*

Proof: We show that the type is a Lascar strong type: Assume that $(a,\alpha), (b,\beta)$ both realize p. We want to show that they have the same Lascar strong type over $\emptyset$. Any two elements of $M \setminus Q^M$ can be mapped with an automorphism of M to such elements (a',α') and (b',β') that $a' \neq 0 \neq b'$, and thus we can assume that $a \neq 0 \neq b$. Similarly we may assume that $\alpha = \beta$. Furthermore, we can find $a_i, i < \omega$ such that both $a, a_i, i < \omega$ and $b, a_i, i < \omega$ are free in G. Hence it is easy to construct two intersecting indiscernible (over $\emptyset$) sequences such that (a,α) and (b,β) are elements in these sequences. Hence we get that $\mathrm{Lstp}((a,\alpha)/\emptyset) = \mathrm{Lstp}((b,\beta)/\emptyset)$.

Before show that p is regular we show that the type $E(v,(a,\alpha))$ (where v is a free variable) forks over $\emptyset$. Again we may assume that $a \neq 0$. Let $(b,\beta) \models p$ be such that $E((b,\beta),(a,\alpha))$. Let $(a_i)_{i<\omega}$, be a free sequence in G with $a_0 = a$. Then $((a_i,\alpha))_{i<\omega}$ is strongly $\emptyset$-indiscernible but $\neg E((b,\beta),(a_i,\alpha))$ and hence

$$\mathrm{tp}^{\mathrm{w}}((a_0,\alpha)/(b,\beta)) \neq \mathrm{tp}^{\mathrm{w}}((a_i,\alpha)/(b,\beta))$$

for any $i \neq 0$. Hence $(a,\alpha) \not\downarrow_{\emptyset} (b,\beta)$ for any (b,β) with $E((b,\beta),(a,\alpha))$.

Then we show that p is regular: Let $M' \preccurlyeq_{\mathbb{K}} M$ be of the form $(G' \times \lambda)$, where $\lambda \in \kappa$ and G' is a subspace of G, and let $(a,\alpha),(a,\alpha') \models p$ be in $M \setminus M'$ such that

$$(a,\alpha) \downarrow_{\emptyset} M' \text{ and } (a',\alpha') \not\downarrow_{\emptyset} M'.$$

Notice that the existence of M' follows from the fact that the Löwenheim-Skolem number is $\aleph_0$ and that a union of an increasing sequence of elementary submodels of M is an elementary submodel.

To show that p is regular, we must show that

$$(a,\alpha) \downarrow_{\emptyset} (a',\alpha').$$

First we claim that $a \notin G'$. If $a \in G'$, then $E((a,\alpha),(a,\beta))$ for some $(a,\beta) \in M'$. but as we showed before, this implies that $(a,\alpha) \not\downarrow_{\emptyset} M'$, a contradiction. Hence $a \notin G'$.

Then we claim that $a' \in G'$. To prove the claim it is enough to show that if $b, b' \notin G'$ and $\beta, \beta' > 0$ there is an automorphism $f \in \mathrm{Aut}(M/M')$ such that $f((b,\beta)) = (b',\beta')$. But this is clear.

We have shown that $a \notin G'$, $a' \in G'$ and $\alpha' \geq \lambda$. Now we can easily find $M'' \preccurlyeq_{\mathbb{K}} M$ containing $M' \cup (a,\alpha)$ such that $(a',\alpha') \notin M''$. Finally we find that for any M'' with $M' \preccurlyeq_{\mathbb{K}} M'' \preccurlyeq_{\mathbb{K}} M$ we have that

$$(a',\alpha') \not\downarrow_{M'} M'' \text{ if and only if } (a',\alpha') \in M''.$$

Then forking calculus (monotonicity, symmetry and transitivity) gives that p is regular. The direction $\Leftarrow$ follows from reflexivity, since $\mathrm{tp}^{\mathrm{w}}((a',\alpha')/M')$ is unbounded. We show $\Rightarrow$. We know that there is a free extension of $\mathrm{tp}^{\mathrm{w}}((a',\alpha')/M')$ to M'', and since $a' \in G'$, any realization of this extension must be E-equivalent to (a',α'). Clearly if $(a',\beta),(a',\beta') \notin M''$, there is an automorphism $f \in \mathrm{Aut}(M/M')$ such that $f((a',\beta)) = (a',\beta')$. This gives that if $(a',\alpha') \notin M''$, (a',α') realizes the free extension. We have shown $\Rightarrow$. Then we have shown that p is regular.

Finally we show that for $(a,\alpha) \models p$, $\mathrm{cl}_{\emptyset}((a,\alpha)) = ((a,\alpha)/E)$. We already showed that $((a,\alpha)/E) \subseteq \mathrm{cl}_{\emptyset}((a,\alpha))$. The other direction again follows from that any $(b,\beta),(b',\beta') \models p$ not equivalent to (a,α) can be permuted by an automorphism fixing (a,α). Since there is a free extension of p to $\{(a,\alpha)\}$, any such (b,β) must realize it. □

Now we move to study the structure M/E. Clearly any automorphism of M/E can be extended to an automorphism of M, so it makes sense to talk about Galois types of elements of M/E.

Lemma 7.4.

$$H = \{g \in Sym(P/E) : \forall \text{ finite } X \subset P/E \text{ and finite } C \subseteq G \text{ there is } f \in \mathrm{Aut}(M/C \times \{0\}) \text{ s.t. } \forall x \in X\ g(x/E) = f(x)/E\ \}.$$

Proof: The other direction is clear, so we need to show that the group defined above is included in H. Assume that $g \in Sym(P/E)$ is as above. Then for all finite $a_1, ..., a_n \in P/E$ and finite $C \subset G$ we have that for Galois types,

$$\mathrm{tp}^{\mathrm{g}}(a_1, ..., a_n/C \times \{0\}) = \mathrm{tp}^{\mathrm{g}}(g(a_1), ..., g(a_n)/C \times \{0\}).$$

But then since M/E is homogeneous by Lemma 7.2, there is an automorphism of M/E fixing $G \times \{0\}$ and mapping a to $g(a)$ for each $a \in P/E$. Clearly this automorphism is induced by some automorphism of M. □

For each $a \in G$ let f_a and g_a be in $Sym(P/E)$ such that

$$f_a((b, \alpha)/E) = (a + b, \alpha)/E \text{ and } g_a((b, \alpha)/E) = (a - b, \alpha)/E.$$

Then clearly both $f_a, g_a \in H$, since the two automorphisms in $\mathrm{Aut}(M/G \times \{0\})$ mapping $(b, \alpha) \mapsto (b + a, \alpha)$ and $(b, \alpha) \mapsto (a - b, \alpha)$ for $\alpha > 0$ preserve the relation R^M.

Lemma 7.5. $H = \{f_a, g_a : a \in G\}$.

Proof: Let h be in H. We claim that h is of the form f_a or g_a for some $a \in G$. We work in M/E and write a shorthand $(a, b, c) \in R$ instead of $((a, 0), (b, \beta)/E, (c, \gamma)/E) \in R$.

Choose an arbitrary tuple (a, b, c) in R. We define two subrelations $R^+_{(a,b,c)}$ and $R^-_{(a,b,c)}$ of R. For any tuple $(a', b', c') \in R$, we decide whether $(a', b', c') \in R^+_{(a,b,c)}$ and whether $(a', b', c') \in R^-_{(a,b,c)}$ with the following procedure: We know that there is $a'' \in G \times \{0\}$ such that $(a'', b, c') \in R$. We define

$$a^* = a'' \text{ if } (a'' - a, c, c') \in R \text{ and}$$

$$a^* = -a'' \text{ if } (a'' + a, c, c') \in R.$$

In both cases we define that $(a^*, b, c') \in R^+_{(a,b,c)}$. Then finally we decide that

$$(a', b', c') \in R^+_{(a,b,c)} \text{ if } (a' - a^*, b', b) \in R \text{ and}$$

$$(a', b', c') \in R^-_{(a,b,c)} \text{ if } (a' + a^*, b', b) \in R.$$

From this procedure we can calculate that

$$\begin{aligned} (a', b', c') \in R^+_{(a,b,c)} \text{ if and only if } \quad & (c = b + a \text{ and } c' = b' + a') \\ & \text{or } (c = b - a \text{ and } c' = b' - a'), \\ (a', b', c') \in R^-_{(a,b,c)} \text{ if and only if } \quad & (c = b + a \text{ and } c' = b' - a') \\ & \text{or } (c = b - a \text{ and } c' = b' + a'). \end{aligned}$$

Then we let R^+ and R^- be $R^+_{(a,b,c)}$ and $R^-_{(a,b,c)}$ for some (a,b,c) such that $c = b+a$. By the previous calculation, R^+ and R^- do not depend on the choice of (a,b,c). Note that although $R^+_{(a,b,c)}$ and $R^-_{(a,b,c)}$ are definable from (a,b,c), the relations R^+ and R^- are not definable, since we do not know whether $c = b+a$ or $c = b-a$. However, if f is an automorphism of $\mathrm{Aut}(M/G \times \{0\})$ and $c = b+a$, we get that if

$$f(c) = f(b) + a, \text{ then } f \text{ preserves both } R^+ \text{ and } R^- \text{ and}$$

$$f(c) = f(b) - a, \text{ then } f \text{ switches the relations } R^+ \text{ and } R^-.$$

We have shown that if $h \in H$ is induced by an automorphism fixing $G \times \{0\}$ pointwise, it can either preserve both R^+ and R^- or switch them. Let us study the first case: Let b, c be in P. There are unique a, a' such that $h(b) = b+a$ and $c = b+a'$. Since h preserves R^+, also $h(c) = h(b) + a'$, and a calculation gives that $h(c) = c + a$. Hence h must be of the form f_a. In the latter case a similar calculation, using that there are a, a' such that $h(b) = a-b$ and $c = b+a'$, gives that h must be of the form g_a for some a. □

Finally, we get that H is isomorphic to $G \rtimes (\mathbb{Z}/2\mathbb{Z})$ with the operation $(a,x)(b,y) = (a + b^x, x+y)$, where $b^x = b$, if $x = 0$, and $b^x = -b$, if $x = 1$. Then H is not abelian. However, $H^0 = G$, where H^0 is by definition the intersection of all $\emptyset$-invariant, normal subgroups of H with bounded index.

7.2. Pappian projective planes

Then we introduce the second example. So called *Pappian* projective planes are known to interpret a field on the points of any any given line of the plane, see the book by R. P. Burn [5]. We can define such a class as an finitary abstract elementary class under *substructure*, although such planes are not elementarily equivalent and hence cannot be thought as an elementary class. However, to gain joint embedding, we must require all members of the class to embed a given projective plane, and this fixes the characteristic of the field interpreted. To prove amalgamation and that the class is closed under chains we use the result that the field really is interpreted and each plane is definable over the field. After that we notice that the monster model of the class is saturated with respect to first order types and also ω-stable. Hence the finitary class is superstable and simple. Furthermore, the all points of a given line are the realizations of a regular Lascar strong type p. We can get the requirements of Theorem 6.5 to hold in the case $n = 2$ and conclude that a field can be interpreted on the realizations of p. It might be possible to prove the same result without using the conclusion in the proof, but that might be almost as hard as to prove the conclusion the traditional way. Hence we only mention this example as a remark related to the history of geometric model theory, not as a reasonable application of the theorem. The model theory of projective planes has been studied for example in Baldwin [1].

Example 7.6 (Pappian projective planes). *Let $L = \{P, L, \in\}$ where P is the predicate for 'points', L is the predicate for 'lines' and $\in$ is the incidence relation. We say that an L-structure is a Pappian projective plane if it satisfies the following:*

- **P0** *P and L are a partition of the universe and $\in$ is a relation between elements of P and L.*
- **P1** *There exists four points, no three of which are collinear.*
- **P2** *There exists a unique line containing any two distinct points.*
- **P3** *Any two distinct lines have a unique point of intersection.*
- **P4** *The plane is* Pappian*, see [5].*

Every Pappian plane is also Desarguesian*, see [5].*

Let M be a (finite or) countable Pappian projective plane. Let $\mathbb{K}$ be the class of those Pappian projective planes which embed M and let $\preccurlyeq_{\mathbb{K}}$ be the substructure relation. Then $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a finitary abstract elementary class.

The monster model of $\mathbb{K}$ is a projective plane $\mathbb{P}^2(K)$ of an algebraically closed field K. Furthermore, it is homogeneous and saturated with respect to first order types, simple and ω-stable (and hence superstable).

Let k, l be lines in $\mathbb{P}^2(K)$ and ∞ their point of intersection. The types
$p = \{x \in l : x \neq \infty\}$ and $q = \{x \in k : x \neq \infty\}$
are unbounded, quasiminimal (and hence regular) Lascar strong types over the finite set $A = \{k, l, \infty\}$. Furthermore, for any independent sequence a_1, a_2 of realizations of p and any countable set C of realizations of q we have

$$\dim(a_1, a_2/A \cup C) = 2.$$

However, for some independent sequence a_1, a_2, a_3 of realizations of p there is a countable set C of realizations of q such that

$$\dim(a_1, a_2, a_3/A \cup C) = 2.$$

We sketch the proofs for the claims in Example 7.6. For this we will use fact that on any line l of a Pappian projective plane M, given three points P_0, P_1 and P_∞ on l, we can define two operations $+$ and $\times$ such that $K = (l \setminus \{P_\infty\}, +, 0, \times, 1)$ is a field with 0,1 interpreted as P_0, P_1.

Furthermore, if $M \subset N$ are two Pappian projective planes and l, P_0, P_1, P_∞ are chosen in M, the field defined in M is a subfield of the field defined in N.

Further still, M can be seen as the projective plane $\mathbb{P}^2(K)$ over the field K, where the set of points of M is [2]

$$\{(x, y, z)/E : (x, y, z) \in K^3 \setminus \{0\}\}$$

where E is the equivalence relation $(x, y, z)E(x', y', z')$ if $x' = \lambda x$, $y' = \lambda y$ and $z' = \lambda z$ for some non-zero $\lambda \in K$ and each line is of the form

$$[l, m, n] = \{(x, y, z) : (l, m, n) \in K^3 \setminus \{0\} \text{ where } lx + my + nz = 0\}.$$

---

[2] See *homogeneous coordinates* in [5].

Furthermore, the incidence relation is just $\in$. Conversely, for a given field K the projective plane $\mathbb{P}^2(K)$ is a Pappian projective plane.

First we need to show that the class is a finitary AEC. Especially, we need to show that it is closed under unions of chains and that is satisfies the amalgamation property. The other properties needed are trivial, since our notion of $\mathbb{K}$-elementary substructure is just substructure and joint embedding follows from amalgamation and the fact there is a prime model in the class.

First, we want to amalgamate M_1 and M_2 over M_0. We choose a line $l \in M_0$ and three points $P_0, P_1, P_\infty \in l \cap M_0$. Then we define fields K_0, K_1 and K_2 in M_0, M_1 and M_2 respectively, using the chosen line and points. Then we can amalgamate the fields K_1 and K_2 over K_0, and this induces an amalgamation of M_1 and M_2 over M_0.

Similarly, when M_i, $i < \alpha$, is an increasing chain of Pappian projective planes, we choose a line and three points in M_0 and define a field K_i in each M_i respectively. The union $\bigcup_{i<\alpha}$ is of the form $\mathbb{P}^2(K)$ for $K = \bigcup_{i<\alpha} K_i$ and hence in $\mathbb{K}$.

Finally, since $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a finitary abstract elementary class, we can study the monster model of $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. We find that the monster model is of the form $\mathbb{P}^2(K)$ for some large, algebraically closed field K. This model is homogeneous and saturated with respect to first order types. Furthermore, it is ω-stable and hence forking gives an independence calculus with the properties listed in Theorem 2.2. Hence by Theorem 4.9 of [9] forking is equal to our notion of independence $\downarrow$. We conclude that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is simple and $\aleph_0$-stable and hence superstable in the sense of Definition 2.1.

Let us fix lines k and l and their point of intersection ∞. Define the types

$$p = \{x \in l : x \neq \infty\} \text{ and } q = \{x \in k : x \neq \infty\}$$

as types over the set $\{l, k, \infty\}$. These types are clearly unbounded. We prove that they are regular Lascar strong types. It is enough to prove that p is a regular Lascar strong type. Furthermore, by Lemma 6.6 it is enough to show p is quasiminimal, that is, for every bounded B, every $\{l, k, \infty\} \cup B$-invariant subset of the set of realizations of p is bounded or co-bounded.

Since any two lines can be mapped to any other two lines with an automorphism, we may assume that $\infty = \{(0, k, 0) : k \in K \setminus \{0\}\}$, $l = [-1, 0, 0] = \{(0, y, z) : y, z \in K\}$ and $k = [0, 0, 1] = \{(x, y, 0) : x, y \in K\}$.

To show that p is quasiminimal, let B be bounded. We may assume that B is a set of points. Let $K_0 \subset K$ be a bounded subfield so that all coordinates of points in B are in K_0.

Assume that $S \subset l \setminus \{\infty\}$ and both S and $l \setminus S$ are unbounded. Then there must exist coordinates $(1, y, 0) \in S$ and $(1, y', 0) \in l \setminus S$ such that both y and y' are transcendental elements over the field K_0. There exists an automorphism of K fixing K_0 and mapping y to y'. This automorphism extends to an automorphism of $\mathbb{P}^2(K)$ taking $(1, y, 0) \in S$ to $(1, y', 0) \in l \setminus S$ and fixing each of B, l, k and ∞. Hence S cannot be $\{l, k, \infty\} \cup B$-invariant, and we have shown that p is quasiminimal.

We check the condition for dimensions. It is known that any two distinct points of $l\backslash\{\infty\}$ can be mapped to any other two distinct points of $l\backslash\{\infty\}$ by an automorphism fixing l setwise and k pointwise. Hence for any two independent a_1, a_2 of realizations of p and any bounded set C of realizations of q we have $\dim(a_1, a_2/A \cup C) = 2$. However, if a_1, a_2, a_3 are distinct points in $l\backslash\{\infty\}$, we can draw lines l_1, l_2 and l_3 intersecting l in the points a_1, a_2, a_3 respectively, which all intersect in some point $P \notin k \cup l$. Let b_1, b_2 and b_3 be the points where the lines intersect k. Now if f is an automorphism fixing the points a_1 and a_2 and the points b_1, b_2 and b_3 on k, it must fix P and the line l, and hence also the point a_3. We find that $\dim(a_1, a_2, a_3/A \cup \{b_1, b_2, b_3\}) \leq \dim(a_1, a_2/A \cup \{b_1, b_2, b_3\}) \leq 2$. Then the condition 2. of Theorem 6.5 follows, since p is quasiminimal.

Finally we will show that if $\mathbf{G}$ is the group constructed in the proof of Theorem 6.5 for this example, then $\mathbf{G}$ is (isomorphic to) $K \rtimes K^*$. Assume that $k = [0, 0, 1]$ and $l = [-1, 0, 0]$ are as above. For all $(a, b) \in K \rtimes K^*$, $(x, y, z) \mapsto (bx, by + az, z)$ induces an automorphism F_{ab} of $\mathbb{P}^2(K)$ such that if fixes l and every element of k. Also an easy calculation shows that $(a, b) \mapsto F_{ab} \restriction l$ is an action and of rank 2 (in fact, these are the automorphisms that move any two distinct points of $l - \{\infty\}$ to any other two distinct points of $l - \{\infty\}$). Thus $K \rtimes K^*$ is a subgroup of G. As seen above, the action of G is 2-determined and so $G = K \rtimes K^*$.

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